Geom Dedicata (2013) 164:227–229
DOI 10.1007/s10711-012-9769-2

ORIGINAL PAPER

A note on the extremal bodies of constant width for the Minkowski measure

Hailin Jin · Qi Guo

Received: 23 July 2012 / Accepted: 4 August 2012 / Published online: 14 August 2012

Abstract In a previous paper, we showed that for all convex bodies K of constant width in $\mathbb{R}^n$, $1 \leq \mathrm{as}_\infty(K) \leq \frac{n+\sqrt{2n(n+1)}}{n+2}$, where $\mathrm{as}_\infty(\cdot)$ denotes the Minkowski measure of asymmetry, with the equality holding on the right-hand side if K is a completion of a regular simplex, and asked whether or not the completions of regular simplices are the only bodies for the equality. A positive answer is given in this short note.

Keywords Asymmetry measures · Constant width · Completion · Meissner's body

Mathematics Subject Classification (1991) 52A38

1 Introduction and preliminary

In paper [4], we proved the following theorem.

Theorem *If K is a convex bodies of constant width in $\mathbb{R}^n$, then $1 \leq \mathrm{as}_\infty(K) \leq \frac{n+\sqrt{2n(n+1)}}{n+2}$, where $\mathrm{as}_\infty(\cdot)$ denotes the Minkowski measure of asymmetry. Moreover, the equality holds on the left-hand side precisely iff K is an Euclidean ball and the upper bounds are attained when K are completions of a regular simplex.*

Then, we asked whether or not the completions of regular simplices are the only bodies for the equality.

Project supported by The NSF of Jiangsu Higher Education (08KJD110016).

H. Jin (✉)
Department of Mathematics, Shanghai University, Shanghai, China
e-mail: ahjhl@sohu.com

Q. Guo
Department of Mathematics, Suzhou University of Science and Technology, Suzhou, China
e-mail: guoqi@mail.usts.edu.cn

We give a positive answer to the above question in this short note. Concretely, we show the following theorem.

Main Theorem *For convex body K of constant width in $\mathbb{R}^n$, if* $\mathrm{as}_\infty(K) = \frac{n+\sqrt{2n(n+1)}}{n+2}$, *then K is a completion of an n-dimensional regular simplex.*

In this paper, $\mathbb{R}^n$ denotes the usual n-dimensional Euclidean space with the canonical inner product $\langle \cdot, \cdot \rangle$. A compact convex set $C \subset \mathbb{R}^n$ is called a *convex body* if it has non-empty interior (int for brivity). The family of all convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{K}^n$. In general, we refer the reader to [7] for standard notation.

Given a convex body $C \in \mathcal{K}^n$ and $x \in \mathrm{int}(C)$, for a hyperplane H through x and the pair H_1, H_2 of support hyperplanes of C parallel to H, let $r(H, x)$ be the ratio, not less than 1, in which H divides the distance between H_1 and H_2. Denote

$$r(C, x) = \max\{r(H, x) : H \ni x\}.$$

Then the *Minkowski measure* $\mathrm{as}_\infty(C)$ *of asymmetry* of C is defined as (see [2,3,5])

$$\mathrm{as}_\infty(C) = \min_{x \in \mathrm{int}(C)} r(C, x).$$

A point $x \in \mathrm{int}(C)$ satisfying $r(C, x) = \mathrm{as}_\infty(C)$ is called a *critical point* (of C). The set of all critical points of C is denoted by C^*.

$C \in \mathcal{K}^n$ is said to be of *constant width* if its width function, i.e., the support function of $C + (-C)$, is a constant (see [1]). Equivalently, C is of constant width iff each boundary point of C is incident with (at least) one diameter (=chord of maximal length) of C. The family of all convex bodies of constant width in $\mathbb{R}^n$ is denoted by $\mathcal{W}^n$.

$C \in \mathcal{K}^n$ is said to be complete if there is no $C' \in \mathcal{K}^n$ with $C \subset C', C \neq C'$ and $d(C) = d(C')$. It is known that a convex body is complete iff it is of constant width (Meissner Theorem [1]).

If $K \in \mathcal{K}^n$ then any complete set C with $K \subset C$ and $d(K) = d(C)$ is called a completion of K. It is also known that every convex bodies has at least one completion [1], e.g. the (unique) completion of a regular triangle is the well-known Reuleaux triangle. The (unique) completion of a regular polygon with odd sides is called a regular Reuleaux polygon, and regular tetrahedrons have two completions both of which are called Meissner tetrahedron (or Meissner body). In general, a convex body may have many completions.

2 Proof of main theorem

In principle, our proof is just an observation of some earlier work of other authors. In the following, we denote by $S_r(p)$ a sphere with center p and radius r, and $r(K)$, $R(K)$ denote the radia of insphere and circumsphere of K respectively.

The following theorem is called Melzak's theorem (see [6])

Theorem 1 *Let $K \in \mathcal{W}^n$ with width ω. Then K has a unique circumsphere $S_{R(K)}(p)$, a unique insphere $S_{r(K)}(q)$ and*

(1) $(1 - \sqrt{n/2(n+1)})\omega \leq r(K),\ \ R(K) \leq \sqrt{n/2(n+1)}\omega$;
(2) $r(K) + R(K) = \omega$;
(3) $p = q$.

Moreover, both equalties in (1) *hold at the same time, and the equality occurs iff K contains a regular n-simplex with diameter ω.*

Corollary 1 *Let $K \in \mathcal{W}^n$ with width ω. Then*

$$R(K) = \sqrt{n/2(n+1)}\omega$$

iff K contains a regular n-simplex with diameter ω, i.e. K is a completion of a regular simplex with diameter ω.

Theorem 2 [4] *Let $K \in \mathcal{W}^n$ with width ω, then* $\mathrm{as}_\infty(K) = \frac{R(K)}{r(K)}$.

Proof of Main Theorem If $\mathrm{as}_\infty(K) = \frac{n+\sqrt{2n(n+1)}}{n+2}$, then $\frac{R(K)}{r(K)} = \frac{R(K)}{\omega - R(K)} = \frac{n+\sqrt{2n(n+1)}}{n+2}$, so $R(K) = \sqrt{n/2(n+1)}\omega$. By Corollary 1, K is a completion of a regular simplex with diameter ω. □

Therefore we get the following theorem.

Theorem 3 *If K is a convex bodies of constant width in $\mathbb{R}^n$, then* $1 \leq \mathrm{as}_\infty(K) \leq \frac{n+\sqrt{2n(n+1)}}{n+2}$. *The equality holds on the left-hand side iff K is an Euclidean ball and on the right-hand precisely iff K is a completion of a regular simplex.*

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