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ORIGINAL PAPER

A new proof of the Log-Brunn–Minkowski inequality

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Abstract Recently Böröczky, Lutwak, Yang and Zhang have proved the log-Brunn–Minkowski inequality which is stronger than the classical Brunn–Minkowski inequality for two origin-symmetric convex bodies in the plane. This paper presents a new proof of this inequality and proves the uniqueness of the cone-volume measure by using the log-Minkowski inequality.

Keywords L_0-Minkowski addition · Log-Minkowski inequality · Log-Brunn–Minkowski inequality · Minkowski mixed-volume inequality

Mathematical Subject Classification (2000) 52A40

1 Introduction

Let $\mathcal{K}^n$ be the set of all convex bodies, i.e., compact convex sets with non-empty interior, in the n dimensional Euclidean space $\mathbb{E}^n$. The volume of a set $M \subset \mathbb{E}^n$, is denoted by $V(M)$. For two convex bodies $K, L \in \mathcal{K}^n$, the Minkowski addition is denoted $K + L = \{x + y : x \in K, y \in L\}$, and the classical Brunn–Minkowski inequality states that

$$V(K+L)^{\frac{1}{n}} \geq V(K)^{\frac{1}{n}} + V(L)^{\frac{1}{n}}, \tag{1.1}$$

with equality if and only if K and L are homothetic. It became an extremely powerful tool in convex geometry with significant applications to various other areas of mathematics. Gardner's article [8] and Schneider's classic text [26] are excellent survey and source of references.

In the early 1960s, Firey [6] extended the Minkowski addition of convex bodies to L_p-Minkowski addition for each $p \geq 1$ (now known as Firey-Minkowski L_p-addition). Fur-

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thermore, he also established the L_p-Brunn–Minkowski inequality (an inequality that is also known as the Brunn–Minkowski–Firey inequality), which states as follows:

$$V(K +_p L)^{\frac{p}{n}} \geq V(K)^{\frac{p}{n}} + V(L)^{\frac{p}{n}}, \tag{1.2}$$

with equality if and only if K and L are dilatates. In the mid 1990s, it was shown in [14,15], that when Firey-Minkowski L_p-addition is combined with volume the result is an embryonic L_p-Brunn–Minkowski theory. This theory has expanded rapidly. (See e.g.[16–21]).

Recently, Böröczky, Lutwak, Yang and Zhang in [4] defined the L_0-Minkowski addition of convex bodies, for $0 \leq \lambda \leq 1$,

$$(1-\lambda) \cdot K +_0 \lambda \cdot L = \bigcap_{u \in S^{n-1}} \{x \in \mathbb{E}^n : x \cdot u \leq h_K(u)^{1-\lambda} h_L(u)^{\lambda}\}, \tag{1.3}$$

where h_K and h_L are the support functions of K and L and conjectured that for origin-symmetric convex bodies K and L in $\mathbb{E}^n$ and $0 \leq \lambda \leq 1$,

$$V((1-\lambda) \cdot K +_0 \lambda \cdot L) \geq V(K)^{1-\lambda} V(L)^{\lambda}. \tag{1.4}$$

They call (1.4) the log-Brunn–Minkowski inequality and note that while it is not true for general convex bodies, it implies the classical Brunn–Minkowski inequality for origin-symmetric convex bodies. In [4], (1.4) is proved when $n = 2$, and it is also shown that for all n, (1.4) is equivalent to the log-Minkowski inequality

$$\int_{S^{n-1}} \log\left(\frac{h_L}{h_K}\right) dV_K \geq V(K) \log\left(\frac{V(L)}{V(K)}\right)^{\frac{1}{n}}, \tag{1.5}$$

where V_K is the cone-volume measure of K.

In fact, Böröczky, Lutwak, Yang and Zhang in [4] first proved the uniqueness of the cone-volume measure, then used the uniqueness of the cone-volume measure to prove the log-Brunn-Minkowski inequality. The cone-volume measure of a convex body is defined as if K is a convex body in $\mathbb{E}^n$ that contains the origin in its interior, then the cone-volume measure, V_K, of K is a Borel measure on the unit sphere S^{n-1} defined for a Borel set $\omega \in S^{n-1}$, by

$$V_K(\omega) = \frac{1}{n} \int_{x \in \nu_K^{-1}(\omega)} x \cdot \nu_K(x) d\mathcal{H}^{n-1}(x), \tag{1.6}$$

where $\nu_K : \partial K \to S^{n-1}$ is the Gauss map of K, defined on ∂K, the set of points of ∂K that have a unique outer unit normal, and $\mathcal{H}^{n-1}$ is $(n-1)$-dimensional Hausdorff measure. In recent years, cone-volume measures have appeared in e.g. [3,5,10,12,13,23–25].

In this paper, we present a new proof for the log-Minkowski inequality in the plane, and then prove the uniqueness of the cone-volume measure of a convex body in the plane by using the log-Minkowski inequality.

2 Preliminaries

In this section, we will review some basic facts in convex geometry, which are useful for our later discussion. Good general references for the theory of convex bodies are provided by the books of Gardner [9], Gruber [11], Leichtweiss [22], Schneider [26], and Thompson [29]. Firstly, we will give the definition of a compact convex set and its support function.

If $K \subseteq \mathcal{K}^n$, its *support function* $h(K, \cdot) : \mathbb{E}^n \to \mathbb{E}$ is defined by

$$h_K(u) = \max\{x \cdot u : x \in K\}. \tag{2.1}$$

A convex body K in $\mathbb{E}^n$ is origin-symmetric if and only if

$$h_K(u) = h_K(-u), \tag{2.2}$$

for all $u \in S^{n-1}$.

Let K be a convex body in $\mathbb{E}^n$ that contains the origin in its interior, then the cone-volume measure, V_K, of K is a Borel measure on the unit sphere S^{n-1} defined for a Borel set $\omega \in S^{n-1}$, by

$$V_K(\omega) = \frac{1}{n} \int\limits_{x \in \nu_K^{-1}(\omega)} x \cdot \nu_K(x) d\mathcal{H}^{n-1}(x). \tag{2.3}$$

There are formulas

$$V_K = \frac{1}{n} h_K S_K,$$

and

$$V(K) = \frac{1}{n} \int\limits_{S^{n-1}} h_K(u) dS_K(u). \tag{2.4}$$

Let $K, L \subseteq \mathcal{K}^n$ and if h_K , h_L are the support functions of K and L, the Minkowski addition can be equivalently defined as the compact convex set such that

$$h_{K+L} = h_K + h_L. \tag{2.5}$$

The Minkowski–Steiner formulas state that

$$V(K + tL) = \sum_{i=0}^{n} \binom{n}{i} V_i(K, L) t^i, \tag{2.6}$$

and

$$V_i(K + tL, L) = \sum_{j=0}^{n-i} \binom{n-i}{j} V_{i+j}(K, L) t^j, \tag{2.7}$$

where $t \geq 0$,

and

$$V_i(K, L) = V(\underbrace{K, \cdots, K}_{n-i}, \underbrace{L, \cdots, L}_{i}),$$

is called the the i-th mixed volume of K and L. In particular, we have $V_0(K, L) = V(K)$, $V_n(K, L) = V(L)$ and $V_i(K, L) = V_{n-i}(L, K)$.

Differentiating both sides of (2.6) with respect to t, and using (2.7), we have

$$V'(K + tL) = \sum_{i=1}^{n} i \binom{n}{i} V_i(K, L) t^{i-1} = n V_1(K + tL, L). \tag{2.8}$$

In the planar case, write $V_1(K, L)$ as $V(K, L)$. Then the formula (2.6) becomes

$$V(K + tL) = V(K) + 2V(K, L)t + V(L)t^2, \tag{2.9}$$

where $V(K, L) = V(L, K)$. So

$$V'(K + tL) = 2V(K, L) + 2V(L)t = 2V(K + tL, L). \tag{2.10}$$

3 The Log-Brunn–Minkowski inequality

The following lemma, proved in [4], will be needed.

Lemma 3.1 *If K, L are origin-symmetric plane convex bodies, then*

$$\int_{S^1} \frac{h_K(u)}{h_L(u)} dV_K(u) \le \frac{V(K)V(L, K)}{V(L)}. \tag{3.1}$$

with equality if and only if K and L are dilates, or K and L are parallelograms with parallel sides.

Theorem 3.1 *If K and L are planar origin-symmetric convex bodies, then*

$$\int_{S^1} \log\left(\frac{h_L}{h_K}\right) dV_K \ge V(K)\log\left(\frac{V(L)}{V(K)}\right)^{\frac{1}{2}}. \tag{3.2}$$

with equality if and only if K and L are dilates, or K and L are parallelograms with parallel sides.

Proof Let

$$F(t) = \int_{S^1} \log\left(\frac{h_{L+tK}}{h_K}\right) dV_K - V(K)\log\left(\frac{V(L+tK)}{V(K)}\right)^{\frac{1}{2}}. \tag{3.3}$$

Now, differentiating $F(t)$ with respect to t, we have

$$\begin{aligned}
F'(t) &= \frac{d}{dt}\left(\int_{S^1} \log\left(\frac{h_{L+tK}}{h_K}\right) dV_K - V(K)\log\left(\frac{V(L+tK)}{V(K)}\right)^{\frac{1}{2}}\right)\\
&= \frac{d}{dt}\left(\int_{S^1} \log\left(\frac{h_L + th_K}{h_K}\right) dV_K - V(K)\log\left(\frac{V(L) + 2V(L, K)t + V(K)t^2}{V(K)}\right)^{\frac{1}{2}}\right)\\
&= \int_{S^1} \frac{h_K}{h_L + th_K} dV_K - \frac{V(K)\left(V(L, K) + V(K)t\right)}{V(L) + 2V(L, K)t + V(K)t^2}\\
&= \int_{S^1} \frac{h_K(u)}{h_{L+tK}(u)} dV_K - \frac{V(K)V(L+tK, K)}{V(L+tK)}.
\end{aligned}$$

Thus, from Lemma 3.1 we get

$$F'(t) = \int_{S^1} \frac{h_K(u)}{h_{L+tK}(u)} dV_K - \frac{V(K)V(K, L+tK)}{V(L+tK)} \le 0. \tag{3.4}$$

This shows that $F(t)$ is decreasing on $[0, +\infty)$.
Since

$$F(t) = \int_{S^1} \log\left(\frac{h_{L+tK}}{h_K}\right) dV_K - \int_{S^1} \log\left(\frac{V(L+tK)}{V(K)}\right)^{\frac{1}{2}} dV_K$$
$$= \int_{S^1} \log\left(\frac{h_{L+tK}}{h_K}\sqrt{\frac{V(K)}{V(L+tK)}}\right) dV_K. \tag{3.5}$$

By mean value theorem for integrals there exists $\xi \in S^1$ such that

$$F(t) = V(K)\log\left(\frac{h_{L+tK}(\xi)}{h_K(\xi)}\sqrt{\frac{V(K)}{V(L+tK)}}\right)$$
$$= V(K)\log\left(\frac{h_L(\xi)+th_K(\xi)}{h_K(\xi)}\sqrt{\frac{V(K)}{V(L)+2V(L,K)t+V(K)t^2}}\right).$$

Then

$$\lim_{t\to+\infty} F(t) = \lim_{t\to+\infty} V(K)\log\left(\frac{h_L(\xi)+th_K(\xi)}{h_K(\xi)}\sqrt{\frac{V(K)}{V(L)+2V(L,K)t+V(K)t^2}}\right)$$
$$= V(K)\lim_{t\to+\infty}\log\left(\frac{h_L(\xi)+th_K(\xi)}{h_K(\xi)}\sqrt{\frac{V(K)}{V(L)+2V(L,K)t+V(K)t^2}}\right)$$
$$= V(K)\cdot\log(1) = 0. \tag{3.6}$$

Thus

$$F(t) \geq 0, \tag{3.7}$$

in particular,

$$F(0) \geq 0. \tag{3.8}$$

This yields the desired inequality (3.2).

We will now establish the equality conditions in (3.2). To that end, suppose:

$$F(t) = \int_{S^1} \log\left(\frac{h_{L+tK}}{h_K}\right) dV_K - V(K)\log\left(\frac{V(L+tK)}{V(K)}\right)^{\frac{1}{2}} = 0. \tag{3.9}$$

Therefore,

$$F'(t) = \int_{S^1} \frac{h_K(u)}{h_{L+tK}(u)} dV_K - \frac{V(K)V(K, L+tK)}{V(L+tK)} = 0. \tag{3.10}$$

Conversely, if $F'(t) = 0$. Then, there exists a constant C such that $F(t) = C$.
However, by (3.6), this means that

$$F(t) = \int_{S^1} \log\left(\frac{h_{L+tK}}{h_K}\right) dV_K - V(K)\log\left(\frac{V(L+tK)}{V(K)}\right)^{\frac{1}{2}} = C = 0. \tag{3.11}$$

From the above discussion, we have $F(t) = 0$ if and only if $F'(t) = 0$.

Thus, from Lemma 3.1 we get $F(t) = 0$ if and only if K and $L + tK$ are dilates, or K and $L + tK$ are parallelograms with parallel sides.

In particular, when $t = 0$, We can see that $F(0) = 0$ if and only if K and L are dilates, or K and L are parallelograms with parallel sides.

It is easily seen that the equality holds in (3.2) if and only if K and L are dilates, or K and L are parallelograms with parallel sides. □

In [4], Böröczky, Lutwak, Yang and Zhang also showed that the following lemma is true.

Lemma 3.2 *For origin symmetric convex bodies in* $\mathbb{E}^n$, *the log-Brunn–Minkowski inequality* (1.4) *and the log-Minkowski inequality* (1.5) *are equivalent.*

Theorem 3.2 *If* K *and* L *are origin-symmetric convex bodies in the plane, then for all real* $\lambda \in [0, 1]$,

$$V((1-\lambda)\cdot K +_0 \lambda \cdot L) \geq V(K)^{1-\lambda} V(L)^{\lambda}. \tag{3.12}$$

When $\lambda \in (0, 1)$, *equality in the inequality holds if and only if* K *and* L *are dilates or* K *and* L *are parallelograms with parallel sides.*

4 Uniqueness question for planar cone-volume measures

Given a finite Borel measure on the unit sphere, under what necessary and sufficient conditions is the measure the cone-volume measure of a convex body? This is the existence question for the unsolved log-Minkowski problem. It requires solving a Monge-Ampère equation and is connected with some important curvature flows (see e.g. [1,2,7,27]). The uniqueness question for the log-Minkowski problem asks under what conditions can two different bodies have identical cone-volume measures. The uniqueness question appears to be more difficult than the existence question. In the plane, Gage [7] showed that within the class of origin-symmetric plane convex bodies that are smooth and have positive curvature, the cone-volume measure determines the convex body uniquely. For even discrete measures, the uniqueness question for the log-Minkowski problem, for plane convex bodies, was treated by Stancu [27]. The uniqueness question for the log-Minkowski problem for arbitrary origin-symmetric plane convex bodies was settled by Böröczky, Lutwak, Yang and Zhang [5]. For convex bodies in $\mathbb{E}^n$, the problem remains open.

The following theorem was established by Gage [7] when the convex bodies are smooth and have positive curvature. When the convex bodies are polytopes it is due to Stancu [28]. The proof of the general case is due to Böröczky, Lutwak, Yang and Zhang [4]. Here, we will prove it by the log-Minkowski inequality.

Theorem 4.1 *If* K *and* L *are planar origin-symmetric convex bodies that have the same cone-volume measure, then either* $K = L$ *or else* K *and* L *are parallelograms with parallel sides.*

Proof Assume that $K \neq L$.

Since

$$V_K = V_L,$$

it follows that $V(K) = V(L)$. Thus, since $K \neq L$, the bodies can not be dilates.

By the log-Minkowski inequality (3.2), we have

$$\int_{S^1} \log(h_L)\, dV_K \geq \int_{S^1} \log(h_K)\, dV_K, \tag{4.1}$$

and

$$\int_{S^1} \log(h_K)\, dV_L \geq \int_{S^1} \log(h_L)\, dV_L, \tag{4.2}$$

with equality, in either inequality, if and only if K and L are parallelograms with parallel sides. Using (4.1), $V_K = V_L$, (4.2) and $V_L = V_K$, we get

$$\begin{aligned}\int_{S^1} \log(h_L)\, dV_K &\geq \int_{S^1} \log(h_K)\, dV_K = \int_{S^1} \log(h_K)\, dV_L \\ &\geq \int_{S^1} \log(h_L)\, dV_L = \int_{S^1} \log(h_L)\, dV_K \end{aligned} \tag{4.3}$$

Thus, we have equalities in both inequalities of (4.1) and (4.2) By the equality conditions of (4.1) and (4.2), we conclude that K and L are parallelograms with parallel sides.

5 Open problems

The following conjecture was made by Böröczky, Lutwak, Yang and Zhang in [4].

Conjecture 1 *If K and L are origin-symmetric convex bodies in $\mathbb{E}^n$, then*

$$\int_{S^{n-1}} \log\left(\frac{h_L}{h_K}\right) dV_K \geq V(K) \log\left(\frac{V(L)}{V(K)}\right)^{\frac{1}{n}}. \tag{5.1}$$

We conjecture that the following result is true.

Conjecture 2 *If K, L are origin-symmetric convex bodies in $\mathbb{E}^n$, then*

$$\int_{S^{n-1}} \frac{h_K}{h_L} dV_K \leq \frac{V(K)V_1(L, K)}{V(L)}. \tag{5.2}$$

Remark 1 In fact, if Conjecture 2 is true, by (2.8) and using methods similar to the proof of Theorem 3.1, we can prove that Conjecture 1 is also true.

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