Geom Dedicata (2014) 170:63–69
DOI 10.1007/s10711-013-9867-9

ORIGINAL PAPER

Symplectic mean curvature flows in Kähler surfaces with positive holomorphic sectional curvatures

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Received: 22 November 2012 / Accepted: 13 May 2013 / Published online: 24 May 2013

Abstract In this paper, we mainly study the mean curvature flow in Kähler surfaces with positive holomorphic sectional curvatures. We prove that if the ratio of the maximum and the minimum of the holomorphic sectional curvatures is less than 2, then there exists a positive constant δ depending on the ratio such that $\cos\alpha \geq \delta$ is preserved along the flow.

Keywords Symplectic mean curvature flow · Kähler angle · Holomorphic sectional curvature · Holomorphic curve

Mathematics Subject Classification (2000) Primary 53C44 ; Secondary 53C21

1 Introduction

Mean curvature flows were studied by many authors, for example Huisken [14,15], Ecker and Huisken [6], Huisken and Sinestrari [16], Carlo Ilmanen [17], Neves [18], Smoczyk [19], Wang [21], White [22], etc.

In this paper we mainly concentrated on the symplectic mean curvature flows, which were studied by Chen and Tian [4], Chen and Li [2], Chen et al. [3], Wang [21], Han and Li [8–10], Han and Sun [13], and Han et al. [11,12]. The basic fact is that the symplectic property is preserved by the mean curvature flow if the ambient space M is Kähler–Einstein, or if the ambient Kähler surface evolves along the Kähler–Ricci flow [10].

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Let $(M, J, \overline{\omega}, \bar{g})$ be a Kähler surface. For a compact oriented real surface Σ which is smoothly immersed in M, the Kähler angle [5] α of Σ in M was defined by

$$\omega|_{\Sigma} = \cos\alpha d\mu_{\Sigma}$$

where $d\mu_{\Sigma}$ is the area element of Σ in the induced metric from $\bar{g}$. We say that Σ is a symplectic surface if $\cos\alpha > 0$; Σ is a holomorphic curve if $\cos\alpha \equiv 1$.

Given an immersed $F_0 : \Sigma \to M$, we consider a one-parameter family of smooth maps $F_t = F(\cdot, t) : \Sigma \to M$ with corresponding images $\Sigma_t = F_t(\Sigma)$ immersed in M and F satisfies the mean curvature flow equation:

$$\begin{cases} \frac{\partial}{\partial t} F(x, t) = H(x, t) \\ F(x, 0) = F_0(x), \end{cases} \tag{1.1}$$

where $H(x, t)$ is the mean curvature vector of Σ_t at $F(x, t)$ in M.

Choose an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ on $(M, \bar{g})$ along Σ_t such that $\{e_1, e_2\}$ is the basis of Σ_t and the symplectic form ω_t takes the form

$$\omega_t = \cos\alpha u_1 \wedge u_2 + \cos\alpha u_3 \wedge u_4 + \sin\alpha u_1 \wedge u_3 - \sin\alpha u_2 \wedge u_4, \tag{1.2}$$

where $\{u_1, u_2, u_3, u_4\}$ is the dual basis of $\{e_1, e_2, e_3, e_4\}$. Then along the surface Σ_t the complex structure on M takes the form ([2])

$$J = \begin{pmatrix} 0 & \cos\alpha & \sin\alpha & 0 \\ -\cos\alpha & 0 & 0 & -\sin\alpha \\ -\sin\alpha & 0 & 0 & \cos\alpha \\ 0 & \sin\alpha & -\cos\alpha & 0 \end{pmatrix}. \tag{1.3}$$

Recall the evolution equation of the Kähler angle along the mean curvature flow deduced in [10],

Theorem 1.1 *The evolution equation for* $\cos\alpha$ *along* Σ_t *is*

$$\left(\frac{\partial}{\partial t} - \Delta\right)\cos\alpha = \left|\overline{\nabla} J_{\Sigma_t}\right|^2 \cos\alpha + \sin^2\alpha Ric(Je_1, e_2). \tag{1.4}$$

Here

$$\left|\overline{\nabla} J_{\Sigma_t}\right|^2 = \left|h^4_{1k} + h^3_{2k}\right|^2 + \left|h^4_{2k} - h^3_{1k}\right|^2 \geq \frac{1}{2}|H|^2. \tag{1.5}$$

We want to see whether the symplectic property is preserved along the mean curvature flow. In the case that M is a Kähler–Einstein surface, we have $Ric(Je_1, e_2) = \bar{\rho}\cos\alpha$, where $\bar{\rho}$ is the scalar curvature of M, so the symplectic property is preserved. If the ambient Kähler surface evolves along the Kähler–Ricci flow, Han and Li [10] derived the evolution equation for $\cos\alpha$ and consequently they showed that the symplectic property is also preserved. In this paper, we find another condition to assure that along the flow, at each time the surface is symplectic. Note that we don't require M to be Einstein. Denote the minimum and maximum of holomorphic sectional curvatures of M by k_1 and k_2. We state our main theorem as follows:

Main Theorem *Suppose M is a Kähler surface with positive holomorphic sectional curvatures. Set $\lambda = \frac{k_2}{k_1}$. If the flow satisfies either*

I. $1 \leq \lambda < \frac{11}{7}$ *and* $\cos\alpha(\cdot, 0) \geq \delta > \frac{53(\lambda-1)}{\sqrt{(53\lambda-53)^2+(48-24\lambda)^2}}$,

or

II. $\frac{11}{7} \leq \lambda < 2$ *and* $\cos\alpha(\cdot, 0) \geq \delta > \frac{8\lambda-5}{\sqrt{(8\lambda-5)^2+(12-6\lambda)^2}}$,

then along the flow

$$\left(\frac{\partial}{\partial t} - \Delta\right)\cos\alpha \geq \left|\overline{\nabla} J_{\Sigma_t}\right|^2 \cos\alpha + C\sin^2\alpha, \tag{1.6}$$

where C is a positive constant depending only on k_1, k_2 and δ. As a corollary, $\min_{\Sigma_t}\cos\alpha$ *is increasing with respect to t. In particular, at each time t, Σ_t is symplectic. Therefore, we call this flow the symplectic mean curvature flow.*

Since we obtain (1.6), many theorems in "symplectic mean curvature flows in Kähler–Einstein surfaces" still hold in our case. For example,

Arguing as in [5] by strong maximum principle, we have

Corollary 1.2 *I. Suppose M is a Kähler surface with positive holomorphic sectional curvatures and $1 \leq \lambda < \frac{11}{7}$, then every symplectic minimal surface satisfying*

$$\cos\alpha > \frac{53(\lambda-1)}{\sqrt{(53\lambda-53)^2+(48-24\lambda)^2}}$$

in M is a holomorphic curve.

II. Suppose M is a Kähler surface with positive holomorphic sectional curvatures and $\frac{11}{7} \leq \lambda < 2$, then every symplectic minimal surface satisfying

$$\cos\alpha > \frac{8\lambda-5}{\sqrt{(8\lambda-5)^2+(12-6\lambda)^2}}$$

in M is a holomorphic curve.

Arguing exactly in the same way as in [2] or [21], we have

Theorem 1.3 *Under the same condition of the Main Theorem, the symplectic mean curvature flow has no type I singularity at any $T > 0$.*

2 Curvature tensor, sectional curvature and holomorphic sectional curvature

Denote the curvature tensor of M by K. Set $K(X) = K(X, JX, X, JX)$ and $K(X, Y) = K(X, Y, X, Y)$, where X, Y are arbitrary vector fields on M. It is known that (c.f. [1,20]) we can express the sectional curvatures by holomorphic sectional curvatures.

Theorem 2.1 *The sectional curvatures of M can be determined by the holomorphic sectional curvatures by*

$$\begin{aligned} K(X, Y) = \frac{1}{32}[&3K(X+JY) + 3K(X-JY) - K(X+Y) - K(X-Y) \\ &-4K(X) - 4K(Y)]. \end{aligned} \tag{2.1}$$

Using (2.1), it is easy to check that,

Theorem 2.2 *For any vector fields X, Y and Z on M,*

$$\begin{aligned}K(X,Y,X,Z) &= \frac{1}{2}[K(Y+Z,X)-K(X,Y)-K(X,Z)]\\ &= \frac{1}{64}[3K(Y+Z+JX)+3K(Y+Z-JX)-K(Y+Z+X)\\ &\quad -K(Y+Z-X)-3K(Y+JX)-3K(Y-JX)-3K(Z+JX)\\ &\quad -3K(Z-JX)-4K(Y+Z)+K(Y+X)+K(Y-X)\\ &\quad +K(Z+X)+K(Z-X)+4K(X)+4K(Y)+4K(Z)]. \qquad (2.2)\end{aligned}$$

Denote the minimum and the maximum of sectional curvatures by K_{min} and K_{max}, respectively, we have the following estimates.

Theorem 2.3 *K_{min} and K_{max} satisfy*

$$K_{max} \leq \frac{3}{2}k_2 - \frac{1}{2}k_1 \qquad (2.3)$$

and

$$K_{min} \geq \frac{3}{4}k_1 - \frac{1}{2}k_2 \qquad (2.4)$$

Proof Given any point $p \in M$ and any two unit orthogonal vectors X and Y at p, we can find two vectors Z and W such that $\{X, Y, Z, W\}$ form an orthonormal basis of T_pM. Suppose $JX = yY + zZ + wW$, then

$$\langle X+JY, X+JY\rangle = 2-2y, \qquad (2.5)$$

and

$$\langle X-JY, X-JY\rangle = 2+2y. \qquad (2.6)$$

Assume the Kähler form is anti-self-dual, it was shown in [12] that, $y^2+z^2+w^2=1$ and J has the form

$$J = \begin{pmatrix} 0 & y & z & w\\ -y & 0 & w & -z\\ -z & -w & 0 & y\\ -w & z & -y & 0\end{pmatrix}. \qquad (2.7)$$

Combining (2.1) with (2.6) and (2.6), we get

$$\begin{aligned}K(X,Y) &\leq \frac{1}{32}\Big[3(2-2y)^2k_2+3(2+2y)^2k_2-2^2k_1-2^2k_1-4k_1-4k_1\Big]\\ &= \frac{1}{4}\Big[(3+3y^2)k_2-2k_1\Big]\\ &\leq \frac{1}{4}(6k_2-2k_1)\\ &= \frac{3}{2}k_2-\frac{1}{2}k_1,\end{aligned}$$

and similarly

$$\begin{aligned} K(X,Y) &\geq \frac{1}{4}[(3+3y^2)k_1 - 2k_2] \\ &\geq \frac{1}{4}(3k_1 - 2k_2) \\ &= \frac{3}{4}k_1 - \frac{1}{2}k_2. \end{aligned}$$

This proves the theorem. □

3 Proof of the Main Theorem

In this section, we will prove the Main Theorem of this paper.

Proof of the Main Theorem In order to prove this theorem, we need to estimate $Ric(Je_1, e_2)$. Using two different methods, we get two available estimates. We now deduce the first one.

$$\begin{aligned} Ric(Je_1, e_2) &= K(Je_1, e_1, e_2, e_1) + K(Je_1, e_3, e_2, e_3) + K(Je_1, e_4, e_2, e_4) \\ &= K(\cos\alpha e_2 + \sin\alpha e_3, e_1, e_2, e_1) + K(\cos\alpha e_2 + \sin\alpha e_3, e_3, e_2, e_3) \\ &\quad + K(\cos\alpha e_2 + \sin\alpha e_3, e_4, e_2, e_4) \\ &= cos\alpha R_{22} + \sin\alpha(K_{3121} + K_{3424}), \end{aligned} \tag{3.1}$$

where

$$R_{22} = K_{2121} + K_{2323} + K_{2424}. \tag{3.2}$$

By (2.1), we have

$$\begin{aligned} K_{2121} = \frac{1}{32}[&3K(e_1 + Je_2) + 3K(e_1 - Je_2) - K(e_1 + e_2) - K(e_1 - e_2) \\ &-4K(e_1) - 4K(e_2)]. \end{aligned}$$

By our choice of the complex structure (1.3), we get

$$\langle e_1 + Je_2, e_1 + Je_2 \rangle = 2 - 2\cos\alpha,$$

and

$$\langle e_1 - Je_2, e_1 - Je_2 \rangle = 2 + 2\cos\alpha.$$

Hence K_{2121} can be estimated by k_1 and k_2,

$$\begin{aligned} K_{2121} &\geq \frac{1}{32}[3(2-2\cos\alpha)^2k_1 + 3(2+2\cos\alpha)^2k_1 - 2^2k_2 - 2^2k_2 - 4k_2 - 4k_2] \\ &= \frac{1}{4}[(3+3\cos^2\alpha)k_1 - 2k_2]. \end{aligned} \tag{3.3}$$

Similarly, we get

$$K_{2323} \geq \frac{1}{4}(3k_1 - 2k_2), \tag{3.4}$$

and

$$K_{2424} \geq \frac{1}{4}[(3+3\sin^2\alpha)k_1 - 2k_2]. \tag{3.5}$$

Putting (3.3), (3.4) and (3.5) into (3.2), we obtain that

$$R_{22} \geq 3k_1 - \frac{3}{2}k_2. \tag{3.6}$$

Using (2.2) and (1.3), we can also estimate K_{3121} and K_{3424}. We have

$$K_{3121} \geq \frac{1}{32}[(53 + 48 \sin\alpha \cos\alpha)k_1 - 53k_2], \tag{3.7}$$

and

$$K_{3424} \geq \frac{1}{32}[(53 - 48 \sin\alpha \cos\alpha)k_1 - 53k_2]. \tag{3.8}$$

Adding (3.7) and (3.8) yields

$$K_{3121} + K_{3424} \geq -\frac{53}{16}(k_2 - k_1). \tag{3.9}$$

By a similar computation in the opposite direction, we get

$$|K_{3121} + K_{3424}| \leq \frac{53}{16}(k_2 - k_1). \tag{3.10}$$

Therefore by (3.1), (3.6), (3.10) and short time existence of the mean curvature flow, we have

$$\begin{aligned} Ric(Je_1, e_2) &\geq \cos\alpha(3k_1 - \frac{3}{2}k_2) - \sqrt{1-\cos^2\alpha}\frac{53}{16}(k_2 - k_1) \\ &= \left(3\cos\alpha + \frac{53}{16}\sqrt{1-\cos^2\alpha}\right)k_1 - \left(\frac{3}{2}\cos\alpha + \frac{53}{16}\sqrt{1-\cos^2\alpha}\right)k_2. \end{aligned} \tag{3.11}$$

If $1 \leq \lambda < 2$ and $\cos\alpha > \frac{53(\lambda-1)}{\sqrt{(53\lambda-53)^2+(48-24\lambda)^2}}$, then the RHS of (3.11) is positive.

Another estimate follows directly from Theorem 2.3 and Berger inequality (c.f. [7]) that

$$|K_{3121} + K_{3424}| \leq |K_{3121}| + |K_{3424}| \leq K_{max} - K_{min} \leq 2k_2 - \frac{5}{4}k_1. \tag{3.12}$$

Putting the above estimate into (3.1) yields

$$\begin{aligned} Ric(Je_1, e_2) &\geq \cos\alpha\left(3k_1 - \frac{3}{2}k_2\right) - \sqrt{1-\cos^2\alpha}\left(2k_2 - \frac{5}{4}k_1\right) \\ &= \left(3\cos\alpha + \frac{5}{4}\sqrt{1-\cos^2\alpha}\right)k_1 - \left(\frac{3}{2}\cos\alpha + \frac{5}{4}\sqrt{1-\cos^2\alpha}\right)k_2. \end{aligned} \tag{3.13}$$

It follows that if $1 \leq \lambda < 2$ and $\cos\alpha > \frac{8\lambda-5}{\sqrt{(8\lambda-5)^2+(12-6\lambda)^2}}$, then the RHS of (3.11) is positive. Note that

$$\frac{53(\lambda-1)}{\sqrt{(53\lambda-53)^2+(48-24\lambda)^2}} \leq \frac{8\lambda-5}{\sqrt{(8\lambda-5)^2+(12-6\lambda)^2}}$$

for $1 \leq \lambda < \frac{11}{7}$, and

$$\frac{53(\lambda-1)}{\sqrt{(53\lambda-53)^2+(48-24\lambda)^2}} \geq \frac{8\lambda-5}{\sqrt{(8\lambda-5)^2+(12-6\lambda)^2}}$$

for $\frac{11}{7} \leq \lambda < 2$, we get the conclusion. □

Acknowledgments The research was supported by NSFC 11071236, 11131007 and 10421101.

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