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ORIGINAL PAPER

Hamiltonian L-stability of Lagrangian translating solitons

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Abstract In this paper, we study the Hamiltonian L-stability of Lagrangian translating solitons to the mean curvature flow. We prove that any Lagrangian translating soliton is Hamiltonian L-stable.

Keywords Translating soliton · Lagrangian translating soliton · Hamiltonian L-stable

Mathematics Subject Classification (2000) 53C44 (primary) · 53C21 (secondary)

1 Introduction

An n-dimensional submanifold Σ^n of $\mathbf{R}^{n+p}$ is called a self-shrinker if it is the time $t = -1$ slice of a self-shrinking mean curvature flow that disappears at (0, 0), i.e., of a mean curvature flow satisfying $\Sigma_t = \sqrt{-t}\,\Sigma_{-1}$. We can also consider a self-shrinker as a submanifold that satisfies

$$H = -\frac{1}{2}x^{\perp}.$$

An n-dimensional submanifold Σ^n of $\mathbf{R}^{n+p}$ is called a translating soliton if there is a constant vector T so that $\Sigma_t = \Sigma + tT$ is a solution to the mean curvature flow. We can also consider a translating soliton as a submanifold that satisfies

$$H = T^{\perp}.$$

According to the blow up rate of the second fundamental form, Huisken [6] classified the singularities of mean curvature flows into two types: Type I and Type II. Any Type I singularity of the mean curvature flow must be a self-shrinker [6]. Type II singularity is one

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class of eternal solutions, which is defined for $-\infty < t < \infty$. One of the most important examples of Type II singularity is the translating soliton [5,7].

In this paper, we mainly study the stability (in some sense) of translating solitons. This was motivated by the work of Colding–Minicozzi [4], where they introduced the concept of F-stability of an n-dimension self-shrinker in $\mathbf{R}^{n+1}$.

Given $x_0 \in \mathbf{R}^{n+1}$ and $t_0 > 0$, F_{x_0,t_0} is defined by

$$F_{x_0,t_0}(\Sigma) = (4\pi t_0)^{-\frac{n}{2}} \int_\Sigma e^{-\frac{|x-x_0|^2}{4t_0}} d\mu. \tag{1.1}$$

In [4], Colding–Minicozzi proved that self-shrinkers are the critical points for the $F_{0,1}$ functional by computing the first variation formula of $F_{0,1}$. They also computed the second variation formula, and defined F-stability of a self-shrinker by modding out translations in space and time. They showed that the round sphere and hyperplanes are the only F-stable self-shrinkers in $\mathbf{R}^{n+1}$.

In 2012, Andrews et al. [1], Arezzo and Sun [2] and Lee and Lue [8] independently generalized some of Colding–Minicozzi's work [4] from the hypersurface case to higher codimensional cases. They computed the first and second variation formulas for the F-functional, and studied F-stability of self-shrinkers in higher codimension.

Recently, motivated by an observation by Oh [10], Li and Zhang [9] and Yang [12] studied the Lagrangian F-stability and Hamiltonian F-stability of Lagrangian self-shrinkers, and proved characterization theorems for Hamiltonian F-stability of Lagrangian self-shrinkers, which characterize the Hamiltonian F-stablity by the eigenvalues and eigenspaces of the drifted Laplacian.

With the above known results for self-shrinkers, it is natural to think that translating solitons might also have some similar properties. In fact, translating solitons are also critical points for an F-functional (cf. [3,11,13]). The F-functional is defined by

$$F(\Sigma) = \int_\Sigma e^{\langle T,x\rangle} d\mu.$$

Note that the F-functional (1.1) for a self-shrinker is finite as long as the self-shrinker has polynomial volume growth, while here $F(\Sigma)$ is usually infinity if Σ is a translating soliton, since any translating soliton is noncompact and $e^{\langle T,x\rangle} \to \infty$ very quickly as $x \to \infty$. This makes it hard to get as many corresponding results in the translating soliton case as in the self-shrinker case.

However, if we require variation vector fields to have compact supports, one can still compute the first and second variation formulas of F, and consider L-stability (see Definition 2.1) of translating solitons. In [11], Shahriyari defined L-stability of translating surfaces in $\mathbf{R}^3$, and proved that any translating graph in $\mathbf{R}^3$ is L-stable. In [3], Arezzo and Sun computed variation formulas of the F-functional for general conformal solitons, which include translating solitons as special cases. They proved that the grim-reaper cylinder $\mathbf{R}^{n-1} \times \Gamma$ in $\mathbf{R}^{n+1}$ is an L-stable translating soliton, where Γ is the grim reaper in $\mathbf{R}^2$.

Especially, if a translating soliton is also a Lagrangian submanifold of the Euclidean space, we call it a Lagrangian translating soliton. In this paper, we consider Hamiltonian L-stability (see Definition 2.3) of Lagrangian translating solitons. Our main theorem is

Theorem 1.1 *Any Lagrangian translating soliton is Hamiltonian L-stable.*

Since we have this theorem, an interesting question is whether this theorem has some application that could help us study the Lagrangian mean curvature flow. Besides, we are also interested in the L-stability of translating solitons in the hypersurface case.

2 Variation formulas and Hamiltonian L-stability

2.1 Variation formulas and L-stability

In this subsection, we sketch the first and second variation formulas of the F-functional as well as the definition of the L-stability of translating solitons (see also [3,11]). Recall that the F-functional is defined by

$$F(\Sigma) = \int_{\Sigma} e^{\langle T,x\rangle} d\mu.$$

The first variation formula of F is

Lemma 2.1 *Let $\Sigma_s \subset \boldsymbol{R}^{n+p}$ be a compactly supported variation of Σ with normal variation vector field V, then*

$$\frac{\partial}{\partial s}(F(\Sigma_s)) = \int_{\Sigma} \langle T^{\perp} - H, V\rangle e^{\langle T,x\rangle} d\mu_{\Sigma}. \tag{2.1}$$

Proof From the first variation formula (for area), we know that

$$(d\mu)' = -\langle H, V\rangle d\mu.$$

It follows that

$$\frac{\partial}{\partial s}(F(\Sigma_s)) = \int_{\Sigma} e^{\langle T,x\rangle}\langle T, V\rangle d\mu - \int_{\Sigma} e^{\langle T,x\rangle}\langle H, V\rangle d\mu = \int_{\Sigma}\langle T^{\perp} - H, V\rangle e^{\langle T,x\rangle} d\mu.$$

This proves the lemma. □

It follows that

Proposition 2.2 *Σ is a critical point for F if and only if $H = T^{\perp}$.*

The second variation formula at a critical point is

Theorem 2.3 *Suppose that Σ is a critical point for F. If Σ_s is a compactly supported normal variation of Σ, and*

$$\partial_s\Big|_{s=0}\Sigma_s = V,$$

then setting $F'' = \partial_{ss}\Big|_{s=0}(F(\Sigma_s))$ gives

$$F'' = \int_{\Sigma} -\langle V, LV\rangle e^{\langle T,x\rangle} d\mu, \tag{2.2}$$

where

$$\begin{aligned} LV &= \Delta^{\perp} V + \nabla^{\perp}_{T^T} V + \big\langle \langle A, V\rangle, A\big\rangle \\ &= \left(\Delta V^{\alpha} + \langle T, \nabla V^{\alpha}\rangle + g^{ik}g^{jl}V^{\beta}h^{\beta}_{ij}h^{\alpha}_{kl}\right) e_{\alpha}. \end{aligned}$$

Proof We sketch the proof here. Letting primes denote derivatives with respect to s at $s = 0$, differentiating (2.1) gives

$$
\begin{aligned}
F'' &= \int_\Sigma \left\{ \frac{\partial}{\partial s}\Big|_{s=0} (\langle T - H, V\rangle) + \langle T^\perp - H, V\rangle^2\Big|_{s=0} \right\} e^{\langle T,x\rangle} \\
&= \int_\Sigma \left\{ -\langle H', V\rangle + \langle T - H, V'\rangle \right\} e^{\langle T,x\rangle}.
\end{aligned} \tag{2.3}
$$

Similar to the derivation of the second variation formula for the area, we have

$$\langle H', V\rangle = \langle \Delta^\perp V + g^{ik} g^{jl} V^\beta h^\beta_{ij} h^\alpha_{kl} e_\alpha, V\rangle. \tag{2.4}$$

On the other hand, since $\langle [V, T^T], V\rangle = 0$, it follows that

$$
\begin{aligned}
\langle T - H, V'\rangle &= \langle T - H, \overline{\nabla}^T_V V\rangle = \langle T, \overline{\nabla}^T_V V\rangle = \langle T^T, \overline{\nabla}_V V\rangle = -\langle \overline{\nabla}_V T^T, V\rangle \\
&= -\langle \overline{\nabla}_{T^T} V, V\rangle = -\langle \nabla^\perp_{T^T} V, V\rangle.
\end{aligned} \tag{2.5}
$$

Putting (2.4) and (2.5) into (2.3) gives (2.2). This proves the theorem. □

Thus we are led to the following definition.

Definition 2.1 We say a translating soliton Σ is L-stable if for every compactly supported variations Σ_s with $\Sigma_0 = \Sigma$, $F'' = \int_\Sigma -\langle V, LV\rangle e^{\langle T,x\rangle} \geq 0$.

2.2 Properties of $\mathcal{L}$ and L

Notice that $T = T^T + H$, where $T^T = \nabla\langle T, x\rangle$, the linear operator defined by

$$\mathcal{L}v = \Delta v + \langle T, \nabla v\rangle = e^{-\langle T,x\rangle} div_\Sigma \left(e^{\langle T,x\rangle} \nabla v \right) \tag{2.6}$$

is self-adjoint in a weighted L^2 space. More precisely, we have the following lemma which follows immediately from Stokes' theorem and (2.6).

Lemma 2.4 *If $\Sigma \subset \boldsymbol{R}^{n+p}$ is a submanifold of $\boldsymbol{R}^{n+p}$, u is a C^1 function with compact support, and v is a C^2 function, then*

$$\int_\Sigma u(\mathcal{L}v) e^{\langle T,x\rangle} = -\int_\Sigma \langle \nabla v, \nabla u\rangle e^{\langle T,x\rangle} \tag{2.7}$$

The next corollary is an extension of Lemma 2.4, which follows by choosing cut-off functions and applying the dominated convergence theorem. The proof is similar to that of Corollary 3.10 in [4].

Corollary 2.5 *Suppose that $\Sigma \subset \boldsymbol{R}^{n+p}$ is a complete submanifold of $\boldsymbol{R}^{n+p}$ without boundary. If u, v are C^2 functions with*

$$\int_\Sigma (|u\nabla v| + |\nabla u||\nabla v| + |u\mathcal{L}v|) e^{\langle T,x\rangle} < \infty,$$

then

$$\int_\Sigma u(\mathcal{L}v) e^{\langle T,x\rangle} = -\int_\Sigma \langle \nabla v, \nabla u\rangle e^{\langle T,x\rangle}.$$

Now we compute some equalities that may be useful in the future. Denote by x^A $(A = 1, 2, \ldots, x + p)$ the coordinate functions of Σ in $\mathbf{R}^{n+p}$, i.e. x^A is the A-th component of the position vector x, then

Proposition 2.6 *If $\Sigma^n \subset \boldsymbol{R}^{n+p}$ is a translating soliton, then for every constant vector field y,*

$$Ly^\perp = 0. \tag{2.8}$$

Especially, choosing $y = T$, we get

$$LH = 0. \tag{2.9}$$

Moreover, we have

$$\mathcal{L}x^A = T^A. \tag{2.10}$$

Proof Fix $p \in \Sigma$ and choose an orthonormal frame $\{e_i\}$ such that $\nabla_{e_i} e_j(p) = 0$, $g_{ij} = \delta_{ij}$ in a neighborhood of p. We have

$$\nabla^\perp_{e_i} y^\perp = \nabla^\perp_{e_i}(y - \langle y, e_j\rangle e_j) = -\langle y, e_j\rangle h^\alpha_{ij} e_\alpha. \tag{2.11}$$

Especially, choosing $y = T$, we have

$$\nabla^\perp_{e_i} H = \nabla^\perp_{e_i} T^\perp = -\langle T, e_j\rangle h^\alpha_{ij} e_\alpha, \tag{2.12}$$

i.e.,

$$H^\alpha_{,i} = -\langle T, e_j\rangle h^\alpha_{ij}. \tag{2.13}$$

Taking another covariant derivative at p, it gives

$$\begin{aligned}
\nabla^\perp_{e_k}\nabla^\perp_{e_i} y^\perp &= - e_k\langle y, e_j\rangle h^\alpha_{ij} e_\alpha - \langle y, e_j\rangle h^\alpha_{ij,k} e_\alpha \\
&= - \langle y, h^\beta_{kj} e_\beta\rangle h^\alpha_{ij} e_\alpha - \langle y, e_j\rangle h^\alpha_{ik,j} e_\alpha,
\end{aligned} \tag{2.14}$$

where we used (2.11), $\nabla_{e_k} e_j(p) = 0$, and the Codazzi equation in the last equality. Taking the trace of (2.14) and using $H = T^\perp$, we conclude that

$$\begin{aligned}
\Delta^\perp y^\perp &= - \langle y, h^\beta_{ij} e_\beta\rangle h^\alpha_{ij} e_\alpha - \langle y, e_j\rangle H^\alpha_{,j} e_\alpha \\
&= - y^\beta h^\beta_{ij} h^\alpha_{ij} e_\alpha + \langle y, e_j\rangle\langle T, e_i\rangle h^\alpha_{ij} e_\alpha \\
&= - y^\beta h^\beta_{ij} h^\alpha_{ij} e_\alpha - \langle T, e_i\rangle \nabla^\perp_{e_i} y^\perp \\
&= - y^\beta h^\beta_{ij} h^\alpha_{ij} e_\alpha - \nabla^\perp_{T^T} y^\perp.
\end{aligned}$$

This proves (2.8).

Since $\Delta x = H$ and $H = T^\perp$, we have

$$\Delta x^A = \langle H, E_A\rangle = \langle T^\perp, E_A\rangle = \langle T, E^\perp_A\rangle = \langle T, E_A\rangle - \langle T, E^T_A\rangle = T^A - \langle T, E^T_A\rangle.$$

Hence

$$\mathcal{L}x^A = \Delta x^A + \langle T, \nabla x^A\rangle = \Delta x^A + \langle T, (E_A)^T\rangle = T^A,$$

This proves (2.10). □

2.3 Hamiltonian L-stability

In this subsection, we will define Hamiltonian L-stability of Lagrangian translating solitons. Recall the definition of Hamiltonian variations on a Lagrangian submanifold.

Definition 2.2 [10] Let $(M, \bar{\omega})$ be a symplectic manifold M. Let $\Sigma \subset M$ be a Lagrangian submanifold and V be a vector field along Σ. V is called a Hamiltonian variation if the one form $i^*(V \rfloor \bar{\omega})$ on Σ is exact.

The Hamiltonian variation has an equivalent definition.

Lemma 2.7 [10] *A normal variation V on Σ is Hamiltonian if and only if*

$$V = J\nabla f,$$

where f is a function on Σ and ∇ is the gradient on Σ with respect to the induced metric.

Now we define Hamiltonian L-stability of Lagrangian translating solitons.

Definition 2.3 We say a Lagrangian translating soliton Σ is Hamiltonian L-stable if for every compactly supported Hamiltonian variations Σ_s with $\Sigma_0 = \Sigma$, $F'' = \int_\Sigma -\langle V, LV\rangle e^{\langle T,x\rangle} \geq 0$.

3 Proof of the main theorem

Note that the normal bundle brings much difficulty to the study of L-stability of translating solitons in the general higher codimension case. However, in [10], Oh studied Hamiltonian stability of minimal Lagrangian submanifolds in Kähler–Einstein manifolds, and characterized Hamiltonian stability by a condition on the first eigenvalue of Δ acting on functions. The key point of Oh's proof is an observation that for a minimal Lagrangian submanifold of a Kähler–Einstein manifold, the set of Hamiltonian variations is an invariant subspace of the Jacobi operator. This idea was recently used to study Hamiltonian F-stability of Lagrangian self-shrinkers [9,12]. It is natural to think that this property also holds for Lagrangian translating solitons. This property inspired us to show the following equality, which well characterizes how the operator L acts on Hamiltonian variations.

Theorem 3.1 *Suppose $\Sigma^n \subset \boldsymbol{C}^n$ is a Lagrangian translating soliton. Then for every function f on Σ,*

$$LJ\nabla f = J\nabla \mathcal{L} f. \tag{3.1}$$

This implies that the set of Hamiltonian variations is an invariant subspace of the operator L.

Proof Fix a point p. We choose a local orthonormal basis $\{e_i\}_{i=1}^n$ of $T\Sigma$ such that $\nabla_{e_i} e_j(p) = 0$. Then since Σ is Lagrangian, $\{e_{n+i} = Je_i\}_{i=1}^n$ is a local orthonomal basis of $N\Sigma$. In the following we compute at the point p. It is easy to compute that

$$\begin{aligned} LJ\nabla f &= \Delta^\perp (J\nabla f) + \nabla^\perp_{T^T}(J\nabla f) + h^{n+k}_{il} f_k h^{n+j}_{il} Je_j \\ &= \left(f_{jii} + \langle T, e_k\rangle f_{jk} + f_k h^{n+l}_{ik} h^{n+l}_{ij} \right) Je_j, \end{aligned}$$

where in the last equality we used the Lagrangian property $h_{il}^{n+k} = h_{ik}^{n+l}$. On the other hand,

$$\begin{aligned} J\nabla\mathcal{L}f &= J\nabla\left(\Delta f + T^T f\right) \\ &= f_{iij}Je_j + J\nabla\langle T^T, \nabla f\rangle \\ &= \left(f_{iji} - f_i R_{jkik} + e_j\langle T^T, \nabla f\rangle\right) Je_j \\ &= \left(f_{jii} - f_i h_{ij}^{n+l} h_{kk}^{n+l} + f_i h_{jk}^{n+l} h_{ik}^{n+l} + \langle\nabla_{e_j} T^T, \nabla f\rangle + \langle T^T, \nabla_{e_j}\nabla f\rangle\right) Je_j \\ &= \Big(f_{jii} - f_i h_{ij}^{n+l}\langle T^\perp, e_{n+l}\rangle + f_k h_{ik}^{n+l} h_{ij}^{n+l} + \langle\overline{\nabla}_{e_j} T, \nabla f\rangle - \langle\overline{\nabla}_{e_j} T^\perp, \nabla f\rangle \\ &\quad + f_{jk}\langle T, e_k\rangle\Big) Je_j \\ &= \left(f_{jii} - f_i h_{ij}^{n+l}\langle T^\perp, e_{n+l}\rangle + f_k h_{ik}^{n+l} h_{ij}^{n+l} + \left\langle T^\perp, \overline{\nabla}_{e_j}(f_k e_k)\right\rangle + \langle T, e_k\rangle f_{jk}\right) Je_j \\ &= \left(f_{jii} - f_i h_{ij}^{n+l}\langle T^\perp, e_{n+l}\rangle + f_k h_{ik}^{n+l} h_{ij}^{n+l} + f_k\langle T^\perp, h_{jk}^{n+l} e_{n+l}\rangle + \langle T, e_k\rangle f_{jk}\right) Je_j \\ &= \left(f_{jii} + \langle T, e_k\rangle f_{jk} + f_k h_{ik}^{n+l} h_{ij}^{n+l}\right) Je_j, \end{aligned}$$

where in the third equality we used the Ricci formula; in the fourth equality we used the Gauss equation; and in the fifth equality we used the translating soliton equation $H = T^\perp$. This proves the theorem. □

Now we prove our main theorem.

Theorem 3.2 *Any Lagrangian translating soliton is Hamiltonian L-stable.*

Proof Recall that the second variation formula for F is

$$F'' = \int_\Sigma -\langle V, LV\rangle e^{\langle T,x\rangle}. \tag{3.2}$$

Now Assume V is a compactly supported Hamiltonian variation, then there exists a function f, such that $V = J\nabla f$. Putting it into (3.2), and using (3.1), we get

$$F'' = \int_\Sigma -\langle J\nabla f, LJ\nabla f\rangle e^{\langle T,x\rangle} = \int_\Sigma -\langle J\nabla f, J\nabla\mathcal{L}f\rangle e^{\langle T,x\rangle} = \int_\Sigma (\mathcal{L}f)^2 e^{\langle T,x\rangle} \geq 0, \tag{3.3}$$

where the last equality used Lemma 2.4 and the fact that $V = J\nabla f$ is compactly supported implies that $\mathcal{L}f$ is compactly supported. This proves the theorem. □

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