Geom Dedicata (2014) 169:411–418
DOI 10.1007/s10711-013-9864-z

ORIGINAL PAPER

# Harnack estimates for geometric flows, applications to Ricci flow coupled with harmonic map flow

**Hongxin Guo · Tongtong He**

Received: 10 October 2012 / Accepted: 25 April 2013 / Published online: 22 June 2013
© Springer Science+Business Media Dordrecht 2013

**Abstract** We derive Harnack estimates for heat and conjugate heat equations in abstract geometric flows. The main results lead to new Harnack inequalities for a variety of geometric flows. In particular, Harnack inequalities for the Ricci flow coupled with Harmonic map flow are obtained.

**Keywords** Ricci flow · Conjugate heat equation · Harnack estimate

**Mathematics Subject Classification (1991)** Primary 53C44

## 1 Introduction

Assume that $M$ is an $n$-dimensional compact manifold endowed with a one-parameter family of Riemannian metrics $g(t)$ evolving along the general flow equation

$$\frac{\partial g(t,x)}{\partial t} = -2\alpha(t,x) \tag{1.1}$$

which exists on $[0, T)$. Here $\alpha(t,x)$ is a one-parameter family of smooth symmetric two tensors on $M$. In particular when $\alpha = \mathrm{Rc}$ Eq. (1.1) is Hamilton's Ricci flow. Let

$$A(t,x) \doteqdot g^{ij}\alpha_{ij}$$

be the trace of $\alpha$ with respect to the time-dependent metric $g(t)$.

H. Guo (✉) · T. He
School of Mathematics and Information Science, Wenzhou University, Wenzhou 325035, Zhejiang, China
e-mail: guo@wzu.edu.cn

H. Guo
Mathematics Research Unit, 6 rue Richard Coudenhove-Kalergi, 1359 Luxembourg, Grand-Duchy of Luxembourg

In [9], Reto Müller studied reduced volume for the abstract flow (1.1). Müller defined an interesting quantity for the tensor $\alpha$ by

$$\mathcal{D}_\alpha(V) \doteqdot \frac{\partial A}{\partial t} - \Delta A - 2|\alpha|^2 + 2\,(\mathrm{Rc} - \alpha)\,(V, V) + \langle 4\,\mathrm{Div}(\alpha) - 2\nabla A, V\rangle \tag{1.2}$$

where Div is the divergence operator defined by $\mathrm{Div}(\alpha)_k = g^{ij}\nabla_i \alpha_{jk}$ in local coordinates.

Under the assumption that $\mathcal{D}_\alpha$ is nonnegative, Müller obtained monotonicity of the reduced volumes. For any vector field $V$, $\mathcal{D}_\alpha(V)$ is nonnegative in the following flows: static manifold with nonnegative Ricci curvature, Hamilton's Ricci flow (in fact $\mathcal{D} = 0$ in this case), List's extended Ricci flow [8], Müller's Ricci flow coupled with Harmonic map flow [10] and Lorenzian mean curvature flow when the ambient space has nonnegative sectional curvature. See [9] for details.

In a recent preprint [5], the authors proved monotonicity of the entropy and lowest eigenvalue in abstract flow (1.1) when $\mathcal{D}_\alpha \geq 0$.

The purpose of this note is to prove Harnack inequalities in the abstract setting with $\mathcal{D}_\alpha \geq 0$. In Sect. 2 we derive Harnack estimates for the conjugate heat equation, while in Sect. 3 for the forward heat equation with potential.

As applications, we apply our abstract formulations to the Ricci flow coupled with harmonic map flow and obtain Harnack estimates for this flow.

## 2 Harnack for the conjugate heat equation

Assume $u$ is a positive solution to the conjugate heat equation

$$\frac{\partial u}{\partial t} = -\Delta u + Au \tag{2.1}$$

where $\Delta$ is the time-dependent Laplace–Beltrami operator with respect to $g(t)$. For the derivative of $\Delta$ we have

$$\left(\frac{\partial}{\partial t}\Delta\right) f = 2\langle \alpha, \nabla\nabla f\rangle + \langle 2\,\mathrm{Div}(\alpha) - \nabla A, \nabla f\rangle \tag{2.2}$$

where $f$ is any smooth function on $M$. The formula can be found in standard textbooks, for instance [3].

Let

$$P \doteqdot 2\Delta \log u + |\nabla \log u|^2 - A + \frac{2n}{\tau} \tag{2.3}$$

where $\tau \doteqdot T - t$.

**Lemma 2.1** *Along the flow* (1.1)*, $P$ satisfies*

$$\begin{aligned}\frac{\partial P}{\partial \tau} &= \Delta P + 2\langle \nabla P, \nabla \log u\rangle - \frac{2P}{\tau} + 2\left|\nabla\nabla \log u - \alpha + \frac{1}{\tau} g\right|^2 \\ &\quad + \frac{2}{\tau}|\nabla \log u|^2 + \frac{2A}{\tau} + \mathcal{D}_\alpha(-\nabla \log u).\end{aligned} \tag{2.4}$$

*Proof* For any positive solution $u$ to the conjugate heat Eq. (2.1) one has

$$\frac{\partial}{\partial t}\log u = \frac{\partial_t u}{u} = -\frac{\Delta u}{u} + A = -\Delta \log u - |\nabla \log u|^2 + A.$$

Notice that $\tau = T - t$ and we have

$$\begin{aligned}\frac{\partial}{\partial \tau}(\Delta \log u) &= -\frac{\partial \Delta}{\partial t}\log u - \Delta\left(\frac{\partial \log u}{\partial t}\right)\\ &= -2\langle \alpha, \nabla\nabla \log u\rangle - \langle 2\,\mathrm{Div}\,\alpha - \nabla A, \nabla \log u\rangle\\ &\quad + \Delta\left(\Delta \log u + |\nabla \log u|^2 - A\right)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial \tau}\left(|\nabla \log u|^2\right) &= -2\alpha(\nabla \log u, \nabla \log u)\\ &\quad + 2\langle \nabla\left(\Delta \log u + |\nabla \log u|^2 - A\right), \nabla \log u\rangle.\\ \frac{\partial P}{\partial \tau} &= 2\frac{\partial}{\partial \tau}\Delta \log u + \frac{\partial}{\partial \tau}|\nabla \log u|^2 - \frac{\partial A}{\partial \tau} - \frac{2n}{\tau^2}\\ &= -4\langle \alpha, \nabla\nabla \log u\rangle - \langle 4\,\mathrm{Div}\,\alpha - 2\nabla A, \nabla \log u\rangle\\ &\quad + \Delta\left(2\Delta \log u + 2|\nabla \log u|^2 - 2A\right) - 2\alpha(\nabla \log u, \nabla \log u)\\ &\quad + 2\langle \nabla\left(\Delta \log u + |\nabla \log u|^2 - A\right), \nabla \log u\rangle - \frac{\partial A}{\partial \tau} - \frac{2n}{\tau^2}\\ &= -4\langle \alpha, \nabla\nabla \log u\rangle - \langle 4\,\mathrm{Div}\,\alpha - 2\nabla A, \nabla \log u\rangle\\ &\quad + \Delta\left(P + |\nabla \log u|^2 - A\right) - 2\alpha(\nabla \log u, \nabla \log u)\\ &\quad + 2\langle \nabla\left(\Delta \log u + |\nabla \log u|^2 - A\right), \nabla \log u\rangle - \frac{\partial A}{\partial \tau} - \frac{2n}{\tau^2}\end{aligned}$$

and then by the Bochner formula

$$\Delta|\nabla \log u|^2 = 2|\nabla\nabla \log u|^2 + 2\,\mathrm{Rc}\,(\nabla \log u, \nabla \log u) + 2\langle \nabla(\Delta \log u), \nabla \log u\rangle$$

we have

$$\begin{aligned}\frac{\partial P}{\partial \tau} &= 2|\nabla\nabla \log u|^2 - 4\langle \alpha, \nabla\nabla \log u\rangle - \langle 4\,\mathrm{Div}\,\alpha - 2\nabla A, \nabla \log u\rangle + \Delta P\\ &\quad + 2\,(\mathrm{Rc} - \alpha)\,(\nabla \log u, \nabla \log u) + 2\langle \nabla P, \nabla \log u\rangle - \frac{\partial A}{\partial \tau} - \frac{2n}{\tau^2} - \Delta A\\ &= 2\left|\nabla\nabla \log u - \alpha + \frac{1}{\tau}g\right|^2 - 2|\alpha|^2 - \frac{2n}{\tau^2} + \frac{4}{\tau}A - \frac{4}{\tau}\Delta \log u\\ &\quad - \langle 4\,\mathrm{Div}\,\alpha - 2\nabla A, \nabla \log u\rangle + \Delta P\\ &\quad + 2\,(\mathrm{Rc} - \alpha)\,(\nabla \log u, \nabla \log u) + 2\langle \nabla P, \nabla \log u\rangle - \frac{\partial A}{\partial \tau} - \frac{2n}{\tau^2} - \Delta A\\ &= \Delta P + 2\langle \nabla P, \nabla \log u\rangle + 2\left|\nabla\nabla \log u - \alpha + \frac{1}{\tau}g\right|^2\\ &\quad - \frac{2}{\tau}P + \frac{2}{\tau}A + \frac{2}{\tau}|\nabla \log u|^2 - \frac{\partial A}{\partial \tau} - 2|\alpha|^2 - \Delta A\\ &\quad + 2\,(\mathrm{Rc} - \alpha)\,(\nabla \log u, \nabla \log u) - \langle 4\,\mathrm{Div}\,\alpha - 2\nabla A, \nabla \log u\rangle\end{aligned}$$

Notice that $\frac{\partial A}{\partial \tau} = -\frac{\partial A}{\partial t}$ and by the definition of $\mathcal{D}$ we see the last five terms of the above is nothing but $\mathcal{D}_\alpha(-\nabla \log u)$. □

**Theorem 2.2** *Under the same assumptions as in Lemma 2.1, if for $t \in [0, T)$, $\frac{2}{\tau}|V|^2 + \frac{2}{\tau}A + \mathcal{D}_\alpha(V) \geq 0$, in particular if*

$$\mathcal{D}_\alpha(V) \geq 0, \quad \text{and} \quad A \geq 0$$

*then*

$$2\Delta \log u + |\nabla \log u|^2 - A + \frac{2n}{\tau} \geq 0. \tag{2.5}$$

*Moreover, for any two points $(x_1, t_1), (x_2, t_2) \in M \times (0, T)$ with $t_1 < t_2$ one has*

$$u(x_2, t_2) \leq u(x_1, t_1) \left(\frac{T - t_1}{T - t_2}\right)^n \exp\left(\frac{1}{2}\int_{t_1}^{t_2} \left(|\gamma'(t)|^2 + A\right) dt\right) \tag{2.6}$$

*where $\gamma(s) : [t_1, t_2] \to M$ is a smooth curve connecting $x_1$ and $x_2$ with $\gamma(t_i) = x_i$, $i = 1, 2$.*

*Proof* Under the assumption that $\frac{2}{\tau}|V|^2 + \frac{2}{\tau}A + \mathcal{D}_\alpha(V) \geq 0$ we can conclude from Eq. (2.4) that

$$\frac{\partial P}{\partial \tau} \geq \Delta P + 2\langle \nabla P, \nabla \log u\rangle - \frac{2P}{\tau}$$

Notice that for $\tau$ sufficiently small we have $P > 0$ and by the maximum principle we know that $P \geq 0$ for all $\tau \in (0, T)$. This proves (2.5).

As standard, integrating (2.5) we have (2.6). Indeed, along a smooth curve $\gamma$ we have

$$\begin{aligned}
\frac{d}{dt} \log u\,(\gamma(t), t) &= \langle \nabla \log u, \gamma'\rangle + \frac{\partial \log u}{\partial t} \\
&= \langle \nabla \log u, \gamma'\rangle - \Delta \log u - |\nabla \log u|^2 + A \\
&\leq \langle \nabla \log u, \gamma'\rangle - \frac{1}{2}|\nabla \log u|^2 + \frac{A}{2} + \frac{n}{T - t} \\
&\leq \frac{1}{2}\left(|\gamma'|^2 + A\right) + \frac{n}{T - t}
\end{aligned}$$

and moreover

$$\begin{aligned}
\log \frac{u(x_2, t_2)}{u(x_1, t_1)} &\leq \int_{t_1}^{t_2} \left(\frac{1}{2}\left(|\gamma'|^2 + A\right) + \frac{n}{T - t}\right) dt \\
&= n \log \frac{T - t_1}{T - t_2} + \frac{1}{2}\int_{t_1}^{t_2} \left(|\gamma'|^2 + A\right) dt
\end{aligned}$$

and this proves the classical Harnack inequality (2.6). □

When the metric is static, i.e. $\alpha = 0$ we know that $\mathcal{D}$ is nonnegative when $(M, g)$ has nonnegative Ricci curvature. Thus for positive solutions to the heat equation one has (2.5) and (2.6) which are however weaker than the Li-Yau Harnack.

In the case of Ricci flow where $\alpha = \text{Rc}$ and $\mathcal{D}_\alpha(V) = 0$, (2.5) and (2.6) have been independently proved by Cao [1] and Kuang-Zhang [7] for nonnegative scalar curvature.

In the following we show new Harnack inequalities in the Ricci flow coupled with harmonic map flow. Suppose that $(N, \gamma)$ is a compact static Riemannian manifold, $a(t)$ a non-negative and non-increasing function depending only on time, and $\varphi(t)\colon M \to N$ a family of 1-parameter smooth maps. Then $(g(t), \varphi(t))$ is called a solution to Müller's Ricci flow coupled with harmonic map flow with coupling function $a(t)$, if it satisfies

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Rc} + 2a(t)\nabla\varphi \otimes \nabla\varphi$$
$$\frac{\partial \varphi}{\partial t} = \tau_g \varphi \tag{2.7}$$

where $\tau_g$ denotes the tension field of the map $\varphi$ with respect to the evolving metric $g(t)$.

**Corollary 2.3** *Assume that* $(M, g(t)$ *is a solution to* (2.7) *with*

$$R(0) - a(0)|\nabla\varphi|^2_{t=0} \geq 0$$

*then for any positive solution to*

$$\frac{\partial u}{\partial t} = -\Delta u + \left(R - a(t)|\nabla\varphi|^2\right) u$$

*we have*

$$2\Delta \log u + |\nabla \log u|^2 - \left(R - a(t)|\nabla\varphi|^2\right) + \frac{2n}{\tau} \geq 0. \tag{2.8}$$

*Moreover, for any two points* $(x_1, t_1), (x_2, t_2) \in M \times (0, T)$ *with* $t_1 < t_2$ *one has*

$$u(x_2, t_2) \leq u(x_1, t_1) \left(\frac{T - t_1}{T - t_2}\right)^n \exp\left(\frac{1}{2} \int_{t_1}^{t_2} \left(|\gamma'(t)|^2 + R - a(t)|\nabla\varphi|^2\right) dt\right) \tag{2.9}$$

*where* $\gamma(s) : [t_1, t_2] \to M$ *is a smooth curve connecting* $x_1$ *and* $x_2$ *with* $\gamma(t_i) = x_i$, $i = 1, 2$.

*Proof* $\mathcal{D}$ is nonnegative in the Ricci flow coupled with harmonic map flow, and nonnegativity of $R - a(t)|\nabla\varphi|^2$ is preserved by the flow (see [10] for details). Thus the assumption in Theorem 2.2 is satisfied and the conclusions follow. □

*Remark 2.4* As pointed out to us by the referee, (2.8) was independently proved by Zhu in [11] by a direct computation. While we here conclude (2.8) from the much more general result in Theorem 2.2.

## 3 Harnack for the heat equation with potential

In this section we consider the forward heat equation. Assume $u$ is a positive solution to

$$\frac{\partial u}{\partial t} = \Delta u + Au \tag{3.1}$$

In this section, we shall use another notation also introduced by Müller in [9]. To understand more about the quantity $\mathcal{D}$, Müller introduced

$$\mathcal{H}_\alpha(V) \doteqdot \frac{\partial A}{\partial t} + \frac{A}{t} - 2\langle \nabla A, V\rangle + 2\alpha\,(V, V) \tag{3.2}$$

If the flow is Hamilton's Ricci flow, then

$$\mathcal{H}(V) = \frac{\partial R}{\partial t} + \frac{R}{t} - 2\langle \nabla R, V\rangle + 2\,\mathrm{Rc}\,(V, V)$$

which is nonnegative if $g(0)$ has nonnegative curvature operator by Hamilton's trace Harnack inequality for the Ricci flow [6].

Following Cao-Hamilton's work in the Ricci flow [2], we define

$$H \doteqdot 2\Delta \log u + |\nabla \log u|^2 + 3A + \frac{2n}{t} \tag{3.3}$$

and prove that

**Lemma 3.1** *Along the flow* (1.1)*, for any positive solution to* (3.1) *H satisfies*

$$\begin{aligned}\frac{\partial H}{\partial t} = \Delta H + 2\langle \nabla H, \nabla \log u\rangle - \frac{2H}{t} + 2\left|\nabla\nabla \log u + \alpha + \frac{1}{t}g\right|^2 \\ + \frac{2}{t}|\nabla \log u|^2 + 2\mathcal{H}_\alpha(\nabla \log u) + \mathcal{D}_\alpha(\nabla \log u).\end{aligned} \tag{3.4}$$

*Proof* For any positive solution $u$ to Eq. (3.1) one has

$$\frac{\partial}{\partial t}\log u = \frac{\partial_t u}{u} = \frac{\Delta u}{u} + A = \Delta \log u + |\nabla \log u|^2 + A.$$

Notice that

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta \log u) =& \frac{\partial \Delta}{\partial t}\log u + \Delta\left(\frac{\partial}{\partial t}\log u\right)\\ =& 2\langle \alpha, \nabla\nabla \log u\rangle + \langle 2\,\mathrm{Div}(\alpha) - \nabla A, \nabla \log u\rangle \\ &+ \Delta\left(\Delta \log u + |\nabla \log u|^2 + A\right)\end{aligned}$$

and

$$\frac{\partial}{\partial t}|\nabla \log u|^2 = 2\alpha(\nabla \log u, \nabla \log u) + 2\langle \nabla\left(\Delta \log u + |\nabla \log u|^2 + A\right), \nabla \log u\rangle$$

We have

$$\begin{aligned}\frac{\partial H}{\partial t} =& 2\frac{\partial}{\partial t}\Delta \log u + \frac{\partial}{\partial t}|\nabla \log u|^2 + 3\frac{\partial A}{\partial t} - \frac{2n}{t^2}\\ =& 4\langle \alpha, \nabla\nabla \log u\rangle + \langle 4\,\mathrm{Div}(\alpha) - 2\nabla A, \nabla \log u\rangle + \Delta\left(H + |\nabla \log u|^2 - A\right)\\ &+ 2\alpha(\nabla \log u, \nabla \log u) + 2\langle \nabla\left(\Delta \log u + |\nabla \log u|^2 + A\right), \nabla \log u\rangle\\ &+ 3\frac{\partial A}{\partial t} - \frac{2n}{t^2}\\ =& \Delta H + 2|\nabla\nabla \log u|^2 + 4\langle \alpha, \nabla\nabla \log u\rangle + 2\,\mathrm{Rc}(\nabla \log u, \nabla \log u)\\ &+ 2\alpha(\nabla \log u, \nabla \log u) + 2\langle \nabla\left(2\Delta \log u + |\nabla \log u|^2 + A\right), \nabla \log u\rangle\\ &+ 3\frac{\partial A}{\partial t} - \frac{2n}{t^2} + \langle 4\,\mathrm{Div}(\alpha) - 2\nabla A, \nabla \log u\rangle - \Delta A\\ =& \Delta H + 2\left|\nabla\nabla \log u + \alpha + \frac{1}{t}g\right|^2 - 2|\alpha|^2 - \frac{2n}{t^2} - \frac{4\Delta \log u}{t} - \frac{4A}{t}\\ &+ 2\,\mathrm{Rc}(\nabla \log u, \nabla \log u) + 2\alpha(\nabla \log u, \nabla \log u) + 2\langle \nabla(H - 2A), \nabla \log u\rangle\\ &+ 3\frac{\partial A}{\partial t} - \frac{2n}{t^2} + \langle 4\,\mathrm{Div}(\alpha) - 2\nabla A, \nabla \log u\rangle - \Delta A\end{aligned}$$

$$= \Delta H + 2\left|\nabla\nabla \log u + \alpha + \frac{1}{t}g\right|^2 - 2|\alpha|^2 - \frac{2H}{t} + \frac{2}{t}|\nabla \log u|^2 + \frac{2A}{t}$$
$$+ 2\,\mathrm{Rc}(\nabla \log u, \nabla \log u) + 2\alpha(\nabla \log u, \nabla \log u) + 2\langle \nabla H, \nabla \log u\rangle$$
$$- 4\langle \nabla A, \nabla \log u\rangle + 3\frac{\partial A}{\partial t} + \langle 4\,\mathrm{Div}(\alpha) - 2\nabla A, \nabla \log u\rangle - \Delta A$$
$$= \Delta H + 2\langle \nabla H, \nabla \log u\rangle - \frac{2H}{t} + 2\left|\nabla\nabla \log u + \alpha + \frac{1}{t}g\right|^2 + \frac{2}{t}|\nabla \log u|^2$$
$$+ 2\frac{\partial A}{\partial t} + \frac{2A}{t} - 4\langle \nabla A, \nabla \log u\rangle + 4\alpha(\nabla \log u, \nabla \log u)$$
$$+ \frac{\partial A}{\partial t} - \Delta A - 2|\alpha|^2 + 2\,(\mathrm{Rc} - \alpha)\,(\nabla \log u, \nabla \log u)$$
$$+ \langle 4\,\mathrm{Div}(\alpha) - 2\nabla A, \nabla \log u\rangle$$
$$= \Delta H + 2\langle \nabla H, \nabla \log u\rangle - \frac{2H}{t} + 2\left|\nabla\nabla \log u + \alpha + \frac{1}{t}g\right|^2 + \frac{2}{t}|\nabla \log u|^2$$
$$+ 2\mathcal{H}_\alpha(\nabla \log u) + \mathcal{D}_\alpha(\nabla \log u)$$

□

**Theorem 3.2** *Under the same assumptions as in Lemma 3.1, if* $2\mathcal{H}_\alpha(V) + \mathcal{D}_\alpha(V) + \frac{2}{t}|V|^2 \geq 0$*, in particular if* $\mathcal{H}_\alpha(V) \geq 0$ *and* $D_\alpha(V) \geq 0$ *then*

$$2\Delta \log u + |\nabla \log u|^2 + 3A + \frac{2n}{t} \geq 0 \tag{3.5}$$

*and for any two points* $(x_1, t_1), (x_2, t_2) \in M \times (0, T)$ *with* $t_1 < t_2$ *we have*

$$u(x_1, t_1) \leq u(x_2, t_2)\left(\frac{t_2}{t_1}\right)^n \exp\left(\frac{1}{2}\int_{t_1}^{t_2} \left(|\gamma'(t)|^2 + A\right) dt\right) \tag{3.6}$$

*where* $\gamma(s) : [t_1, t_2] \to M$ *is a smooth curve connecting* $x_1$ *and* $x_2$ *with* $\gamma(t_i) = x_i$, $i = 1, 2$.

*Proof* The proof is analogous to the proof of Theorem 2.2. We omit details. □

*Remark 3.3* 1. In the static case, where $A = 0$ Eqs. (2.5) and (3.5) give the same estimate.
2. In the Ricci flow, Hamilton has already proved nonnegativity of $\mathcal{H}$ under nonnegative curvature operator assumption [6]. Thus assuming that $(M, g(0))$ has nonnegative curvature operator, one has Eqs. (3.5) and (3.6). This has been proved in [2].
3. For the Ricci flow coupled with Harmonic map flow (2.7), under assumption that $\mathcal{H}_\alpha(V)$ is nonnegative, Harnack estimates (3.5) and (3.6) hold. We note that Fang [4] has proved Eq. (3.5) under the assumption that $\mathcal{H}_\alpha(V) \geq 0$ in List's extended Ricci flow, which is a special case of the Ricci flow coupled with harmonic map flow (2.7). It would be interesting to see whether one can prove nonnegativity of $\mathcal{H}$ under reasonable assumptions by generalizing Hamilton's arguments of the trace Harnack for the Ricci flow [6].

**Acknowledgments** Research supported by NSF of China (Grants No. 11001203 and 11171143) and Zhejiang Provincial Natural Science Foundation of China (Project No. LY13A010009). HG thanks Professors Robert Philipowski and Anton Thalmaier for very helpful discussions. HG and TH thank the anonymous referee for pointing out the reference [11] to them, and for the expertise suggestions they have followed to improve the original manuscript significantly.

## References


1. Cao, X.: Differential Harnack estimates for backward heat equations with potentials under the Ricci flow. J. Funct. Anal. **255**(4), 1024–1038 (2008)
2. Cao, X., Hamilton, R.S.: Differential Harnack estimates for time-dependent heat equations with potentials. Geom. Funct. Anal. **19**(4), 989–1000 (2009)
3. Chow, B., Lu, P., Ni, L.: Hamilton's Ricci flow. Graduate Studies in Mathematics, vol. 77. American Mathematical Society; Science Press, Providence, RI; New York (2006)
4. Fang, S.: Differential Harnack inequalities for heat equations with potentials under the Bernhard List's flow. Geom. Dedicata. doi:10.1007/s10711-011-9690-0
5. Guo, H., Philipowski, R., Thalmaier, A.: Entropy and lowest eigenvalue on evolving manifolds. Pacific J. Math., to appear
6. Hamilton, R.S.: The Harnack estimate for the Ricci flow. J. Differ. Geom. **37**(1), 225–243 (1993)
7. Kuang, S., Zhang, Q.S.: A gradient estimate for all positive solutions of the conjugate heat equation under Ricci flow. J. Funct. Anal. **255**(4), 1008–1023 (2008)
8. List, B.: Evolution of an extended Ricci flow system. Comm. Anal. Geom. **16**(5), 1007–1048 (2008)
9. Müller, R.: Monotone volume formulas for geometric flows. J. Reine Angew. Math. **643**, 39–57 (2010)
10. Müller, R.: Ricci flow coupled with harmonic map flow. Ann. Sci. Ec. Norm. Super. **45**(1), 101–142 (2012)
11. Zhu, A.: Differential Harnack inequalities for backward heat equation with potential under the Harmonic-Ricci flow. J. Math. Anal. Appl. **406**(2), 502–510 (2013)