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Mean curvature decay in symplectic and lagrangian translating solitons

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Abstract In this note, we prove that the infimum of the norm of the mean curvature vector on a symplectic translating soliton or an almost-calibrated Lagrangian translating soliton must be zero.

Keywords Translating soliton · Symplectic · Lagrangian

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1 Introduction

In recent years, symplectic mean curvature flow and Lagrangian mean curvature flow have attracted much attention. Chen and Li [1,2] and Wang [15] proved that there is no finite time Type-I singularity for symplectic mean curvature flow and almost-calibrated Lagrangian mean curvature flow. Therefore, it is important to study the properties of Type-II singularity. It is well known that [7,10,11,16] translating solitons play important role in classifying Type-II singularity of mean curvature flow. Thus, we need to study translating solitons to symplectic and Lagrangian mean curvature flows.

Recall that a surface Σ^n in $\mathbf{R}^{n+k}$ is called a *translating soliton* (or *translator*) of the mean curvature flow, if it satisfies

$$\mathbf{T}^{\perp} = \mathbf{H}, \tag{1.1}$$

where $\mathbf{H}$ is the mean curvature vector of Σ in $\mathbf{R}^{n+k}$. Let $\mathbf{V}$ be the tangent part of $\mathbf{T}$. Then we have

$$\mathbf{T} = \mathbf{V} + \mathbf{H}. \tag{1.2}$$

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There are several results on symplectic and Lagrangian translating solitons. In [8], Han and Li proved that if the Kähler angle on a symplectic translating soliton is not too large, then it must be a plane. Using the equations in [8] and maximum principle, we proved that ([9]) any symplectic translating soliton with nonpositive normal curvature must be a plane. Neves and Tian ([12]) proved that almost-calibrated Lagrangian translating soliton must be a plane under some assumptions.

In this note, we continue to study symplectic and almost-calibrated Lagrangian translating solitons. We first give one decay property about the mean curvature vector:

Main Theorem 1 *Suppose Σ is a complete symplectic translating soliton in $\boldsymbol{C}^2$ with $\cos\alpha \geq \delta > 0$ and quadratic area growth. Then:*

$$\inf_{\Sigma} |\boldsymbol{H}|^2 = 0. \tag{1.3}$$

Main Theorem 1 will follow immediately from the following stronger result:

Theorem 3.1 *Suppose Σ is a complete symplectic translating soliton in $\boldsymbol{C}^2$ with $\cos\alpha \geq \delta > 0$ and quadratic area growth. Then:*

$$\inf_{\Sigma} |\overline{\nabla} J|^2 = 0, \tag{1.4}$$

where J is the complex structure on $\boldsymbol{C}^2$ and $\overline{\nabla}$ is the connection on $\boldsymbol{C}^2$.

It is known that on blow-up limits, the norm of the mean curvature vector is always uniformly bounded from above. Main Theorem 1 says that any translator which arises as a blow-up limit of symplectic mean curvature flow cannot have a positive lower bound for the norm of the mean curvature vector. Note that in Main Theorem 1, we do not assume a uniform upper bound for the second fundamental form. A similar argument gives us the same result for almost calibrated Lagrangian translating solitons:

Main Theorem 2 *Suppose Σ is a complete almost-calibrated Lagrangian translating soliton in $\boldsymbol{C}^2$ with $\cos\theta \geq \delta > 0$ and quadratic area growth. Then:*

$$\inf_{\Sigma} |\boldsymbol{H}|^2 = 0. \tag{1.5}$$

Recall that in [14], the author showed that any almost-calibrated Lagrangian translating soliton with $\sup |\mathbf{H}|$ small comparable to $|\mathbf{T}|$ must be a flat plane.

As applications of the Main Theorem 1, we can give a nonexistence result of graphic translating solitons in $\mathbf{R}^3$. For hypersurface case, (1.1) is equivalent to the following

$$-\langle \mathbf{T}, \nu \rangle = H, \tag{1.6}$$

where ν and H are the unit outer normal and the mean curvature, respectively. By definition, after a translation and rotation, any translating soliton Σ^n in $\mathbf{R}^{n+1}$ with positive mean curvature can be represented as a graph of some function u. It is easy to see that (1.6) can be written as

$$div\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{1}{\sqrt{1+|Du|^2}}. \tag{1.7}$$

In this case, $\mathbf{T} = (0, \ldots, 0, 1)$.

Using the Main Theorem 1, we can prove that

Corollary 1.1 *When $n = 2$, there is no entire solution to the equation (1.7) with bounded gradient.*

Note that Corollary 1.1 is also implicitly implied by [6].

2 Preliminaries

In this section, we recall some basic facts on symplectic and almost calibrated Lagrangian mean curvature flows and translating solitons.

Suppose M is a Kähler–Einstein surface. Let Σ be a smooth surface in M, and ω, $\langle\cdot,\cdot\rangle$ be the Kähler form and the Kähler metric on M respectively. The Kähler angle α of Σ in M is defined by

$$\omega|_\Sigma = \cos\alpha d\mu_\Sigma,$$

where $d\mu_\Sigma$ is the area element of the induced metric from $\langle\cdot,\cdot\rangle$. We call Σ a *symplectic* surface if $\cos\alpha > 0$, a *Lagrangian* surface if $\cos\alpha = 0$, a *holomorphic* curve if $\cos\alpha = 1$. One question in symplectic geometry is that given a symplectic surface Σ in a Kähler–Einstein surface M, whether there is a symplectic minimal surface in the homotopy class of Σ. One natural approach to this problem is to use the negative gradient flow of the area functional, i.e., the mean curvature flow.

Assume that Σ is a real surface and we consider the immersion

$$F_0 : \Sigma \longrightarrow M$$

of smooth surface Σ in M. Suppose that Σ evolves along the mean curvature in M, then there is a one-parameter family $F_t = F(\cdot,t)$ of immersions which satisfy the mean curvature flow equation:

$$\begin{cases} \frac{d}{dt}F(x,t) = \mathbf{H}(x,t), \\ F(x,0) = F_0(x). \end{cases}$$

Here $\mathbf{H}(x,t)$ is the mean curvature vector of $\Sigma_t = F_t(\Sigma)$ at $F(x,t)$ in M.

Recall that [1] the Kähler angle α of Σ in M satisfies the parabolic equation:

$$\left(\frac{\partial}{\partial t} - \Delta\right)\cos\alpha = |\overline{\nabla} J|^2 \cos\alpha + R\sin^2\alpha\cos\alpha, \tag{2.1}$$

where J is the complex structure of M, $\overline{\nabla}$ is the connection on M, and in local orthonormal frame $|\overline{\nabla} J|^2 = |h^2_{1i} + h^1_{2i}|^2 + |h^2_{2i} - h^1_{1i}|^2$ which depends only on the orientation of Σ and does not depend on the choice of the frame. By direct computation, we have [1]

$$2|\mathbf{A}|^2 \ge |\overline{\nabla} J|^2 \ge \frac{1}{2}|\mathbf{H}|^2, \tag{2.2}$$

where $\mathbf{A}$ is the second fundamental form of Σ in M. If the initial surface is symplectic, i.e., $\cos\alpha(\cdot,0)$ has a positive lower bound, then by applying the parabolic maximum principle to this evolution equation, one concludes that $\cos\alpha$ remains positive as long as the mean curvature flow has a smooth solution. In this case, the mean curvature flow is called *symplectic mean curvature flow*.

Chen and Li [1] and Wang [15] proved that there is no finite time Type-I singularity for symplectic mean curvature flow. Suppose the symplectic mean curvature flow develops Type-II singularity at finite time T. Then applying maximum principle to (2.1), we see that for $t \in [0,T)$,

$$\cos\alpha \ge \delta > 0. \tag{2.3}$$

As Kähler angle is invariant under scaling, we conclude that any blow-up flow of symplectic mean curvature flow must satisfy (2.3). In particular, any translating soliton arising as a blow-up limit of the symplectic mean curvature flow must satisfy (2.3).

In Lagrangian case, we assume that M is a Calabi–Yau manifold of complex dimension 2 with a parallel holomorphic (2,0) form Ω. The fact that a surface Σ in M is Lagrangian implies that

$$\Omega|_\Sigma = e^{i\theta} d\mu_\Sigma = \cos\theta d\mu_\Sigma + i\sin\theta d\mu_\Sigma \tag{2.4}$$

for some θ called the Lagrangian angle which is a multivalued function and is well-defined up to an additive constant $2k\pi$, $k \in \mathbf{Z}$. If $\cos\theta > 0$, then Σ is called *almost-calibrated*. If θ is constant, then Σ is called special Lagrangian.

It is proved in [13] that if the initial surface is Lagrangian, then along the mean curvature flow, at each time the surface is still Lagrangian. The evolution equation of Lagrangian angle is given by

$$\left(\frac{\partial}{\partial t} - \Delta\right)\cos\theta = |H|^2\cos\theta. \tag{2.5}$$

If the initial Lagrangian submanifold Σ_0 is almost-calibrated, then Σ_t is almost-calibrated along a smooth mean curvature flow by the parabolic maximum principle.

Chen and Li [2] and Wang [15] proved that there is no finite time Type-I singularity for almost-calibrated Lagrangian mean curvature flow. Suppose the Lagrangian mean curvature flow develops Type-II singularity at finite time T. Then applying maximum principle to (2.5), we see that for $t \in [0, T)$,

$$\cos\theta \geq \delta > 0. \tag{2.6}$$

As Lagrangian angle is also invariant under scaling, we conclude that any blow-up flow of almost-calibrated Lagrangian mean curvature flow must satisfy (2.6). In particular, any translating soliton arising as a blow-up limit of the almost-calibrated Lagrangian mean curvature flow must satisfy (2.6).

Let Σ be a surface in $\mathbf{C}^2$, we say it has *quadratic area growth*, if there exists $D_0 > 0$, such that for all $R \geq 1$,

$$Area(\Sigma \cap B_R(0)) \leq D_0 R^2, \tag{2.7}$$

where $B_R(0)$ is the ball of radius R in $\mathbf{C}^2$ centered at the origin. From Huisken's monotonicity formula, we see that the blow-up limit of symplectic or almost calibrated Lagrangian mean curvature flow always has quadratic area growth (see, for example, Section 2.1 of [4]).

In [8], Han and Li computed several identities on translating solitons. We recall here some of them that we will use in the following.

Lemma 2.1 *On the translating soliton to the symplectic mean curvature flow, the Kähler angle satisfies the following equation*

$$-\Delta\cos\alpha = |\overline{\nabla} J|^2\cos\alpha + \boldsymbol{V}\cdot\nabla\cos\alpha. \tag{2.8}$$

Lemma 2.2 *On the translating soliton to the Lagrangian mean curvature flow, the Lagrangian angle satisfies the following equation*

$$-\Delta\cos\theta = |H|^2\cos\theta + \boldsymbol{V}\cdot\nabla\cos\theta. \tag{2.9}$$

Lemma 2.3 *On the two dimensional translating soliton in $\boldsymbol{C}^2$, at the points where $|\boldsymbol{V}| \neq 0$,*

$$|\boldsymbol{A}|^2 = |\boldsymbol{H}|^2 + 2\frac{|\nabla\boldsymbol{H}|^2}{|\boldsymbol{V}|^2} + \frac{\boldsymbol{V}\cdot\nabla|\boldsymbol{H}|^2}{|\boldsymbol{V}|^2}. \tag{2.10}$$

Corollary 2.1 *Any stationary symplectic translating soliton in $\boldsymbol{C}^2$ is a plane.*

3 Proof of the main theorems

In this section, we prove the Main Theorems. The proof of Main Theorem 2 is the same as that of Main theorem 1 (with Lemma 2.1 replaced by Lemma 2.2). So we will only prove Main Theorem 1. By (2.2), Main Theorem 1 follows from the following stronger result:

Theorem 3.1 *Suppose* Σ *is a complete symplectic translating soliton in* $\boldsymbol{C}^2$ *with* $\cos\alpha \geq \delta > 0$ *and quadratic area growth. Then:*

$$\inf_{\Sigma} |\overline{\nabla} J|^2 = 0. \tag{3.1}$$

Proof We prove it by contradiction. Suppose there is a symplectic translating soliton Σ in $\mathbf{C}^2$ with $\cos\alpha \geq \delta > 0$, quadratic area growth and

$$\inf_{\Sigma} |\overline{\nabla} J|^2 \equiv a > 0. \tag{3.2}$$

Set $v = \frac{1}{\cos\alpha}$, then

$$1 \leq v \leq \frac{1}{\delta} \equiv D_1. \tag{3.3}$$

By (2.8), we have

$$\Delta v = |\overline{\nabla} J|^2 v + 2v^{-1}|\nabla v|^2 - \langle \mathbf{V}, \nabla v\rangle, \tag{3.4}$$

where Δ and ∇ are the Laplacian and gradient operator on Σ with respect to the induced metric, respectively.

Let ϕ be any cutoff function and p be a positive number to be determined later. Multiplying both sides of (3.4) by $\phi^2 v^p$ and integrating by parts yields

$$\begin{aligned}
&\int_{\Sigma} \phi^2 v^{p+1} |\overline{\nabla} J|^2 d\mu + 2\int_{\Sigma} \phi^2 v^{p-1} |\nabla v|^2 d\mu - \int_{\Sigma} \phi^2 v^p \langle \mathbf{V}, \nabla v\rangle d\mu \\
&\quad = \int_{\Sigma} \phi^2 v^p \Delta v = -p\int_{\Sigma} \phi^2 v^{p-1}|\nabla v|^2 d\mu - 2\int_{\Sigma} \phi v^p \langle \nabla\phi, \nabla v\rangle d\mu.
\end{aligned}$$

Rearranging this equality and using Young's inequality, we obtain

$$\begin{aligned}
(p+2)&\int_{\Sigma} \phi^2 v^{p-1}|\nabla v|^2 d\mu + \int_{\Sigma} \phi^2 v^{p+1} |\overline{\nabla} J|^2 d\mu \\
&= \int_{\Sigma} \phi^2 v^p \langle \mathbf{V}, \nabla v\rangle d\mu - 2\int_{\Sigma} \phi v^p \langle \nabla\phi, \nabla v\rangle d\mu \\
&\leq \int_{\Sigma} \phi^2 v^p |\mathbf{V}||\nabla v| d\mu + 2\int_{\Sigma} \phi v^p |\nabla\phi||\nabla v| d\mu \\
&\leq \varepsilon \int_{\Sigma} \phi^2 v^{p+1} |\mathbf{V}|^2 d\mu + \frac{1}{4\varepsilon}\int_{\Sigma} \phi^2 v^{p-1}|\nabla v|^2 d\mu \\
&\quad + \int_{\Sigma} \phi^2 v^{p-1}|\nabla v|^2 d\mu + \int_{\Sigma} v^{p+1}|\nabla\phi|^2 d\mu,
\end{aligned}$$

which implies that

$$\left(p+1-\frac{1}{4\varepsilon}\right)\int_{\Sigma}\phi^2 v^{p-1}|\nabla v|^2 d\mu+\int_{\Sigma}\phi^2 v^{p+1}(|\overline{\nabla} J|^2-\varepsilon|\mathbf{V}|^2)d\mu\le\int_{\Sigma}v^{p+1}|\nabla\phi|^2 d\mu. \tag{3.5}$$

From $1=|\mathbf{T}|^2=|\mathbf{H}|^2+|\mathbf{V}|^2$ and (3.2), we get that

$$|\overline{\nabla} J|^2-\varepsilon|\mathbf{V}|^2\ge a-\varepsilon.$$

We first choose $\varepsilon=a$ so that $|\overline{\nabla} J|^2-\varepsilon|\mathbf{V}|^2\ge 0$, then take $p=\frac{1}{4a}$ so that $p+1-\frac{1}{4\varepsilon}=1$. Then we obtain from (3.5) that

$$\int_{\Sigma}\phi^2 v^{p-1}|\nabla v|^2 d\mu\le\int_{\Sigma}v^{p+1}|\nabla\phi|^2 d\mu. \tag{3.6}$$

Next, we will choose appropriate cutoff function to deduce that v is a constant function. We will use the logarithmic cutoff argument (see Chapter 1 of [3]). Let $R>1$ be any fixed number. Define the cutoff function ϕ on all of $\mathbf{R}^4$ and then restrict it to the graph of u as follows: Let r denote the distance to the origin in $\mathbf{R}^4$, define

$$\phi=\begin{cases}1, & r^2\le R;\\ 2-2\frac{\log r}{\log R}, & R<r^2\le R^2;\\ 0, & r^2>R^2.\end{cases}$$

Recall that we assume the translating soliton has quadratic area growth. From (2.7), (3.3) and (3.6), we have

$$\begin{aligned}\int_{\Sigma\cap B(0,\sqrt{R})}v^{p-1}|\nabla v|^2 d\mu&\le\int_{\Sigma}\phi^2 v^{p-1}|\nabla v|^2 d\mu\le\int_{\Sigma}v^{p+1}|\nabla\phi|^2 d\mu\\ &\le\frac{4D_1^{p+1}}{(\log R)^2}\sum_{\frac{\log R}{2}\le l\le\log R}\int_{\Sigma\cap(B(0,e^l)\setminus B(0,e^{l-1}))}r^{-2}d\mu\\ &\le\frac{4D_1^{p+1}}{(\log R)^2}\sum_{\frac{\log R}{2}\le l\le\log R}e^{-2(l-1)}D_0e^{2l}\\ &\le\frac{C}{\log R},\end{aligned}$$

where C depends only on D_0, δ and a. As $v\ge 1$, letting $R\to\infty$, we get that v is a constant, i.e., $\cos\alpha$ is a constant on Σ. Therefore, Σ is a holomorphic curve with respect to some complex structure of $\mathbf{C}^2$. In particular, Σ is minimal. By Corollary 2.1, we know that Σ is a flat plane, i.e., $|\mathbf{A}|\equiv 0$. But by (2.2) and (3.2),

$$|\mathbf{A}|\ge\frac{|\overline{\nabla} J|^2}{2}\ge\frac{a}{2}>0.$$

This gives the desired contradiction. □

4 Proof of Corollary 1.1

In this section, we will consider graphic translating solitons in $\mathbf{R}^3$. First we consider general graph in $\mathbf{C}^2$, that is,

$$\begin{aligned} F : U \subset \mathbf{R}^2 &\to \mathbf{R}^4 \\ (x, y) &\mapsto (x, y, f(x, y), g(x, y)). \end{aligned} \tag{4.1}$$

We will compute the Kähler angle of $\Sigma = F(U)$.

By (4.1), we know the basis of the tangent space and normal space of Σ are given by

$$\begin{aligned} e_1 &= F_x = (1, 0, f_x, g_x), \quad e_2 = F_y = (0, 1, f_y, g_y), \\ v_3 &= (-f_x, -f_y, 1, 0), \qquad v_4 = (-g_x, -g_y, 0, 1). \end{aligned}$$

Therefore, the induced metric on Σ is given by

$$(g_{ij})_{1\le i,j,\le 2} = \begin{pmatrix} 1 + f_x^2 + g_x^2 & f_x f_y + g_x g_y \\ f_x f_y + g_x g_y & 1 + f_y^2 + g_y^2 \end{pmatrix},$$

and the inverse matrix is

$$(g^{ij})_{1\le i,j,\le 2} = \frac{1}{\det(g_{ij})} \begin{pmatrix} 1 + f_y^2 + g_y^2 & -f_x f_y - g_x g_y \\ -f_x f_y - g_x g_y & 1 + f_x^2 + g_x^2 \end{pmatrix}.$$

By direct calculation, we have

$$\det(g_{ij}) = 1 + f_x^2 + g_x^2 + f_y^2 + g_y^2 + (f_x g_y - f_y g_x)^2. \tag{4.2}$$

Recall that the standard complex structure in $\mathbf{C}^2$ is given by

$$J_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{4.3}$$

By the definition of Kähler angle, we have

$$\cos\alpha = \frac{\omega_1(e_1, e_2)}{\sqrt{\det(g_{ij})}} = \frac{\langle J_1 e_1, e_2\rangle}{\sqrt{\det(g_{ij})}} = \frac{1 + f_x g_y - f_y g_x}{\sqrt{\det(g_{ij})}}. \tag{4.4}$$

We immediately have

Proposition 4.1 *A graph in $\mathbf{C}^2$ defined by (4.1) is symplectic with respect to the complex structure J_1 if and only if*

$$1 + f_x g_y - f_y g_x > 0. \tag{4.5}$$

In particular, any graph in $\mathbf{R}^3$ (i.e., $g \equiv 0$) is symplectic with respect to the complex structure J_1 in $\mathbf{C}^2$ with

$$\cos\alpha = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}}. \tag{4.6}$$

In particular, we see that any graphic translating soliton in $\mathbf{R}^3$ given by $(x, y, u(x, y))$ is symplectic if we view it as a surface in $\mathbf{R}^4$. The Kähler angle is given by

$$\cos\alpha = \frac{1}{\sqrt{1+|Du|^2}}. \tag{4.7}$$

In this case, u will be a solution of (1.7). We will denote $\Sigma = Graph_u$.

Now we can prove Corollary 1.1 using the Main Theorem 1.

Proof of Corollary 1.1 We prove it by contradiction. Suppose there is an entire solution u to the equation (1.7) defined on the whole $\mathbf{R}^2$ with

$$|Du| \le D_2. \tag{4.8}$$

From (1.7) and (4.7), we see that

$$|\mathbf{H}|^2 = \frac{1}{1+|Du|^2} = \cos^2\alpha \ge \frac{1}{1+D_2^2}. \tag{4.9}$$

Next, we will show that Σ has quadratic area growth. We denote by $\hat{B}(0, r)$ the ball of radius r centered at 0 in the domain plane $\mathbf{R}^2$, while denote by $B(0, r)$ the ball of radius r centered at 0 in $\mathbf{R}^4$. It is obvious that

$$\Sigma \cap B(0, r) \subset Graph_u(\hat{B}(0, r)).$$

Therefore,

$$\begin{aligned} Area(\Sigma \cap B(0, r)) &\le Area(Graph_u(\hat{B}(0, r))) \\ &= \int_{\hat{B}(0,r)} \sqrt{1+|Du|^2}dxdy \\ &\le \sqrt{1+D_2^2}\pi r^2 \equiv C_1 r^2. \end{aligned} \tag{4.10}$$

Combining the above together, we find a complete symplectic translating soliton in $\mathbf{C}^2$ with $\cos\alpha \ge \delta > 0$ for some δ, quadratic area growth, and $|\mathbf{H}|^2 \ge \frac{1}{1+D_2^2} > 0$. This contradicts the Main Theorem 1. □

Remark 4.1 Corollary 1.1 can also be proved directly as follows: Set $v = \sqrt{1+|Du|^2}$. Then by Lemma 3.1 of [5], we can obtain

$$\Delta v = |\mathbf{A}|^2 v + 2v^{-1}|\nabla v|^2 - \langle \mathbf{V}, \nabla v\rangle, \tag{4.11}$$

Then by the similar argument as we gave to prove Theorem 3.1, $|Du|$ is constant. Also by (1.7), $\Delta_0 u = 1$, where Δ_0 is the standard Laplacian on $\mathbf{R}^2$. So by Bochner's formula,

$$0 = \Delta_0|Du|^2 = 2|D^2u|^2 + 2\langle Du, D\Delta_0 u\rangle = 2|D^2u|^2.$$

Thus $\Delta_0 u = 0$, which is a contradiction.

Remark 4.2 Note that the above argument just works for $n = 2$, because we need the translating soliton to have quadratic area growth when we use the logarithmic cutoff argument.

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