Geom Dedicata (2014) 171:397–406
DOI 10.1007/s10711-013-9905-7

The p-hyperbolicity of infinity volume ends and applications

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Received: 6 May 2013 / Accepted: 13 August 2013 / Published online: 23 August 2013
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Abstract In this paper we prove a characterization of p-hyperbolic ends on complete Riemannian manifolds which carries a Sobolev type inequality

Keywords p-Hyperbolicity · Sobolev type inequality · Cheng type inequality

Mathematics Subject Classification (1991) Primary 31C12, 58C40 · Secondary 47J10, 53C26

1 Introduction

Let M^n be a complete noncompact Riemannian manifold. Given $p \geq 1$, we recall that the p-Laplacian operator on M is defined by

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u),$$

for $u \in W^{1,p}_{loc}(M)$. It is the Euler-Lagrange operator associated to the p-energy functional, $E_p(u) := \int_M |\nabla u|^p \, \mathrm{d}M$. This non-linear operator appears naturally in many situations, and we refer the reader to [5,7,16] and the references cited therein for further information. As usual, we say that a function u is p-harmonic if $\Delta_p u = 0$.

Let $E \subset M$ be an *end* of M, that is an unbounded connect component of $M \setminus \Omega$, for some compact subset, $\Omega \subset M$, with smooth boundary. We say that E is p-*parabolic* (see Definition 2.4 of [10] for $p = 2$ and Theorem 2.5 of [19] for the general case) if it does not admit a p-harmonic function, $f : E \to \mathbb{R}$, satisfying:

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$$\begin{cases} f|_{\partial E} = 1; \\ \liminf_{\substack{y \to \infty \\ y \in E}} f(y) < 1. \end{cases}$$

Otherwise, we say that E is a *p-hyperbolic* end of M.

In [12] Li and Wang obtained the following characterization of the ends of complete manifolds.

For simplicity, we omit the volume element of integrals.

Theorem A (Corollary 4 of [12]) *Let E be an end of a complete manifold. Suppose that, for some constants $\nu \geq 1$ and $C > 0$, E satisfies a Sobolev-type inequality of the form*

$$\left(\int_E |u|^{2\nu} \right)^{\frac{1}{\nu}} \leq C \int_E |\nabla u|^2, \tag{1}$$

for all compactly supported Sobolev function $u \in W_0^{1,2}(E)$. Then E must either have finite volume or be 2-hyperbolic.

In our first result, we extend the above theorem for p-hyperbolic ends. Namely

Theorem 1.1 *Let E be an end of a complete Riemannian manifold. Assume that for some constants, $1 < p \leq q < \infty$ and $C > 0$, E satisfies a Sobolev-type inequality of the form*

$$\left(\int_E |u|^{q} \right)^{\frac{p}{q}} \leq C \int_E |\nabla u|^p, \tag{2}$$

for all $u \in W_0^{1,p}(E)$. Then E must either have finite volume or be p-hyperbolic.

To prove this theorem we apply the techniques developed in [12] and a lemma due to Cacciopolli (see Lemma 2.1 in Sect. 2). Some application for Cheng's type inequalities are given in the Sect. 5

Our next result is characterization of p-hyperbolic ends in the context of submanifolds as recently obtained in [4]. Bellow, let us denote by H the mean curvature vector field of an isometric immersion $x : M^m \to \bar{M}$ and by $||H||_{L^q(E)}$ its Lebesgue L^q-norm on $E \subset M$.

Theorem 1.2 *Let $x : M^m \to \bar{M}$, with $m \geq 3$, be an isometric immersion of a complete non-compact manifold M in a manifold $\bar{M}$ with nonpositive sectional radial curvature. Given, $1 < p < m$, let E be an end of M such that the mean curvature vector satisfies $\|H\|_{L^q(E)} < \infty$, for some $q \in [p, m]$. Then E must either have finite volume or be p-hyperbolic.*

As a direct consequence, we have:

Corollary 1.1 *Let $x : M^m \to \bar{M}$, with $m \geq 3$, be a minimal isometric immersion of a complete manifold M in a manifold $\bar{M}$ with nonpositive sectional radial curvature. Then, each end of M is p-hyperbolic, for each $p \in (1, m)$.*

The main tool in the proof of Theorem 1.2 is the Hofmann-Spruck inequality [6] and its refinement given in [2].

2 Preliminaries on p-harmonic function

In this section we prove two basic results which will be used to prove Theorems 1.1 and 1.2 as well for Cheng's inequalities in Sect. 5. We first refine a technical lemma due to Caccioppoli (see Lemma 2.9 of [14]).

Lemma 2.1 (Caccioppoli) *Let $\Omega \subset M$ be a compact set and let Γ be a connect component of $\partial\Omega$. Given $p > 1$, if u is a weak solution for the p-Laplace equation in Ω such that u vanishes on Γ, then*

$$\int_\Omega \varphi^p |\nabla u|^p \leq p^p \int_\Omega u^p |\nabla \varphi|^p,$$

for all smooth function φ such that $0 \leq \varphi \leq 1$ and φ equals zero in $\partial\Omega \setminus \Gamma$.

Proof Since $\Delta_p u = 0$ weakly in Ω and $\varphi^p u$ vanishes on $\partial\Omega$ we have

$$\int_\Omega \langle \nabla(\varphi^p u), |\nabla u|^{p-2} \nabla u \rangle = 0.$$

Thus, using Hölder inequality,

$$\begin{aligned} \int_\Omega \varphi^p |\nabla u|^p &= -p \int_\Omega \varphi^{p-1} u \, \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi \rangle \\ &\leq p \int_\Omega |\varphi \nabla u|^{p-1} |u \nabla \varphi| \leq p \left(\int_\Omega \varphi^p |\nabla u|^p \right)^{(p-1)/p} \left(\int_\Omega |u|^p |\nabla \varphi|^p \right)^{1/p}. \end{aligned}$$

This completes the proof of the lemma. □

The next lemma is a well known result for the Laplacian operator and the proof follows closely the one in [9]. We include the proof here for the sake of completeness.

Lemma 2.2 *Let M be a complete noncompact Riemannian manifold. If M has a polynomial volume growth, then $\lambda_{1,p}(M) = 0$.*

Proof By hypothesis, there exist $C > 0$ and $k \geq 0$ such that

$$V(r) := Vol(B_r) \leq C\, r^k,$$

for all $r > 0$ big enough. On the other hand, from the variational characterization of $\lambda_{1,p}(M)$ we have

$$\lambda_{1,p}(M) \int_M |\varphi|^p \leq \int_M |\nabla \varphi|^p,$$

for any $\varphi \in W_0^{1,p}(M)$. Given $x \in M$, let us denote by $r(x)$ the distance function on M from a fixed point. So, given $r > 0$, if we choose

$$\varphi(x) = \begin{cases} 1 & \text{on } B_r, \\ \dfrac{2r - r(x)}{r} & \text{on } B_{2r} \setminus B_r, \\ 0 & \text{on } M \setminus B_{2r}, \end{cases}$$

we obtain

$$\lambda_{1,p}(M)V(r) \le r^{-p}V(2r), \tag{3}$$

for all $r > 0$. Assuming, by contradiction, that $\lambda_{1,p}(M)$ is positive and applying the volume growth assumption to $V(2r)$ we get $V(r) \le Cr^{k-p}$, for $r > 0$ big enough.

Iterating this argument $\left[\dfrac{k}{p}\right]$ times we obtain $V(r) \le Cr^a$, with $a < p$. Now, we use the inequality (3) to obtain

$$\lambda_{1,p}(M)V(r) \le Cr^{a-p}.$$

Letting $r \to \infty$, we conclude that $V(M) = 0$, which is a contradiction. □

3 Proof of Theorem 1.1

Given $r > 0$, let B_r be a geodesic ball in M centered at some point $p \in M$. We set $E_r = E \cap B_r$ and $\partial E_r = E \cap \partial B_r$.

Let f_r be the solution of the following Dirichlet problem

$$\begin{cases} \Delta_p f_r = 0 & \text{in } E_r, \\ f_r = 1 & \text{in } \partial E, \\ f_r = 0 & \text{in } \partial E_r. \end{cases}$$

By the arguments used in the proof of Lemma 2.7 in [19] $f_r \in C^{1,\alpha}_{loc}(E_r) \cap C(\bar{E}_r)$, $0 < f_r < 1$ in E_r, it is increasing and converges (locally uniformly) to a p-harmonic function f with $f \in C^{1,\alpha}_{loc}(E) \cap C(\bar{E})$ satisfying $0 < f \le 1$ and $f = 1$ on ∂E.

For a fixed $0 < r_0 < r$ such that $E_{r_0} \neq \emptyset$, let φ be a nonnegative cut-off function satisfying the properties that

$$\begin{cases} \varphi = 1 & \text{on } E_r \setminus E_{r_0}, \\ \varphi = 0 & \text{on } \partial E, \\ |\nabla \varphi| \le C. \end{cases}$$

Applying the inequality (2) of the assumption and using the fact that f_r is p-harmonic, we obtain

$$\begin{aligned} \left(\int_{E_r} |\varphi f_r|^p \right)^{p/q} &\le C \int_{E_r} |\nabla(\varphi f_r)|^p = C \int_{E_r} |\varphi \nabla f_r + f_r \nabla \varphi|^p \\ &\le C_1 \int_{E_r} |\varphi \nabla f_r|^p + |f_r \nabla \varphi|^p \\ &\le C_2 \int_{E_r} |f_r|^p |\nabla \varphi|^p \\ &\le C_3 \int_{E_r} |f_r|^p, \end{aligned}$$

where we have used that $(a+b)^p \le C(a^p+b^p)$, for a fixed constant $C = 2^{p-1}$, and every positive numbers a, b in the second inequality, Cacciopoli's Lemma, 2.1, in the third inequality and $|\nabla\varphi| \le C$, in the last inequality.

In particular, for a fixed r_1 satisfying $r_0 < r_1 < r$, we have

$$\left(\int_{E_{r_1}\setminus E_{r_0}} f_r^q\right)^{p/q} \le C_3 \int_{E_{r_0}} f_r^p.$$

If E is p-parabolic, then the limiting function f is identically 1. Letting $r\to\infty$, we obtain

$$(V_E(r_1) - V_E(r_0))^{p/q} \le C_3\, V_E(r_0),$$

where $V_E(r)$ denotes the volume of the set E_r. Since $r_1 > r_0$ is arbitrary, this implies that E has finite volume. This conclude proof of the theorem. □

4 Proof of Theorem 1.2

Let f_r be the sequence given above and f its limit. Let us suppose, by contradiction, that $f\equiv 1$ and $\mathrm{vol}(E)$ is infinite. This implies that, given any $L>1$, there exists $r_1 > r_0$ such that $\mathrm{vol}(E_{r_1} - E_{r_0}) > 2L$. Since $f_r \to 1$ uniformly on compact subsets, there exists $r_2 > r_1$ such that $f_r^{\frac{pm}{m-p}} > \frac{1}{2}$ everywhere in E_{r_1}, for all $r > r_2$. Thus, defining $h(r) := \int_{E_r - E_{r_0}} f_r^{\frac{pm}{m-p}}$, with $r > r_0$, we obtain

$$h(r) \ge \int_{E_{r_1}-E_{r_0}} f_r^{\frac{pm}{m-p}} > L, \tag{4}$$

for all $r > r_2$. In particular, we have that $\lim\limits_{r\to\infty} h(r) = \infty$.

Now, for each $r > r_0$, let $\varphi = \varphi_r \in C_0^\infty(E)$ be a cut-off function satisfying:

$$\begin{cases} 0 \le \varphi \le 1 \text{ everywhere in } E; \\ \varphi \equiv 1 \text{ in } E_r - E_{r_0}. \end{cases}$$

By modified Hoffmann-Spruck Inequality [6] or [2] we have

$$S^{-1}\left(\int_{E_r} (\varphi f_r)^{\frac{pm}{m-p}}\right)^{\frac{m-p}{m}} \le \int_{E_r} |\nabla(\varphi f_r)|^p + \int_{E_r} (\varphi f_r)^p |H|^p,$$

where S is a positive constant and $p\in(1,m)$.

Using that $f_r\varphi$ vanishes on ∂E_r and the Cacciopoli's Lemma 2.1 we obtain

$$S^{-1}\left(\int_{E_r} (\varphi f_r)^{\frac{pm}{m-p}}\right)^{\frac{m-p}{m}} \le C\left(\int_{E_r} f_r^p |\nabla\varphi|^p + \int_{E_r} (\varphi f_r)^p |H|^p\right),$$

where $C = 1 + p^p$. Thus, since $0 \le \varphi \le 1$ in E and $\varphi \equiv 1$ in $E_r - E_{r_0}$, we obtain

$$(SC)^{-1}h(r)^{\frac{m-p}{m}} \le (SC)^{-1}\left(\int_{E_r}(\varphi f_r)^{\frac{pm}{m-p}}\right)^{\frac{m-p}{m}} \le \int_{E_{r_0}} f_r^p|\nabla\varphi|^p + \int_{E_r} f_r^p|H|^p. \tag{5}$$

First, assume that $\|H\|_{L^p(E)}$ is finite. Then, since $0 \le f_r \le 1$, we have

$$(SC)^{-1}h(r)^{\frac{m-p}{m}} \le \int_{E_{r_0}} |\nabla\varphi|^p + \int_E |H|^p.$$

Thus, $\lim_{r\to\infty} h(r) < \infty$, which is a contradiction. Now, assume that $\|H\|_{L^q(E)}$ is finite, for some $p < q \le m$. Note that $\frac{m}{m-p} \le \frac{q}{q-p}$. Since $0 \le f_r \le 1$ and $h(r) > 1$, for all $r > r_2$, we have:

$$\begin{cases} f_r^{\frac{pq}{q-p}} \le f_r^{\frac{pm}{m-p}}; \\ h(r)^{\frac{q-p}{q}} \le h(r)^{\frac{m-p}{m}}, \text{ for all } r > r_2. \end{cases}$$

Thus, using Hölder Inequality, we have

$$\begin{aligned}\int_{E_r-E_{r_0}} f_r^p|H|^p &\le \|H\|^p_{L^q(E_r-E_{r_0})}\left(\int_{E_r-E_{r_0}} f_r^{\frac{pq}{q-p}}\right)^{\frac{q-p}{q}} \\ &\le \|H\|^p_{L^q(E-E_{r_0})}h(r)^{\frac{m-p}{m}},\end{aligned} \tag{6}$$

for all $r > r_2$.

Choose $r_0 > 0$ large enough so that $\|H\|^p_{L^q(E-E_{r_0})} < \frac{1}{2SC}$. Using (5) and (6) we get:

$$(SC)^{-1}h(r)^{\frac{m-p}{m}} \le \int_{E_{r_0}} |\nabla\varphi|^p + \int_{E_{r_0}} |H|^p + \frac{(SC)^{-1}}{2}h(r)^{\frac{m-p}{m}}.$$

This shows that $\lim_{r\to\infty} h(r) < \infty$, which is a contradiction and Theorem 1.2 are proved.

5 Cheng's theorems for the p-Laplacian

Now we describe how we can apply Theorem 1.1 to obtain new Cheng's type inequalities. For that, we use the Li-Wang approach as in [13].

Given a regular domain $\Omega \subset M$ let $\lambda_1(\Omega)$ be the first Dirichlet eigenvalue of the Laplacian operator. That is,

$$\lambda_1(\Omega) = \inf\left\{\frac{\int_\Omega |\nabla\varphi|^2}{\int_\Omega \varphi^2} : \varphi \in W_0^{1,2}(\Omega) \setminus \{0\}\right\}.$$

We recall that the *bottom of the spectrum* of M is given by

$$\lambda_1(M) = \lim_{i\to\infty} \lambda_1(\Omega_i),$$

where $\{\Omega_i\}_i$ is an exhaustion of M, and this definition does not depend on the exhaustion.

Let B_r^M denote a geodesic ball on M with radius $r > 0$ and centered at some point of M. The classical Cheng's comparison theorem asserts that, if $\mathrm{Ric}_M \geq -(n-1)$, then $\lambda_1(B_r^M) \leq \lambda_1(B_r^{\mathbb{H}^n})$, where $\mathbb{H}^n$ denotes the n-dimensional hyperbolic space $\mathbb{H}^n$. One of the consequences is a sharp upper bound for the bottom of the spectrum on a complete manifold with Ricci curvature bounded from below. Precisely

Theorem B (Cheng [3]) *Let M^m be a complete noncompact Riemannian manifold such that the Ricci curvature of M has a lower bound given by*

$$Ric_M \geq -(m-1).$$

Then, the bottom of the spectrum of the Laplacian must satisfy the upper bound

$$\lambda_1(M) \leq \frac{(m-1)^2}{4} = \lambda_1(\mathbb{H}^m).$$

The Cheng's theorem still holds for the p-Laplacian operator.

An eigenfunction for the Dirichlet problem of the p-Laplacian on $\Omega \subset M$ is a nonzero function u such that

$$\begin{cases} \Delta_p u + \lambda |u|^{p-2} u = 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

for some number $\lambda \in \mathbb{R}$.

We shall denote by $\lambda_{1,p}(\Omega)$, the smallest eigenvalue of Δ_p in Ω for the Dirichlet problem. It is well known that $\lambda_{1,p}(\Omega)$ has a variational characterization, analogous to the first eigenvalue of the Laplacian (see [17])

$$\lambda_{1,p}(\Omega) = \inf \left\{ \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p} : u \in W_0^{1,p}(\Omega) \setminus \{0\} \right\}.$$

Using standard comparison ideas, Matei [17] generalized Cheng's result for the p-Laplacian operator, with $p \geq 2$.

Using Theorem A and the growth rate of the volume of 2-hyperbolic ends with positive spectrum (Theorem 1.4 of [11]), Li and Wang proved Cheng's comparison theorem for Kähler manifolds under an assumption on the bisectional curvature. Latter, Kong, Li and Zhou [8] solved the case of quaternionic Kähler manifolds.

Here we use Theorem 1.1 and the volume estimates of Buckley and Koskela in [1] to prove Cheng's inequalities for the p-Laplacian on Kähler and Kähler quaternionic manifolds and thus, we complete the picture for theses cases. More precisely

Theorem 5.1 *Let M^{2m} be a complete noncompact Kähler manifold, of real dimension $2m$, such that the bisectional curvature of M has a lower bound given by $\mathrm{BK}_M \geq -1$. Then, for each $p > 1$, the bottom of the spectrum of the p-Laplacian must satisfy the upper bound*

$$\lambda_{1,p}(M) \leq \frac{4^p m^p}{p^p}.$$

Moreover, this estimate is sharp since equality is achieved by the complex hyperbolic space form $\mathbb{CH}^{2m}$.

Remark 1 Munteanu ([18]) has obtained a Cheng's comparison theorem for Kähler manifolds under the weaker assumption on Ricci curvature when $p = 2$. However, the techniques we used in this note do not work in that case.

Following [8], we are able to obtain Cheng's comparison theorem for quaternionic Kähler manifolds, under a weaker hypothesis on the scalar curvature.

Theorem 5.2 *Let M^{4m} be a complete noncompact quaternionic Kähler manifold, of real dimension $4m$, such that the scalar curvature of M has a lower bound given by*

$$\mathrm{S}_M \geq -16m(m+2).$$

Then, for each $p > 1$, the bottom of the spectrum of the p-Laplacian must satisfy the upper bound

$$\lambda_{1,p}(M) \leq \frac{2^p(2m+1)^p}{p^p}.$$

Moreover, this estimate is sharp as equality is a achieved by the quaternionic hyperbolic space form $\mathbb{QH}^{4m}$.

Remark 2 We can apply the techniques above to extend the Cheng's comparison theorem of Matei ([17]) for $p > 1$. The Theorems 5.1 and 5.2 can be obtained by using a p-version of Brooks' theorem described in [15] provided the volume of M is infinity.

Below we provide a unified proof of Theorems 5.1 and 5.2.

Without loss of generality, we assume that $\lambda_{1,p}(M)$ is positive. By Theorem 1.1 and Lemma 2.2 we have that M is p-hyperbolic. Now, by Theorem 0.1 in [1] we obtain

$$V(r) \geq C_0 \exp(p\lambda_{1,p}(M)^{1/p} r),$$

for all $r \gg 1$ and some $C_0 > 0$.

We point out that our hypotheses on the curvature imply volume growth estimates for geodesic balls. Namely, $V(r) \leq C \exp(ar)$, where $a = 4m$ in Theorem 5.1 (see [13]) and $a = 2(2m+1)$ in Theorem 5.2 (see [8]).

Therefore, we get

$$C_0 \exp(p\lambda_{1,p}(M)^{1/p} r) \leq C \exp(ar),$$

for all $r \gg 1$. i.e.,

$$\lambda_{1,p}(M)^{1/p} \leq \frac{1}{pr} \ln\left(\frac{C}{C_0}\right) + \frac{a}{p}.$$

Letting $r \to \infty$, we obtain

$$\lambda_{1,p}(M) \leq \left(\frac{a}{p}\right)^p.$$

In particular we have

$$\lambda_{1,p}(\mathbb{CH}^{2m}) \leq \left(\frac{4m}{p}\right)^p \quad \text{and} \quad \lambda_{1,p}(\mathbb{QH}^{4m}) \leq \left(\frac{2(2m+1)}{p}\right)^p. \tag{7}$$

To proof the equality in the space form case we use Theorem 1.1 of [15] applied to the gradient of distance function. Precisely

Lemma 5.1 (Theorem 1.1 of [15]) *Let $\Omega \subset M$ be a domain with compact closure and $\partial\Omega \neq \emptyset$, in a Riemannian manifold, M. Then*

$$\lambda_{1,p}(\Omega) \geq \frac{c(\Omega)^p}{p^p}, \tag{8}$$

where $c(\Omega)$ is the constant given by

$$c(\Omega) := \sup\left\{\frac{\inf_\Omega \operatorname{div} X}{\|X\|_\infty};\quad X \in \mathfrak{X}(\Omega)\right\}.$$

Here $\mathfrak{X}(\Omega)$ denotes the set of all smooth vector fields, X, on Ω with sup norm $\|X\|_\infty = \sup_\Omega \|X\| < \infty$ (where $\|X\| = g(X,X)^{1/2}$) and $\inf_\Omega \operatorname{div} X > 0$.

Now, taking $X = \nabla r$ the gradient of the distance function on M, we obtain $\|X\| = 1$ and $\operatorname{div} X = \Delta r$, and consequently $c(\Omega) \geq \inf_\Omega \Delta r$.

We point out that, in the space form cases we have

$$\Delta^{\mathbb{CH}} r(x) = 2\coth 2r(x) + 2(2m-1)\coth r(x) \quad \text{on } \mathbb{CH}^{2m}$$

and

$$\Delta^{\mathbb{QH}} r(x) = 6\coth 2r(x) + 4(m-1)\coth r(x) \quad \text{on } \mathbb{QH}^{4m}.$$

Thus

$$\inf_\Omega \Delta^{\mathbb{CH}} r(x) \geq 4m \qquad \text{and} \qquad \inf_\Omega \Delta^{\mathbb{QH}} r(x) \geq 2(2m+1)$$

and the result follows from the estimate (8). □

Acknowledgments The authors wish to thank Professor S. Pigola for helpful comments about this paper. M. P. Cavalcante was partially supported by CNPq.

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