Rank two quadratic pairs and surface group representations

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Abstract   Let X be a compact Riemann surface. A quadratic pair on X consists of a holomorphic vector bundle with a quadratic form which takes values in a fixed line bundle. We show that the moduli spaces of quadratic pairs of rank 2 are connected under some constraints on their topological invariants. As an application of our results we determine the connected components of the $SO_0(2, 3)$-character variety of X.

1 Introduction

Let X be a compact Riemann surface of genus $g \geqslant 2$. Many kinds of pairs (V, φ) on X, consisting of a holomorphic vector bundle $V \to X$ and a holomorphic section φ of an

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associated bundle, have been extensively studied. Important examples are Bradlow pairs [3], where $\varphi \in H^0(X, E)$ lives in the fundamental representation and Higgs bundles [19], where $\varphi \in H^0(X, \mathrm{End}(E) \otimes K)$ lives in the adjoint representation (twisted by the canonical bundle K of X). Many more examples can be found in the survey [4].

In this paper we focus on U-*quadratic pairs* (V, γ), where γ is a global section of $S^2 V^* \otimes U$ for a fixed line bundle $U \to X$. These are of interest for at least two reasons. On the one hand they can be viewed as giving rise to bundles of quadrics and hence form a very natural generalization of the linear objects of vector bundles. On the other hand they arise naturally in the study of another kind of linear pairs, namely G-Higgs bundles: these are the appropriate objects for studying character varieties for the fundamental group of X in a real Lie group G through the non-abelian Hodge theory correspondence (see for example [7] for a survey on this topic).

Moduli spaces of quadratic pairs were constructed via GIT and studied by Gómez and Sols in [15] and also by Schmitt in [26]. Moreover, Mundet in the appendix to [15], showed that the stability condition used for constructing moduli is the same one which allows to prove a Hitchin–Kobayashi correspondence for quadratic pairs, relating stability of the quadratic pair to the existence of solutions to certain gauge theoretic equations. This stability condition depends on a real parameter α hence, for each value of this parameter, there is a moduli space which we denote by $\mathcal{N}_\alpha(n, d)$.

In the first part of this paper, we study the number of connected components of the moduli spaces of U-quadratic pairs on X. Our strategy is the one pioneered by Thaddeus [30] and subsequently used in many other cases, e.g., [6]. It consists in studying the variation of the moduli space $\mathcal{N}_\alpha(n, d)$ with the parameter α. As usually happens, when we run over α, the moduli spaces $\mathcal{N}_\alpha(n, d)$ are isomorphic for parameter values in intervals and only change at a discrete set of *critical values*. In these cases, the difference between the moduli spaces are confined to subvarieties, which are called the *flip loci*. For $n = 2$, we describe explicitly these subvarieties and show that they have positive codimension in $\mathcal{N}_\alpha(2, d)$. A necessary condition for the non-emptiness of $\mathcal{N}_\alpha(2, d)$ is $\alpha \leqslant d/2$. Moreover, if $d > d_U = \deg(U)$, then $\mathcal{N}_\alpha(2, d)$ is empty unless $\alpha = d/2$ and $\mathcal{N}_{d/2}(2, d)$ is the moduli space of semistable rank 2 and degree d vector bundles. So we consider only $d < d_U$ (the $d = d_U$ case is special and not considered here). We show that there is an α_m such that the $\mathcal{N}_\alpha(2, d)$'s for $\alpha < \alpha_m$ are all isomorphic. Then, using the theory of the Hitchin system, and in particular the results obtained in [17], we show that $\mathcal{N}_{\alpha_m^-}(2, d)$ is connected, where α_m^- is any value less than α_m. This, together with study of the flip loci, provides a similar conclusion for the connectedness of the other spaces $\mathcal{N}_\alpha(2, d)$, whenever $d_U - d > g - 1$ holds. Our result (Theorem 5.3) states then the following:

Theorem *Let d and d_U be such that $d_U - d > g - 1$. Then, for every $\alpha \leqslant d/2$, the moduli space $\mathcal{N}_\alpha(2, d)$ is connected.*

In the last part of the paper we apply our results to counting the connected components of the character variety

$$\mathcal{R}(\pi_1 X, \mathrm{SO}_0(2, 3)) = \mathrm{Hom}^{\mathrm{red}}(\pi_1 X, \mathrm{SO}_0(2, 3))/\mathrm{SO}_0(2, 3),$$

i.e., the space of reductive representations $\rho \colon \pi_1 X \to \mathrm{SO}_0(2, 3)$ modulo the action by simultaneous conjugation. Such a representation ρ has a topological invariant $c(\rho) = (\tau(\rho), w_2(\rho)) \in \pi_1 \mathrm{SO}_0(2, 3) \cong \mathbb{Z} \times \mathbb{Z}/2$ given by the topological class of the associated flat bundle. Now, for representations of $\pi_1 X$ into any isometry group of a hermitian symmetric space of non-compact type there is an integer invariant, known as the *Toledo invariant*, and

in the present case this invariant is just the first coordinate $\tau(\rho)$ of the topological class. Moreover, the Toledo invariant is bounded by the Milnor–Wood type inequality

$$|\tau(\rho)| \leqslant 2g - 2.$$

For $(a, w) \in \mathbb{Z} \times \mathbb{Z}/2$, denote by $\mathcal{R}_{a,w}(\pi_1 X, \mathrm{SO}_0(2, 3)) \subseteq \mathcal{R}(\pi_1 X, \mathrm{SO}_0(2, 3))$ the subspace of representations ρ such that $c(\rho) = (a, w)$.

Our results on quadratic pairs then lead to the following (Theorem 6.26).

Theorem *For each $(a, w) \in \mathbb{Z} \times \mathbb{Z}/2$ such that $0 < |a| < 2g - 2$, the space $\mathcal{R}_{a,w}(\pi_1 X, \mathrm{SO}_0(2, 3))$ is connected.*

To put this result into perspective, recall that $\mathrm{SO}_0(2, 3)$ is isomorphic to the adjoint form $\mathrm{PSp}(4, \mathbb{R})$ of the real symplectic group $\mathrm{Sp}(4, \mathbb{R})$. It can be seen that a representation $\rho \colon \pi_1 X \to \mathrm{SO}_0(2, 3)$ lifts to $\mathrm{Sp}(4, \mathbb{R})$ if and only if $\tau(\rho) = w_2(\rho) \mod 2$. Moreover, if this is the case, $\tau(\rho)$ coincides with the Toledo invariant of the lifted representation and this in turn coincides with the topological invariant in $\pi_1 \mathrm{Sp}(4, \mathbb{R}) \cong \mathbb{Z}$ of the lifted representation. The connected components of the character variety $\mathcal{R}_a(\pi_1 X, \mathrm{Sp}(4, \mathbb{R}))$ of representations in $\mathrm{Sp}(4, \mathbb{R})$ with topological invariant $a \in \mathbb{Z}$ have been determined in [16] for $|a| = 0$ and $|a| = 2g - 2$, and for the remaining values of $|a|$ by García–Prada and Mundet in [14]. In the case of representations which lift to $\mathrm{Sp}(4, \mathbb{R})$ these results easily lead to the count of connected components for representations in $\mathrm{SO}_0(2, 3)$ (cf. [7]). Thus our Theorem completes the count of the connected components for representations in $\mathrm{SO}_0(2, 3)$ and the novelty lies in the cases $a \neq w \mod 2$.

This paper is organized as follows. In Sect. 2 we recall some basic facts about quadratic pairs. In Sect. 3 we carry out the analysis of the variation of the moduli spaces $\mathcal{N}_\alpha(2, d)$ with the parameter, leaving however the proof of connectedness of $\mathcal{N}_{\alpha_m^-}(2, d)$ for Sect. 4. Then, in Sect. 5, we put our results together to obtain the main connectedness theorem for the moduli of quadratic pairs. Finally, in Sect. 6, we give the application of our results to Higgs bundles and representations of surface groups in the group $\mathrm{SO}_0(2, 3)$.

2 Quadratic pairs

2.1 Quadratic pairs and their moduli spaces

Let X be smooth projective curve over $\mathbb{C}$ of genus $g \geqslant 2$, and let U be a fixed holomorphic line bundle over X. Write

$$d_U = \deg(U)$$

for the degree of U.

Definition 2.1 A U-*quadratic pair* on X is a pair (V, γ), where V is a holomorphic vector bundle over X and γ is a global holomorphic non-zero section of $S^2 V^* \otimes U$, i.e., $\gamma \in H^0(X, S^2 V^* \otimes U)$. The *rank* and *degree* of a quadratic pair are the rank and degree of the underlying vector bundle V, respectively. We say that (V, γ) is of *type* (n, d) if $\mathrm{rk}(V) = n$ and $\deg(V) = d$.

We shall often refer to a U-quadratic pair simply as a *quadratic pair*. Quadratic pairs are sometimes called *conic bundles* in the literature.

Definition 2.2 Two U-quadratic pairs (V, γ) and (V', γ') are *isomorphic* if there is an isomorphism $f : V \to V'$ such that $\gamma' f = ((f^t)^{-1} \otimes 1_U)\gamma$, i.e., such that the following diagram commutes:

$$
\begin{array}{ccc}
V & \xrightarrow{\ \ f\ \ } & V' \\
\gamma \downarrow & & \downarrow \gamma' \\
V^* \otimes U & \xrightarrow[(f^t)^{-1} \otimes 1_U]{} & V'^* \otimes U.
\end{array}
$$

Quadratic pairs of rank $n \leqslant 3$ were studied in [15] by Gómez and Sols. They introduced an appropriate α-semistability condition, depending on a real parameter α, [1] and constructed moduli spaces of S-equivalence classes of α-semistable quadratic pairs using GIT. The construction of the moduli spaces for general rank is due to Schmitt [26,28]. We denote the moduli space of S-equivalence classes of α-semistable U-quadratic pairs on X of rank n and degree d by

$$
\mathcal{N}_{X,\alpha}(n, d) = \mathcal{N}_\alpha(n, d).
$$

There is a Hitchin–Kobayashi correspondence for quadratic pairs. This follows from the general results of [8,22] and [11]; the Appendix to [15] treats the application to the case of quadratic pairs. It says the a quadratic pair supports a solution to a certain natural gauge theoretic equation if and only if it is α-*polystable* (see below for the definition of this concept). Moreover, each S-equivalence class has a unique α-polystable representative and thus S-equivalence of α-polystable pairs reduces to isomorphism. We can therefore also consider $\mathcal{N}_\alpha(n, d)$ as the moduli space of isomorphism classes of α-polystable quadratic pairs.

2.2 U-quadratic pairs of rank 1

Although we will be mainly interested in quadratic pairs of type $(2, d)$, we shall also need the description of the moduli spaces of quadratic pairs on X of rank 1.

The τ-stability condition of [15] is empty for rank 1 pairs, though it is required that $\tau > 0$. We shall take this inequality as our definition of τ-stability of rank 1 pairs, since it allows for a convenient formulation of certain results in Sect. 3 below. In analogy with the rank 2 case below, we prefer to state the definition in terms of the parameter $\alpha = (d - 2\tau)/n$ (this is the parameter c of the appendix to [15]).

Definition 2.3 Fix a real parameter α. A U-quadratic pair (L, δ) of type $(1, d)$ is α-*stable* if $\alpha \leqslant d$.

Remark 2.4 There are no strictly α-semistable quadratic pairs of rank 1.

For quadratic pairs of type $(1, d')$, all the moduli spaces $\mathcal{N}_\alpha(1, d')$ with $\alpha \leqslant d'$ are isomorphic and there is only one so-called *critical value* of α, for which the stability condition changes, namely $\alpha = d'$.

Lemma 2.5 *Let* $\mathcal{N}_\alpha(1, d')$ *be the moduli space of* α-*stable quadratic pairs of type* $(1, d')$. *Then,*

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[1] In fact a different parameter τ is used in [15]. In the cases of interest to us, the precise definition of α-semistability is given below, as well as the relation between the parameters τ and α.

(1) *for all $\alpha > d'$, $\mathcal{N}_\alpha(1, d') = \emptyset$;*
(2) *for all $d' > d_U/2$, $\mathcal{N}_\alpha(1, d') = \emptyset$.*

Proof The first item follows from the stability condition. For the second part, we have that, if $(M, \delta) \in \mathcal{N}_\alpha(1, d')$, then $\delta : M \to M^{-1}U$ is non-zero and holomorphic, so $-2d' + d_U \geqslant 0$. □

Proposition 2.6 *Suppose that d_U is even. If $\alpha \leqslant d_U/2$, then the moduli space $\mathcal{N}_\alpha(1, d_U/2)$ is isomorphic to*

$$S = \{F \in \mathrm{Pic}^{d_U/2}(X) \mid F^2 \cong U\},$$

the set of the 2^{2g} square roots of U.

Proof Let $\alpha \leqslant d_U/2$. If $(M, \delta) \in \mathcal{N}_\alpha(1, d_U/2)$, then $\delta : M \to M^{-1}U$ must be non-zero, hence an isomorphism. Moreover, it is defined up to a non-zero scalar so the map $\mathcal{N}_\alpha(1, d_U/2) \to S$, $(M, \delta) \mapsto M$ is an isomorphism. □

It remains to describe $\mathcal{N}_\alpha(1, d')$ for $\alpha \leqslant d'$ and $d' < d_U/2$. Denote by $\mathrm{Sym}^n(X)$ the nth symmetric product of X, the smooth variety which parametrizes the degree n effective divisors on X.

Proposition 2.7 *If $d' < d_U/2$ and $\alpha \leqslant d'$, then $\mathcal{N}_\alpha(1, d')$ is the 2^{2g}-fold cover of the symmetric product $\mathrm{Sym}^{d_U-2d'}(X)$ obtained by pulling back, via the Abel-Jacobi map, the cover $\mathrm{Pic}(X) \to \mathrm{Pic}(X)$ given by squaring of line bundles.*

Proof Consider the map $\pi : \mathcal{N}_\alpha(1, d') \to \mathrm{Sym}^{d_U-2d'}(X)$, $(M, \delta) \mapsto \mathrm{div}(\delta)$. Given $D \in \mathrm{Sym}^{d_U-2d'}(X)$, $\pi^{-1}(D)$ is isomorphic to the set of square roots of $U(-D)$. The result follows. □

Corollary 2.8 *Let $d' < d_U/2$ and $\alpha \leqslant d'$. Then $\dim \mathcal{N}_\alpha(1, d') = d_U - 2d'$.*

2.3 Stability of quadratic pairs of rank 2

Our main objects of interest are type $(2, d)$ quadratic pairs (V, γ), where V is a holomorphic vector bundle of rank 2 and degree d, and $\gamma \in H^0(X, S^2V^* \otimes U)$. Most of the time we will think of γ as a holomorphic map $\gamma : V \longrightarrow V^* \otimes U$ which is symmetric, i.e., $\gamma^t \otimes 1_U = \gamma$.

Given a rank 2 vector bundle V and a line subbundle $L \subset V$, we denote by $L^\perp$ the kernel of the projection $V^* \to L^{-1}$. It is a line subbundle of V^* and V/L is isomorphic to $(L^\perp)^{-1}$.

The general definition of stability from [8,11,22] specializes as follows in the case of quadratic pairs. It is equivalent to the definition of τ-semistability of Gómez and Sols [15] by taking $\tau = d/2 - \alpha$. To state the definition we shall use the following notation. Let V_1 and V_2 be subbundles of a vector bundle V. Then we denote by $V_1 \otimes_S V_2$ their *symmetrized tensor product*. By this we mean the symmetric part of $V_1 \otimes V_2$ inside the symmetric product S^2V—these bundles can, for example, be constructed from the corresponding representations via principal bundles.

Definition 2.9 Fix $\alpha \in \mathbb{R}$. A U-quadratic pair (V, γ) of type $(2, d)$ is:

• *α-semistable* if $\alpha \leqslant d/2$ and, for every line subbundle $L \subset V$,

  (1) $\deg(L) \leqslant \alpha$ if $\gamma \in H^0(X, (L^\perp)^2 U)$;
  (2) $\deg(L) \leqslant d/2$ if $\gamma \in H^0(X, L^\perp \otimes_S V^* \otimes U)$;

(3) $\deg(L) \leqslant d - \alpha$ if $\gamma \notin H^0(X, L^\perp \otimes_S V^* \otimes U)$.

- α-*stable* if it is α-semistable and strict inequalities hold in (1), (2) and (3) above.
- α-*polystable* if $\alpha \leqslant d/2$ and, for every line subbundle $L \subset V$,

  (1) $\deg(L) \leqslant \alpha$ if $\gamma \in H^0(X, (L^\perp)^2 U)$. Moreover, if $\deg(L) = \alpha$, there is $L' \subset V$ such that $V = L \oplus L'$;
  (2) $\deg(L) \leqslant d/2$ if $\gamma \in H^0(X, L^\perp \otimes_S V^* \otimes U)$. Moreover, if $\deg(L) = d/2$, there is $L' \subset V$ such that $V = L \oplus L'$ and $\gamma' \in H^0(X, L^{-1}L'^{-1}U)$ such that $\gamma = \gamma' \oplus \gamma'$;
  (3) $\deg(L) \leqslant d - \alpha$ if $\gamma \notin H^0(X, L^\perp \otimes_S V^* \otimes U)$. Moreover, if $\deg(L) = d - \alpha$, there is $L' \subset V$ such that $V = L \oplus L'$.

Remark 2.10 The $d/2$-(semi)stability condition for (V, γ) is equivalent to the usual (semi)stability condition for the vector bundle V.

An α-semistable quadratic pair (V, γ) is *strictly α-semistable* if it is not α-stable. From the previous definition, we can separate strictly α-semistable quadratic pairs into three types.

Definition 2.11 A rank 2 α-semistable quadratic pair (V, γ) is strictly α-semistable of *type*:

(A) if there is a holomorphic line bundle $L \subset V$ such that $\gamma \in H^0(X, (L^\perp)^2 U)$ and $\deg(L) = \alpha$;
(B) if there is a holomorphic line bundle $L \subset V$ such that $\gamma \in H^0(X, L^\perp \otimes_S V^* \otimes U)$ and $\deg(L) = d/2$;
(C) if there a holomorphic line bundle $L \subset V$ such that $\deg(L) = d - \alpha$.

Definition 2.12 For a given type $(2, d)$, the values of α for which strictly α-semistable quadratic pairs of type **(A)** or **(C)** exist are called *critical values*, and the other values of α are called *generic values*.

Remark 2.13 For generic α and for a pair (V, γ), if there is no $L \subset V$ such that $\gamma(L) \subset L^\perp U$ and $\deg(L) = d/2$, then (V, γ) is α-semistable if and only if it is α-stable. In particular, if d is odd there are no strictly α-semistable pairs of type **(B).**

Lemma 2.14 *Let (V, γ) be a U-quadratic pair of rank 2 and let L be a line subbundle of V. Then,*

(1) $\gamma \in H^0(X, (L^\perp)^2 U) \Longleftrightarrow \gamma(L) = 0 \Longleftrightarrow \gamma(V) \subset L^\perp U$;
(2) $\gamma \in H^0(X, L^\perp \otimes_S V^* \otimes U) \Longleftrightarrow \gamma(L) \subset L^\perp U$.

Proof This is an exercise in fibrewise linear algebra; see [23] for details.   □

Using this lemma, we can rewrite the α-(poly,semi)stability condition in the following way.

Proposition 2.15 *Let (V, γ) be a rank 2 quadratic pair.*

- *The pair (V, γ) is α-semistable if and only if $\alpha \leqslant d/2$ and, for any line bundle $L \subset V$, the following conditions hold:*

  (1) $\deg(L) \leqslant \alpha$, *if $\gamma(L) = 0$;*
  (2) $\deg(L) \leqslant d/2$, *if $\gamma(L) \subset L^\perp U$;*
  (3) $\deg(L) \leqslant d - \alpha$, *if $\gamma(L) \not\subset L^\perp U$.*

- *The pair (V, γ) is α-stable if and only if it is α-semistable for any line bundle $L \subset V$, the conditions (1), (2) and (3) above hold with strict inequalities.*
- *The pair (V, γ) is α-polystable if and only if $\alpha \leqslant d/2$ and, for any line bundle $L \subset V$, the following conditions hold:*

  (1) $\deg(L) \leqslant \alpha$, *if* $\gamma(L) = 0$. *Moreover, if* $\deg(L) = \alpha$, *there is an* $L' \subset V$ *such that* $V = L \oplus L'$ *and with respect to this decomposition,*

  $$\gamma = \begin{pmatrix} 0 & 0 \\ 0 & \gamma' \end{pmatrix}$$

  with $\gamma' \in H^0(X, L'^{-2}U)$*non-zero;*

  (2) $\deg(L) \leqslant d/2$, *if* $\gamma(L) \subset L^{\perp}U$. *Moreover, if* $\deg(L) = d/2$, *there is* $L' \subset V$ *such that* $V = L \oplus L'$ *and with respect to this decomposition,*

  $$\gamma = \begin{pmatrix} 0 & \gamma' \\ \gamma' & 0 \end{pmatrix}$$

  with $\gamma' \in H^0(X, L^{-1}L'^{-1}U)$*non-zero;*

  (3) $\deg(L) \leqslant d - \alpha$, *if* $\gamma(L) \not\subset L^{\perp}U$. *Moreover, if* $\deg(L) = d - \alpha$, *there is* $L' \subset V$ *such that* $V = L \oplus L'$ *and with respect to this decomposition,*

  $$\gamma = \begin{pmatrix} \gamma' & 0 \\ 0 & 0 \end{pmatrix}$$

  with $\gamma' \in H^0(X, L^{-2}U)$ *non-zero.*

Definition 2.16 Let (V, γ) be a quadratic pair. A subbundle $L \subset V$ is α-destabilizing of type:

(A) if $\deg(L) \geqslant \alpha$ and $\gamma(L) = 0$;
(B) if $\deg(L) \geqslant d/2$ and $\gamma(L) \subset L^{\perp}U$;
(C) if $\deg(L) \geqslant d - \alpha$ and $\gamma(L) \not\subset L^{\perp}U$.

Proposition 2.17 *Let (V, γ) be a quadratic pair and let $\alpha < d/2$.*

(1) *There is at most one α-destabilizing subbundle $L \subset V$ of type **(A)** and at most one α-destabilizing subbundle $M \subset V$ of type **(C)**. Moreover, if such L and M both exist, then $V \cong L \oplus M$.*
(2) *There are at most two distinct α-destabilizing subbundles $L_1, L_2 \subset V$ of type **(B)**. Moreover, if there exist such distinct L_1 and L_2, then $V \cong L_1 \oplus L_2$ and $\gamma(L_1) \subset L_2^{-1}U$.*
(3) *There cannot exist simultaneously α-destabilizing subbundles of type **(A)** and **(B)**.*
(4) *There cannot exist simultaneously α-destabilizing subbundles of type **(C)** and **(B)**.*

Proof Since $\mathrm{rk}(V) = 2$ and γ is holomorphic and non-zero, there is at most one subbundle $L \subset V$ with $\gamma(L) = 0$. This proves the first statement in (1). For the second statement in (1), note that $\deg(M) \geqslant d - \alpha > d/2 = \mu(V)$. Thus the claim about the destabilizing bundle M follows from the uniqueness of destabilizing subbundles of ordinary rank 2 vector bundles (cf. Proposition 10.38 of [21]). If such L and M both exist, then clearly $L \not\cong M$ so the composite $M \to V \to \Lambda^2 V L^{-1}$ is non-zero, and $-\deg(M) + d - \deg(L) \geqslant 0$. But $\deg(M) \geqslant d - \alpha$ and $\deg(L) \geqslant \alpha$, therefore $\Lambda^2 V L^{-1} \cong M$ and $V \cong L \oplus M$.

The proof of (2) is similar. Let $L_1, L_2 \subset V$ be two different destabilizing subbundles of (V, γ) of type **(B)**. Then $L_2 \cong \Lambda^2 V L_1^{-1}$ as before, and $V = L_1 \oplus L_2$. In this case, $L_2^{-1} \cong L_1^{\perp}$

and $\gamma(L_1) \subset L_2^{-1}U$. It is clear that there cannot exist a third subbundle satisfying the same conditions.

The proof of (3) is similar to the proof of the first statement of (1) because $\gamma(L) = 0$ is equivalent to $\gamma(V) \subset L^{\perp}U$. The proof of (4) is analogous to the proof of the second statement in (1), observing that there cannot exist simultaneously a destabilizing subbundle of V and another subbundle with degree $d/2$. Indeed, if there is an $M \subset V$ with $\deg(M) > d/2$ and if $F \subset V$ is different from M then there is a non-zero homomorphism $F \to \Lambda^2 V M^{-1}$, so $\deg(F) \leqslant d - \deg(M) < d/2$.   □

Recall that $d_U = \deg(U)$ and that $\mathcal{N}_\alpha(2, d)$ denotes the moduli space of α-polystable U-quadratic pairs of rank 2 and degree d.

Proposition 2.18

(1) If $d > d_U$, then $\mathcal{N}_\alpha(2, d) = \emptyset$ for all α.
(2) If $d \leqslant d_U$, then $\mathcal{N}_\alpha(2, d) = \emptyset$ for all $\alpha > d/2$.

Proof Let (V, γ) be a quadratic pair of rank 2 and degree $d > d_U$. If $\mathrm{rk}(\gamma) = 2$ (generically), then $\det(\gamma)$ is a non-zero section of $\Lambda^2 V^{-2} U^2$ so $d \leqslant d_U$. Hence, since $\gamma \neq 0$, we must have $\mathrm{rk}(\gamma) = 1$. Take any α and suppose moreover that the pair (V, γ) is α-semistable. Since V is locally free, the sheaf $N = \ker(\gamma) \subset V$ is torsion free. For the same reason, the quotient $V/N \cong \mathrm{im}(\gamma) \subset V^* \otimes U$ is torsion free. Thus N is a line subbundle of V. Let $I \subset V^*$ be such that IU is the saturation of the image sheaf $\mathrm{im}(\gamma)$. From the α-semistability condition,

$$\deg(N) \leqslant \alpha \tag{2.1}$$

and, since $\gamma(I^{\perp}) = 0$,

$$\deg(I) \leqslant \alpha - d. \tag{2.2}$$

On the other hand, γ induces a non-zero map of line bundles $V/N \to IU$, so

$$- d + \deg(N) + \deg(I) + d_U \geqslant 0. \tag{2.3}$$

But, from (2.1) and (2.2), we have

$$- d + \deg(N) + \deg(I) + d_U < 0 \tag{2.4}$$

because $d > d_U$ and $\alpha \leqslant d/2$. From (2.3) and (2.4) we conclude that there is no such (V, γ) and this finishes the proof of the first part.

The second part is immediate, since $\alpha \leqslant d/2$ is part of the definition of α-semistability.
                                                                                                 □

This result deals with the cases $d > d_U$ and any α, and $d \leqslant d_U$ and $\alpha > d/2$. From now on we will restrict ourselves to the study of U-quadratic pairs of type $(2, d)$ with $d < d_U$. When $d = d_U$, the map γ becomes an isomorphism, making this a special case in what concerns the connected components of the moduli space. In the next remark we give a very brief explanation of this phenomenon, which can be seen as somewhat similar to the difference between the situations in Propositions 2.6 and 2.7.

Remark 2.19 If d_U is odd and $d = d_U$, then it will follow from Proposition 3.3 below (see also Fig. 1 in Sect. 3.1) that $\mathcal{N}_\alpha(2, d_U) = \emptyset$. So let us assume that d_U is even and that $d = d_U$. In this case $\gamma : V \to V^* \otimes U$ is an isomorphism. If we choose a square root U' of U, then γ gives rise to a symmetric isomorphism $q : V \otimes U'^* \cong V^* \otimes U'$, i.e., to a non-degenerate

quadratic form on the vector bundle $V \otimes U'^*$. Such a pair, consisting of a vector bundle with a non-degenerate quadratic form, is called an *orthogonal bundle*. There is a stability condition for orthogonal bundles, due to Ramanathan [25], and it can be seen that $(V \otimes U'^*, q)$ is semistable (i.e., the degree of any isotropic subbundle of $V \otimes U'^*$ is less or equal than 0) if and only if (V, γ) is α-semistable for any $\alpha \leqslant d/2$. Hence $\mathcal{N}_\alpha(2, d_U)$ is isomorphic to the moduli space $\mathcal{M}_{O(2,\mathbb{C})}$ of polystable orthogonal bundles and this gives rise to the existence of extra connected components (cf. [7,12,16]).

2.4 Deformation theory of quadratic pairs

The deformation theory of a quadratic pair (V, γ) is governed by the following complex of sheaves on X (see, e.g., Biswas–Ramanan [2]):

$$C^\bullet(V, \gamma) : \text{End}(V) \xrightarrow{\rho(\gamma)} S^2 V^* \otimes U,$$

where

$$\rho(\gamma)(\psi) = -(\psi^t \otimes 1_U)\gamma - \gamma\psi.$$

In particular, the infinitesimal deformation space of a quadratic pair (V, γ) is isomorphic to $\mathbb{H}^1(X, C^\bullet(V, \gamma))$. Moreover, one has a long exact sequence

$$
\begin{aligned}
0 &\longrightarrow \mathbb{H}^0(X, C^\bullet(V, \gamma)) \longrightarrow H^0(X, \text{End}(V)) \longrightarrow H^0(X, S^2 V^* \otimes U) \longrightarrow \\
&\longrightarrow \mathbb{H}^1(X, C^\bullet(V, \gamma)) \longrightarrow H^1(X, \text{End}(V)) \longrightarrow H^1(X, S^2 V^* \otimes U) \longrightarrow \quad (2.5) \\
&\longrightarrow \mathbb{H}^2(X, C^\bullet(V, \gamma)) \longrightarrow 0
\end{aligned}
$$

where the maps $H^i(X, \text{End}(V)) \to H^i(X, S^2 V^* \otimes U)$ are induced by $\rho(\gamma)$. It is immediate from this long exact sequence that the infinitesimal automorphism space (defined for general pairs in [11]) of a quadratic pair (V, γ) can be canonically identified with $\mathbb{H}^0(X, C^\bullet(V, \gamma))$.

Definition 2.20 A quadratic pair (V, γ) is *infinitesimally simple* if the vanishing $\mathbb{H}^0(X, C^\bullet (V, \gamma)) = 0$ holds. A quadratic pair (V, γ) is *simple* if the group $\text{Aut}(V, \gamma)$ of automorphisms of (V, γ) is equal to $\{\pm 1_V\}$.

We have the following useful facts.

Proposition 2.21 (1) *An α-stable quadratic pair is infinitesimally simple.*
(2) *An α-stable quadratic pair (V, γ) represents a smooth point in the moduli space if it is simple and $\mathbb{H}^2(X, C^\bullet(V, \gamma)) = 0$.*

Proof (1) This can be proved in a manner analogous to the corresponding result for G-Higgs bundles (see [11, § 3]).
(2) This follows from the existence of a fine moduli space for α-stable quadratic pairs (see Gómez and Sols [15, Theorem I]) and Theorem 3.1 of Biswas and Ramanan [2]. □

This motivates the following definition.

Definition 2.22 The *expected dimension* of $\mathcal{N}_\alpha(2, d)$ is $\dim \mathbb{H}^1(X, C^\bullet(V, \gamma))$.

Using (2.5), the expected dimension can be calculated as follows:

$$\dim \mathbb{H}^1(X, C^\bullet(V, \gamma)) = \chi(S^2 V^* \otimes U) - \chi(\text{End}(V)) = 3(d_U - d) + g - 1. \quad (2.6)$$

Remark 2.23 If a (local) universal family exists over a component of the moduli space, then this component has the expected dimension. However, for values of α such that no α-stable quadratic pairs exist, the actual dimension of the moduli space can be strictly smaller than the expected dimension (see [5] for an example of this phenomenon, in the Higgs bundle context).

3 Variation of the moduli with the parameter

The purpose of this section is to study the variation of the moduli spaces $\mathcal{N}_\alpha(2, d)$ with the stability parameter α. As in the case of holomorphic triples [6,30] we have critical values α_k—for which the moduli spaces $\mathcal{N}_\alpha(2, d)$ change—and corresponding *flip loci* $\mathcal{S}_{\alpha_k^\pm}(2, d) \subset \mathcal{N}_{\alpha^\pm}(2, d)$, where the change takes place. We shall see that, in contrast with the case of holomorphic triples, there is no symmetry between $\mathcal{S}_{\alpha_k^+}(2, d)$ and $\mathcal{S}_{\alpha_k^-}(2, d)$. This is due to the non-linear nature of quadratic pairs.

3.1 Critical values

We begin by determining precisely the critical values of the parameter α.

Proposition 3.1 *If (V, γ) is an α-semistable pair with $\alpha < d - d_U/2$, then generically* rk$(\gamma) = 2$.

Proof Recall that we always have $\gamma \neq 0$. If rk$(\gamma) = 1$, considering again the line bundles $N = \ker(\gamma) \subset V$ and $I \subset V^*$ such that IU is the saturation of the image sheaf im(γ), we have, as in the proof of Proposition 2.18, that

$$0 \leqslant -d + \deg(N) + \deg(I) + d_U \leqslant 2\alpha - 2d + d_U,$$

i.e., $\alpha \geqslant d - d_U/2$. $\square$

The next result shows that the injectivity parameter $d - d_U/2$ of Proposition 3.1 is also a stabilization parameter, in the sense that after it the moduli spaces $\mathcal{N}_\alpha(2, d)$, for different values of α, are all isomorphic.

Proposition 3.2 *If $\alpha_2 \leqslant \alpha_1 < d - d_U/2$, then a quadratic pair (V, α) is α_1-semistable if and only if it is α_2-semistable, and hence $\mathcal{N}_{\alpha_1}(2, d) \simeq \mathcal{N}_{\alpha_2}(2, d)$.*

Proof Let $(V, \gamma) \in \mathcal{N}_{\alpha_1}(2, d)$. Since $\alpha_2 \leqslant \alpha_1$, the existence of an α_2-destabilizing subbundle implies that it must be of type **(A)**, which in turn implies that rk$(\gamma) = 1$ generically. But this is impossible due to Proposition 3.1, since $\alpha_1 < d - d_U/2$. Hence $\mathcal{N}_{\alpha_1}(2, d) \subseteq \mathcal{N}_{\alpha_2}(2, d)$.

Conversely, if $(V, \gamma) \in \mathcal{N}_{\alpha_2}(2, d)$, then $(V, \gamma) \in \mathcal{N}_{\alpha_1}(2, d)$ unless there is an α_1-destabilizing subbundle of (V, γ). Hence L is such that $d - \alpha_1 < \deg(L) \leqslant d - \alpha_2$, and $\gamma(L) \not\subset L^\perp U$, therefore the composite $L \to V \xrightarrow{\gamma} V^* \otimes U \to L^{-1}U$ is non-zero. Thus

$$-2 \deg(L) + d_U \geqslant 0.$$

On the other hand, $d - \alpha_1 < \deg(L)$ together with $\alpha_1 < d - d_U/2$, gives

$$-2 \deg(L) + d_U < 0.$$

It follows that no such line subbundle L can exist. $\square$

From the definition of α-semistability and from the previous proposition, the following is immediate.

Proposition 3.3 *The critical values of U-quadratic pairs of type $(2, d)$ are the elements of the following set:*

$$\{d/2\} \cup \{[d/2] + k \mid k \in \{d - [d/2] - [d_U/2], \ldots, 0\}\}.$$

Moreover, on each open interval between consecutive critical values,

$$([d/2] + k, \min\{d/2, [d/2] + k + 1\})$$

the α-semistability condition is the same; hence the corresponding moduli spaces are isomorphic.

Notation 3.4 For each integer $d - [d/2] - [d_U/2] \leqslant k \leqslant 0$, we define

$$\alpha_k = [d/2] + k.$$

Also, let

$$\alpha_M = d/2 \quad \text{and} \quad \alpha_m = \alpha_{d-[d/2]-[d_U/2]} = d - [d_U/2],$$

and let α_k^+ denote the value of any parameter between the critical values α_k and α_{k+1}, and let α_k^- denote the value of any parameter between the critical values α_{k-1} and α_k.

Proposition 3.3 means that we can write without ambiguity

$$\mathcal{N}_{\alpha_k^+}(2, d)$$

for the moduli space of α_k^+-semistable U-quadratic pairs of rank 2 and degree d, for any α between the critical values $[d/2] + k$ and $\min\{d/2, [d/2] + k + 1\}$. Note that, with this notation, we always have $\mathcal{N}_{\alpha_k^+}(2, d) = \mathcal{N}_{\alpha_{k+1}^-}(2, d)$.

The information obtained so far on the variation of $\mathcal{N}_\alpha(2, d)$ with α and d is summarized in Fig. 1.

3.2 Flip loci

We shall now study what are the differences between moduli spaces of U-quadratic pairs of type $(2, d)$, which are separated by a critical value of the parameter α.
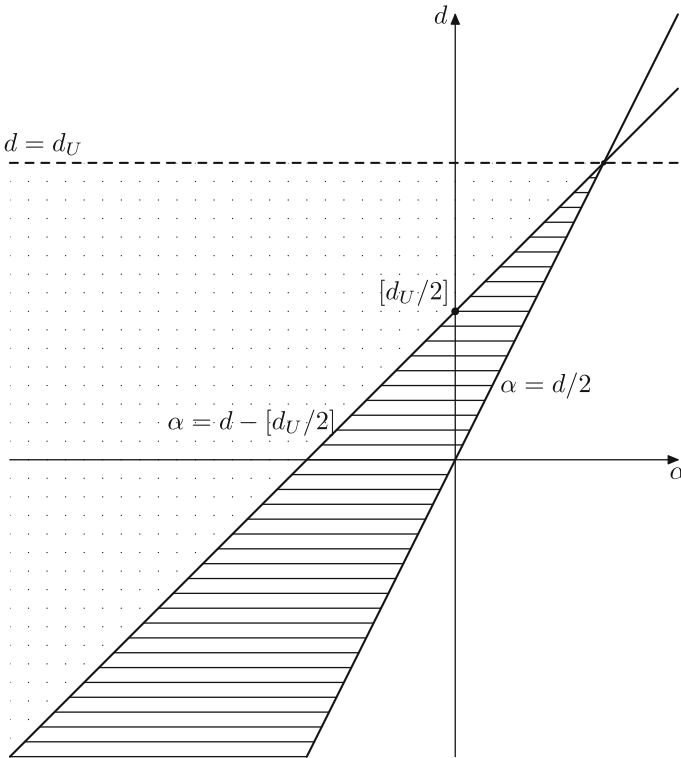
Definition 3.5 For each $k \in \{d - [d/2] - [d_U/2], \ldots, 0\}$, let $\mathcal{S}_{\alpha_k^+}(2, d)$ be the set of pairs of degree d which are α_k^+-semistable but α_k^--unstable, i.e.,

$$\mathcal{S}_{\alpha_k^+}(2, d) = \left\{ (V, \gamma) \in \mathcal{N}_{\alpha_k^+}(2, d) \mid (V, \gamma) \notin \mathcal{N}_{\alpha_k^-}(2, d) \right\}.$$

Similarly, define $\mathcal{S}_{\alpha_k^-}(2, d)$ to be the set of pairs of degree d which are α_k^--semistable but α_k^+-unstable,

$$\mathcal{S}_{\alpha_k^-}(2, d) = \left\{ (V, \gamma) \in \mathcal{N}_{\alpha_k^-}(2, d) \mid (V, \gamma) \notin \mathcal{N}_{\alpha_k^+}(2, d) \right\}.$$

The spaces $\mathcal{S}_{\alpha_k^\pm}(2, d)$ are called the *flip loci* for the critical value α_k.

Fig. 1 Variation regions of $\mathcal{N}_\alpha(2, d)$. Above the line $d = d_U$, $\mathcal{N}_\alpha(2, d) = \emptyset$ as well as on the right of the line $\alpha = d/2$. Also, $\mathcal{N}_\alpha(2, d_U) = \emptyset$ whenever d_U is odd. The *dotted* region, on the left of the line $\alpha = d - [d_U/2]$, is the region where there are no critical values, hence there is no change of the moduli spaces and, also, γ is non-degenerate. The critical values lie in the region between the lines $\alpha = d - [d_U/2]$ and $\alpha = d/2$

As a direct consequence of this definition, we have

$$\mathcal{N}_{\alpha_k^+}(2, d) \smallsetminus \mathcal{S}_{\alpha_k^+}(2, d) = \mathcal{N}_{\alpha_k^-}(2, d) \smallsetminus \mathcal{S}_{\alpha_k^-}(2, d). \tag{3.1}$$

Proposition 3.6 *Any quadratic pair* (V, γ) *in* $\mathcal{S}_{\alpha_k^\pm}(2, d)$ *is* $\alpha_k^\pm$*-stable. Hence, for* $\alpha_k \neq \alpha_M$,

$$\mathcal{S}_{\alpha_k^+}(2, d) = \left\{ (V, \gamma) \in \mathcal{N}_{\alpha_k^+}^s(2, d) \mid (V, \gamma) \notin \mathcal{N}_{\alpha_k^-}(2, d) \right\} \tag{3.2}$$

and

$$\mathcal{S}_{\alpha_k^-}(2, d) = \left\{ (V, \gamma) \in \mathcal{N}_{\alpha_k^-}^s(2, d) \mid (V, \gamma) \notin \mathcal{N}_{\alpha_k^+}(2, d) \right\} \tag{3.3}$$

where $\mathcal{N}_{\alpha_k^\pm}^s(2, d)$ *stands for the stable locus of* $\mathcal{N}_{\alpha_k^\pm}(2, d)$.

Proof If (V, γ) is strictly $\alpha_k^\pm$-semistable then, since $\alpha_k^\pm$ is a generic value, the destabilizing subbundle must be of type **(B)**. Since such a subbundle is destabilizing for all values of α, (3) and (4) of Proposition 2.17 imply that there are no destabilizing subbundles of type **(A)** or **(C)**. The conclusion is now immediate from the definition of $\mathcal{S}_{\alpha_k^\pm}(2, d)$.   □

Proposition 3.7 *Let* $\alpha_k \neq \alpha_M$. *Then:*

(1) $\mathcal{S}_{\alpha_k^+}(2, d)$ is a subvariety of $\mathcal{N}_{\alpha_k^+}^s(2, d)$.

(2) $\mathcal{S}_{\alpha_k^-}(2, d)$ is a subvariety of $\mathcal{N}_{\alpha_k^-}^s(2, d)$.

Proof From (3.2), $\mathcal{S}_{\alpha_k^+}(2, d) \subset \mathcal{N}_{\alpha_k^+}^s(2, d)$. From [15] we know that there is a (universal) family of quadratic pairs parametrized by $\mathcal{N}_{\alpha_k^+}^s(2, d)$. By definition, the restriction of this family to $\mathcal{S}_{\alpha_k^+}(2, d)$ parametrizes the pairs which are not α_k^--semistable. Since α_k^--semistability is an open condition (cf. Proposition 3.1 of [15]), it follows that $\mathcal{S}_{\alpha_k^+}(2, d)$ is Zariski closed in $\mathcal{N}_{\alpha_k^+}^s(2, d)$. This proves (1). The proof of (2) is the same, but now using (3.3).   □

Remark 3.8 In the next two sections we shall see that the flip loci $\mathcal{S}_{\alpha_k^\pm}(2, d)$ are compact and therefore also closed in $\mathcal{N}_{\alpha_k^\pm}(2, d)$. Hence the $\mathcal{S}_{\alpha_k^\pm}(2, d)$ are in fact subvarieties of $\mathcal{N}_{\alpha_k^\pm}(2, d)$.

3.3 The flip locus $\mathcal{S}_{\alpha_k^+}(2, d)$

From (2) of Proposition 2.18, $\mathcal{S}_{\alpha_M^+}(2, d) = \emptyset$, so we shall study the flip loci $\mathcal{S}_{\alpha_k^+}(2, d)$ for the other critical values.

Proposition 3.9 *Let* $(V, \gamma) \in \mathcal{N}_{\alpha_k^+}(2, d)$ *with* $\alpha_k < d/2$. *If* $(V, \gamma) \in \mathcal{S}_{\alpha_k^+}(2, d)$, *then* V *is a non-trivial extension*

$$0 \longrightarrow L \longrightarrow V \longrightarrow M \longrightarrow 0$$

where $L \subset V$ *is a line bundle such that* $\deg(L) = \alpha_k$ *and* $\gamma(L) = 0$. *Moreover,* γ *induces* $\gamma' \in H^0(X, M^{-2}U)$ *such that the quadratic pair* (M, γ') *of type* $(1, d - \alpha_k)$ *is* $(d - \alpha_k^+)$-*stable.*

Proof Let $(V, \gamma) \in \mathcal{S}_{\alpha_k^+}(2, d)$. Then it must be strictly α_k-semistable and, from the definition of $\mathcal{S}_{\alpha_k^+}(2, d)$ and Proposition 2.15, the destabilizing subbundle must be an $L \subset V$ such that $\gamma(L) = 0$ and $\deg(L) = \alpha_k$.

Write

$$0 \longrightarrow L \longrightarrow V \longrightarrow \Lambda^2 V L^{-1} \longrightarrow 0 \tag{3.4}$$

and define $M = \Lambda^2 V L^{-1}$. If we had $V = L \oplus M$, then M would be an α_k^+-destabilizing subbundle of (V, γ) (of type **(C)**), which is not possible. The extension (3.4) is thus non-trivial.

Using the symmetry of γ and the fact that $\gamma(L) = 0$, we see that γ induces a map $\gamma' : M \to M^{-1}U$ and hence we obtain the pair (M, γ') of type $(1, d - \alpha_k)$. From Definition 2.3, it is clearly $(d - \alpha_k^+)$-stable.   □

Proposition 3.10 *Let* $\alpha_k < d/2$. *There is a morphism*

$$\mathcal{S}_{\alpha_k^+}(2, d) \longrightarrow \mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times \mathrm{Jac}^{\alpha_k}(X)$$

with fibre isomorphic to $\mathbb{P}^{d-2\alpha_k+g-2}$.

Proof From Proposition 3.9, we see that there is a map

$$p : \mathcal{S}_{\alpha_k^+}(2, d) \longrightarrow \mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times \mathrm{Jac}^{\alpha_k}(X)$$

defined by

$$p(V, \gamma) = ((M, \gamma'), L).$$

where L is the destabilizing subbundle and M is the quotient bundle, $M \cong \Lambda^2 V L^{-1}$.

Let

$$((M, \gamma'), L) \in \mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times \mathrm{Jac}^{\alpha_k}(X).$$

The fibre of p over $((M, \gamma'), L)$ is given by the isomorphism classes of non-trivial extensions of M by L. Indeed, if V is such an extension then, defining

$$\gamma = (\pi^t \otimes 1_U)\gamma'\pi$$

where $\pi : V \to M$ is the projection, we obtain a quadratic pair (V, γ). This pair is strictly α_k-semistable and α_k^--unstable (with L being the destabilizing subbundle) and as we go from $\mathcal{N}_{\alpha_k^-}(2, d)$ to $\mathcal{N}_{\alpha_k^+}(2, d)$, then (V, γ) gets α_k^+-stable unless V has also a destabilizing subbundle M' such that $\deg(M') = d - \alpha_k$. But then $M' \cong M$ and $V = L \oplus M$, which contradicts the non-triviality of the extension V.

The fibre of p over $((M, \gamma'), L)$ is then the space $\mathbb{P}\,\mathrm{Ext}^1(M, L) \cong \mathbb{P}H^1(X, M^{-1}L)$. Since $\alpha_k < d/2$, $\deg(M^{-1}L) = 2\alpha_k - d < 0$, so $H^0(X, M^{-1}L) = 0$ and

$$\dim H^1(X, M^{-1}L) = d - 2\alpha_k + g - 1 > 0.$$

Hence p is surjective, with fibre isomorphic to $\mathbb{P}^{d-2\alpha_k+g-2}$.

It remains to check that p is a morphism. For that we proceed as follows.

Let $p_X : \mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times X \to X$ be the projection. From Remark 2.4 and Theorem I of [15], there is a universal $p_X^* U$-quadratic pair $(\mathcal{L}_1, \boldsymbol{\gamma}')$ over $\mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times X$. On the other hand, we have the Poincaré line bundle $\mathcal{L}_2$ over $\mathrm{Jac}^{\alpha_k}(X) \times X$. Let

$$\mathrm{pr}_{13} : \mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times \mathrm{Jac}^{\alpha_k}(X) \times X \longrightarrow \mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times X$$

$$\mathrm{pr}_{23} : \mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times \mathrm{Jac}^{\alpha_k}(X) \times X \longrightarrow \mathrm{Jac}^{\alpha_k}(X) \times X$$

and

$$\mathrm{pr}_{12} : \mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times \mathrm{Jac}^{\alpha_k}(X) \times X \longrightarrow \mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times \mathrm{Jac}^{\alpha_k}(X)$$

be the projections. Consider the first direct image sheaf $R^1\mathrm{pr}_{12*}(\mathrm{pr}_{13}^*\mathcal{L}_1^{-1} \otimes \mathrm{pr}_{23}^*\mathcal{L}_2)$. This sheaf is locally free since its fibres have constant dimension (because $\alpha_k < d/2$).

If

$$\widetilde{\mathcal{S}}((\mathcal{L}_1, \boldsymbol{\gamma}'), \mathcal{L}_2) = \mathbb{P}R^1\mathrm{pr}_{12*}(\mathrm{pr}_{13}^*\mathcal{L}_1^{-1} \otimes \mathrm{pr}_{23}^*\mathcal{L}_2)$$

there is then a morphism $\widetilde{\mathcal{S}}((\mathcal{L}_1, \boldsymbol{\gamma}'), \mathcal{L}_2) \to \mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times \mathrm{Jac}^{\alpha_k}(X)$. Moreover, in a similar manner to [20] (see also Proposition 3.2 of [30] and Proposition 5.10 of [13]), one sees that $\widetilde{\mathcal{S}}((\mathcal{L}_1, \boldsymbol{\gamma}'), \mathcal{L}_2)$ is base of a family parametrizing all α_k^+-semistable U-quadratic pairs over X which are α_k^+-stable but α_k^--unstable. Hence, from the universal property of the coarse moduli space $\mathcal{N}_{\alpha_k^+}(2, d)$, there is a morphism $\widetilde{\mathcal{S}}((\mathcal{L}_1, \boldsymbol{\gamma}'), \mathcal{L}_2) \to \mathcal{N}_{\alpha_k^+}(2, d)$ which factors through $\mathcal{S}_{\alpha_k^+}(2, d)$ and yields an isomorphism $\widetilde{\mathcal{S}}((\mathcal{L}_1, \boldsymbol{\gamma}'), \mathcal{L}_2) \cong \mathcal{S}_{\alpha_k^+}(2, d)$ such that

the following diagram commutes:

$$
\begin{array}{ccc}
\widetilde{\mathcal{S}}((\mathcal{L}_1, \boldsymbol{\gamma}'), \mathcal{L}_2) & \xrightarrow{\;\cong\;} & \mathcal{S}_{\alpha_k^+}(2, d) \\
& \searrow & \downarrow p \\
& & \mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times \mathrm{Jac}^{\alpha_k}(X).
\end{array}
$$

So p is a morphism and the result follows. $\qquad\qquad\square$

Since $\mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k)$ is compact, one concludes from this proposition that $\mathcal{S}_{\alpha_k^+}(2, d)$ is compact as well. It follows from Proposition 3.7 (see also Remark 3.8) that it is a subvariety of $\mathcal{N}_{\alpha_k^+}(2, d)$.

From the previous proposition and from Proposition 2.7 (in the case of $\mathcal{S}_{\alpha_m^+}(2, d)$ use instead Proposition 2.6), we have:

Corollary 3.11 *For every $\alpha_k < d/2$, $\dim \mathcal{S}_{\alpha_k^+}(2, d) = d_U - d + 2g - 2$.*

3.4 The flip locus $\mathcal{S}_{\alpha_k^-}(2, d)$

Now we turn our attention to the other flip loci, $\mathcal{S}_{\alpha_k^-}(2, d)$. As in the case of $\mathcal{S}_{\alpha_k^+}(2, d)$, the behaviour of $\mathcal{S}_{\alpha_k^-}(2, d)$ depends on whether $\alpha_k = \alpha_M = d/2$ or not. On the other hand, for $\alpha_k < d/2$, the description of $\mathcal{S}_{\alpha_k^-}(2, d)$ is more involved than that of $\mathcal{S}_{\alpha_k^+}(2, d)$, with several difficulties appearing due to the fact that the α_k-destabilizing subbundle is of type **(C)**.

Let us begin by studying $\mathcal{S}_{\alpha_M^-}(2, d)$ and see why it is a separate case. Indeed,

$$
\mathcal{S}_{\alpha_M^-}(2, d) = \mathcal{N}_{\alpha_M^-}(2, d)
$$

and one cannot compare this flip locus with the others because in this extreme case the cause of the destabilization after $d/2$ is not related with subbundles of V.

We have that α_M^- is any value in the open interval $([d/2], d/2)$ if d is odd, or $(d/2-1, d/2)$ if d is even. So we can write $\alpha_M^- = d/2 - \epsilon$ for sufficiently small $\epsilon > 0$.

Lemma 3.12 *If a U-quadratic pair (V, γ) of type $(2, d)$ is α_M^--semistable then V is semistable.*

Proof Suppose (V, γ) is α_M^--semistable and let $L \subset V$ be a line subbundle. Then:

- $\deg(L) < \alpha_M^- \leqslant d/2$ if $\gamma(L) = 0$ (note that α_M^- is not a critical value, so we could not have $\deg(L) = \alpha_M^-$);
- $\deg(L) \leqslant d/2$ if $\gamma(L) \subset L^\perp U$;
- $\deg(L) < d - \alpha_M^- = d/2 + \epsilon$ if $\gamma(L) \not\subset L^\perp U$ (again we could not have $\deg(L) = d - \alpha_M^-$). Since we can take $\epsilon > 0$ as small as wanted, it follows that $\deg(L) \leqslant d/2$.

In any case, $\deg(L) \leqslant d/2$, and thus V is semistable. $\qquad\qquad\square$

Proposition 3.13 *Let $\mathcal{M}(2, d)$ be the moduli space of rank 2 semistable vector bundles over X. There is a map $\pi : \mathcal{N}_{\alpha_M^-}(2, d) \to \mathcal{M}(2, d)$ which, if $d_U - d > g - 1$, is surjective and the fibre over a stable vector bundle V is $\mathbb{P}H^0(X, S^2 V^* \otimes U)$.*

Proof Using the previous lemma, define $\pi : \mathcal{N}_{\alpha_M^-}(2, d) \to \mathcal{M}(2, d)$ as $\pi(V, \gamma) = V$.

For the map to be surjective, given any semistable vector bundle V in $\mathcal{M}(2, d)$, there must exist a non-zero holomorphic section γ of $S^2 V^* \otimes U$ such that (V, γ) is α_M^--semistable. Suppose that V is stable. Since $d_U - d > g - 1$, we have $\chi(S^2 V^* \otimes U) > 0$, where χ is the Euler characteristic, hence $H^0(X, S^2 V^* \otimes U) \neq 0$ and this yields a pair (V, γ). For any $L \subset V$, $\deg(L) < d/2$. Hence $\deg(L) \leqslant d/2 - 1$ if d is even or $\deg(L) \leqslant [d/2]$. In any case, $\deg(L) \leqslant \alpha_M^-$, hence (V, γ) is α_M^--stable. The image of π contains therefore the open dense subspace of stable vector bundles. Since $\mathcal{N}_{\alpha_M^-}(2, d)$ is compact (cf. Theorem I of [15]) it follows that π is surjective.

To compute the fibre over a stable vector bundle, we only have to note that $(V, \gamma) \cong (V, \gamma')$ if and only if $\gamma = \theta \gamma'$ for some $\theta \in \mathbb{C}^*$. If $(V, \gamma) \cong (V, \gamma')$, then there is an automorphism $\lambda : V \to V$ such that $\gamma'\lambda = ((\lambda^t)^{-1} \otimes 1_U)\gamma$. But, as V is stable, λ is a non-zero scalar so $\gamma = \lambda^2 \gamma'$. On the other hand, if $\gamma = \theta \gamma'$ for some $\theta \in \mathbb{C}^*$ then the scalar automorphism of V given by $\sqrt{\theta} \in \mathbb{C}^*$ is an isomorphism between (V, γ) and (V, γ').   □

We now move on to the description of the flip loci $\mathcal{S}_{\alpha_k^-}(2, d)$ with $\alpha_k < d/2$.

If $(V, \gamma) \in \mathcal{S}_{\alpha_k^-}(2, d)$, then it is α_k^--stable and α_k^+-unstable hence strictly α_k-semistable. The destabilizing subbundle must be a line subbundle $M \subset V$ such that

$$\deg(M) = d - \alpha_k$$

and

$$\gamma(M) \not\subset M^\perp U.$$

Therefore γ induces a non-zero holomorphic map

$$\gamma' : M \longrightarrow V \xrightarrow{\gamma} V^* \otimes U \longrightarrow M^{-1}U \tag{3.5}$$

i.e.,

$$\gamma' \in H^0(X, M^{-2}U).$$

The description of the flip loci $\mathcal{S}_{\alpha_k^-}(2, d)$, with $\alpha_k < d/2$, will be done by carrying out a detailed analysis of this information.

Write V as an extension

$$0 \longrightarrow M \longrightarrow V \longrightarrow L \longrightarrow 0 \tag{3.6}$$

where $L = \Lambda^2 V M^{-1}$.

What we have is already enough to describe $\mathcal{S}_{\alpha_m^-}(2, d)$ for U-quadratic pairs such that d_U is even.

Proposition 3.14 *Suppose that d_U is even. If $(V, \gamma) \in \mathcal{S}_{\alpha_m^-}(2, d)$ then the extension* (3.6) *is trivial,*

$$V = M \oplus L,$$

and with respect to this decomposition,

$$\gamma = \begin{pmatrix} \gamma' & 0 \\ 0 & \gamma'' \end{pmatrix}$$

where γ' is defined in (3.5) *and $\gamma'' \in H^0(X, L^{-2}U) \smallsetminus \{0\}$.*

Proof If $(V, \gamma) \in \mathcal{S}_{\alpha_m^-}(2, d)$, then $\deg(M) = d - \alpha_m = d_U/2$, thus $\deg(M^{-1}U) = d_U/2$. Consider the map

$$\varphi : V \longrightarrow M^{-1}U$$

defined by

$$\varphi = (i^t \otimes 1_U)\gamma,$$

where $i : M \hookrightarrow V$ is the inclusion. Since $\gamma(V) \not\subset M^\perp U$, then $\mathrm{rk}(\varphi) = 1$ generically.

Consider the line subbundle $N = \ker(\varphi)$ of V. We have the induced non-zero map

$$\widetilde{\varphi} : V/N \longrightarrow M^{-1}U$$

i.e., $\widetilde{\varphi} \in H^0(X, (V/N)^{-1}M^{-1}U)$, hence

$$\deg(N) \geqslant d - \deg(M^{-1}U) = d - d_U/2 = \deg(L).$$

On the other hand, since $N \not\cong M$, we have a non-zero map $N \to L$, so

$$\deg(N) \leqslant \deg(L).$$

We conclude that $\deg(N) = \deg(L)$ and that the map $N \to L$ is an isomorphism, $L \cong N$, from which follows that extension (3.6) is trivial:

$$V = M \oplus L.$$

Since $L \cong N = \ker(\varphi)$, we have that $\gamma(L) \subset M^\perp U \cong L^{-1}U$, thus the form of γ with respect to the decomposition $V = M \oplus L$ is

$$\gamma = \begin{pmatrix} \gamma' & 0 \\ 0 & \gamma'' \end{pmatrix}.$$

$\gamma'' \neq 0$ otherwise L would be an α_m^--destabilizing subbundle of (V, γ) of type **(A)**, contradicting the assumption $(V, \gamma) \in \mathcal{S}_{\alpha_m^-}(2, d)$ (or, alternatively, because of Proposition 3.1). $\square$

Corollary 3.15 *If U has even degree d_U, $\mathcal{S}_{\alpha_m^-}(2, d)$ is isomorphic to*

$$\mathcal{N}_{d_U/2^-}(1, d_U/2) \times \mathcal{N}_{(d - d_U/2)^-}(1, d - d_U/2).$$

Proof Given $(V, \gamma) \in \mathcal{S}_{\alpha_m^-}(2, d)$, the pair (V, γ) determines and is determined by the pairs (M, γ') and (L, γ'') obtained in the previous proposition. These are $(d - \alpha_m^+)$-stable and α_m^--stable, respectively, therefore the map $(V, \gamma) \mapsto ((M, \gamma'), (L, \gamma''))$ is an isomorphism between $\mathcal{S}_{\alpha_m^-}(2, d)$ and $\mathcal{N}_{d - \alpha_m^+}(1, d - \alpha_m) \times \mathcal{N}_{\alpha_m^-}(1, \alpha_m)$. $\square$

So, from Proposition 2.6, in this case $\mathcal{S}_{\alpha_m^-}(2, d)$ is isomorphic to

$$S \times \mathcal{N}_{(d - d_U/2)^-}(1, d - d_U/2)$$

where S is the set of square roots of U.

Now we pass to the analysis of $\mathcal{S}_{\alpha_k^-}(2, d)$, with $\alpha_k \neq \alpha_m$, α_M if d_U is even or just $\alpha_k \neq \alpha_M$ if d_U is odd. We start by noticing some constrains of its elements.

In the cases we are now considering, the map $\gamma' : M \to M^{-1}U$ as defined in (3.5) is not an isomorphism. Let D be its divisor

$$D = \mathrm{div}(\gamma') \tag{3.7}$$

and consider the structure sheaf $\mathcal{O}_D$ of X restricted to D (or structure sheaf of the scheme D).

Remark 3.16 If $D = \sum_{i=1}^m n_i p_i$, then, choosing a local coordinate z_i centred at p_i, a global section of $\mathcal{O}_D$ can be written as $\sum_{i=1}^m f_i(z)$ where $f_i(z) = \sum_{k=0}^{n_i-1} a_k z_i^k$. One has then a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}(-D) \longrightarrow \mathcal{O} \xrightarrow{r(D)} \mathcal{O}_D \longrightarrow 0 \qquad (3.8)$$

where, for each open U such that there is only one point $p \in \mathrm{Supp}(D)$ in U,

$$r(D)(U)(s) = r(D)(U)\left(\sum_{k=0}^{\infty} a_k z^k\right) = \sum_{k=0}^{D(p)-1} a_k z^k, \qquad (3.9)$$

for $s \in \mathcal{O}(U)$ such that, in a local coordinate z centred at p, $s(z) = \sum_{k=0}^{\infty} a_k z^k$.

Proposition 3.17 *There is a well defined section*

$$\theta_\gamma \in H^0(D, M^{-1}L^{-1}U). \qquad (3.10)$$

given by restriction of $\gamma|_M$ to D.

Proof For any sheaf $\mathcal{F}$, write $\mathcal{F}|_D$ for $\mathcal{F} \otimes \mathcal{O}_D$. From (3.8), we obtain the short exact sequence of sheaves

$$0 \longrightarrow V^* \otimes M^{-1}U(-D) \longrightarrow V^* \otimes M^{-1}U \xrightarrow{r(D)} V^* \otimes M^{-1}U|_D \longrightarrow 0 \quad (3.11)$$

and we have a map, which we still denote by $r(D)$,

$$r(D) : H^0(X, V^* \otimes M^{-1}U) \longrightarrow H^0(D, V^* \otimes M^{-1}U).$$

Now, $\gamma|_M \in H^0(X, V^* \otimes M^{-1}U)$ so consider $r(D)(\gamma|_M) \in H^0(D, V^* \otimes M^{-1}U)$. But, since $D = \mathrm{div}(\gamma')$, we have in fact that $r(D)(\gamma|_M) \in H^0(D, M^{-1}L^{-1}U)$, so we define

$$\theta_\gamma = r(D)(\gamma|_M) \in H^0(D, M^{-1}L^{-1}U),$$

as claimed. $\square$

If we also denote by $r(D)$ the map in H^0 of the restriction $\Lambda^2 V^{-2}U^2 \to \Lambda^2 V^{-2}U^2|_D$, we see that

$$r(D)(\det(\gamma)) = -\theta_\gamma^2 \in H^0(D, M^{-2}L^{-2}U^2). \qquad (3.12)$$

This section $\theta_\gamma \in H^0(D, M^{-1}L^{-1}U)$, obtained in the previous proposition, will be very important in the description of $\mathcal{S}_{\alpha_k^-}(2, d)$ and the next result is a first instance of this.

Proposition 3.18 *Let $(V, \gamma) \in \mathcal{S}_{\alpha_k^-}(2, d)$. Then $\theta_\gamma = 0$ if and only if extension (3.6) is trivial,*

$$V = M \oplus L,$$

and with respect to this decomposition,

$$\gamma = \begin{pmatrix} \gamma' & 0 \\ 0 & \gamma'' \end{pmatrix},$$

where γ' is defined in (3.5) and $\gamma'' \in H^0(X, L^{-2}U) \smallsetminus \{0\}$.

Proof As in the proof of Proposition 3.14, consider the map $\varphi : V \rightarrow M^{-1}U$ given by $\varphi = (i^t \otimes 1_U)\gamma$, and its kernel N.

We have the induced non-zero map

$$\widetilde{\varphi} : V/N \longrightarrow M^{-1}U$$

i.e., $\widetilde{\varphi} \in H^0(X, (V/N)^{-1}M^{-1}U)$ and let

$$\widetilde{D} = \mathrm{div}(\widetilde{\varphi})$$

so that

$$\deg(\widetilde{D}) = \deg(N) - 2d + \alpha_k + d_U. \tag{3.13}$$

Let p be any point in $\mathrm{Supp}(D)$ and choose a local coordinate z of X centred at p. Locally, we can write $V = M \oplus L$ and, with respect to this decomposition,

$$\gamma(z) = \begin{pmatrix} f_1(z) & f_2(z) \\ f_2(z) & f_3(z) \end{pmatrix}$$

hence

$$\varphi(z) = \begin{pmatrix} f_1(z) & f_2(z) \end{pmatrix}$$

so

$$\widetilde{D}(p) \geqslant \min\{\mathrm{ord}_0\, f_1(z), \mathrm{ord}_0\, f_2(z)\} = \min\{D(p), \mathrm{ord}_0\, f_2(z)\}. \tag{3.14}$$

Since $\theta_\gamma = 0$, then

$$\theta_\gamma(p) = 0 \tag{3.15}$$

but, by the definition of θ_γ in (3.10),

$$\theta_\gamma(p) = r(D)(U)(f_2(z))$$

and from this, (3.9) and (3.15), we see that $\mathrm{ord}_0\, f_2(z) \geqslant D(p)$. It follows from (3.14) that

$$\widetilde{D} \geqslant D \tag{3.16}$$

so

$$\deg(\widetilde{D}) \geqslant \deg(D) = -2\deg(M) + d_U = -2d + 2\alpha_k + d_U.$$

From this and (3.13), we see that

$$\deg(N) \geqslant \alpha_k = \deg(L). \tag{3.17}$$

On the other hand, since $N \not\cong M$, we have a non-zero map $N \rightarrow L$ so (3.17) implies $L \cong N$. Extension (3.6) is hence trivial:

$$V = M \oplus L.$$

From $L \cong N = \ker(\varphi)$, it follows that $\gamma(L) \subset M^\perp U \cong L^{-1}U$, thus the form of γ with respect to the decomposition $V = M \oplus L$ is

$$\gamma = \begin{pmatrix} \gamma' & 0 \\ 0 & \gamma'' \end{pmatrix}.$$

$\gamma'' \neq 0$ otherwise L would be an α_k^--destabilizing subbundle of (V, γ) of type **(A)**, contradicting the assumption $(V, \gamma) \in \mathcal{S}_{\alpha_k^-}(2, d)$. $\square$

We shall write $\mathcal{S}_{\alpha_k^-}(2, d)$ as a disjoint union

$$\mathcal{S}_{\alpha_k^-}(2, d) = \mathcal{S}_{\alpha_k^-}^0(2, d) \sqcup \mathcal{S}_{\alpha_k^-}^1(2, d) \tag{3.18}$$

where

- $\mathcal{S}_{\alpha_k^-}^0(2, d)$ is the space of pairs in $\mathcal{S}_{\alpha_k^-}(2, d)$ with $\theta_\gamma = 0$;
- $\mathcal{S}_{\alpha_k^-}^1(2, d)$ is the space of pairs in $\mathcal{S}_{\alpha_k^-}(2, d)$ with $\theta_\gamma \neq 0$.

Let us now study each of the spaces $\mathcal{S}_{\alpha_k^-}^0(2, d)$ and $\mathcal{S}_{\alpha_k^-}^1(2, d)$.

3.4.1

From Proposition 3.18, $\mathcal{S}_{\alpha_k^-}^0(2, d)$ is precisely the space of pairs in $\mathcal{S}_{\alpha_k^-}(2, d)$ such that extension (3.6) is trivial and γ has the given form. So we have the following corollary.

Corollary 3.19 *Let $\alpha_k \neq \alpha_M$. Then $\mathcal{S}_{\alpha_k^-}^0(2, d)$ is isomorphic to*

$$\mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times \mathcal{N}_{\alpha_k^-}(1, \alpha_k).$$

Proof Given $(V, \gamma) \in \mathcal{S}_{\alpha_k^-}^0(2, d)$, the pair (V, γ) determines and is determined by the pairs (M, γ') and (L, γ'') obtained in the previous proposition. These are $(d - \alpha_k^+)$-stable and α_k^--stable, respectively, therefore the map $(V, \gamma) \mapsto ((M, \gamma'), (L, \gamma''))$ is an isomorphism between $\mathcal{S}_{\alpha_k^-}^0(2, d)$ and $\mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times \mathcal{N}_{\alpha_k^-}(1, \alpha_k)$. $\qquad\square$

Remark 3.20 We have seen that in the case of $\mathcal{S}_{\alpha_m^-}(2, d)$, the section θ_γ is always zero. Hence $\mathcal{S}_{\alpha_m^-}(2, d) = \mathcal{S}_{\alpha_m^-}^0(2, d)$ and therefore the similarity of Corollaries 3.15 and 3.19.

3.4.2 $\mathcal{S}_{\alpha_k^-}^1(2, d)$

We move on to the description of $\mathcal{S}_{\alpha_k^-}^1(2, d)$. From Proposition 3.18, $\mathcal{S}_{\alpha_k^-}^1(2, d)$ is the space of pairs in $\mathcal{S}_{\alpha_k^-}(2, d)$ such that extension (3.6) is non-trivial.

Before going to the analysis of $\mathcal{S}_{\alpha_k^-}^1(2, d)$ we first need the following proposition.

Proposition 3.21 *Let $\alpha_k < d/2$. If $(V, \gamma) \in \mathcal{S}_{\alpha_k^-}(2, d)$, then generically* rk$(\gamma) = 2$.

Proof As always, $\gamma \neq 0$. Suppose that rk$(\gamma) = 1$. Then, det$(\gamma) = 0$, hence, from (3.12), $\theta_\gamma = 0$. From Proposition 3.18, $V = M \oplus L$ and

$$\gamma = \begin{pmatrix} \gamma' & 0 \\ 0 & \gamma'' \end{pmatrix}$$

with $\gamma'' \neq 0$. Since also $\gamma' \neq 0$, it follows that det$(\gamma) \neq 0$ and this is a contradiction with rk$(\gamma) = 1$. $\qquad\square$

Given $((M, \gamma'), L) \in \mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times \text{Jac}^{\alpha_k}(X)$ and recalling that $D = \text{div}(\gamma')$, consider the subvariety

$$\mathcal{C}((M, \gamma'), L)$$

of

$$H^0(D, M^{-1}L^{-1}U) \smallsetminus \{0\} \times H^0(X, M^{-2}L^{-2}U^2) \smallsetminus \{0\}$$

whose elements (q, η) satisfy the equation

$$q^2 + \eta|_D = 0.$$

$\mathbb{C}^*$ acts freely on $\mathcal{C}((M, \gamma'), L)$ as

$$\lambda \cdot (q, \eta) = (\lambda q, \lambda^2 \eta)$$

and we denote the quotient by

$$\mathcal{Q}((M, \gamma'), L) = \mathcal{C}((M, \gamma'), L)/\mathbb{C}^*. \tag{3.19}$$

Proposition 3.22 *Let d_U even and $\alpha_k \neq \alpha_m, \alpha_M$ or d_U odd and $\alpha_k \neq \alpha_M$. Suppose that $d_U - d > g - 1$. Then there is a morphism*

$$\mathcal{S}^1_{\alpha_k^-}(2, d) \longrightarrow \mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times \mathrm{Jac}^{\alpha_k}(X)$$

whose fibre over $((M, \gamma'), L)$ is isomorphic to $\mathcal{Q}((M, \gamma'), L)$ as defined in (3.19).

Proof If $(V, \gamma) \in \mathcal{S}^1_{\alpha_k^-}(2, d)$, we already know that we can write V as the extension (3.6), and that the pair (M, γ') is $(d - \alpha_k^+)$-stable. So we have the map

$$p : \mathcal{S}^1_{\alpha_k^-}(2, d) \longrightarrow \mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times \mathrm{Jac}^{\alpha_k}(X)$$

given by

$$p(V, \gamma) = ((M, \gamma'), L).$$

Let $(V, \gamma) \in p^{-1}((M, \gamma'), L)$ and $D = \mathrm{div}(\gamma')$. Then

$$\theta_\gamma \in H^0(D, M^{-1}L^{-1}U) \smallsetminus \{0\}$$

because $(V, \gamma) \in \mathcal{S}^1_{\alpha_k^-}(2, d)$ and, by the previous proposition,

$$\det(\gamma) \in H^0(X, M^{-2}L^{-2}U^2) \smallsetminus \{0\}.$$

Moreover,

$$\theta_\gamma^2 + \det(\gamma)|_D = 0$$

so we have the map

$$p^{-1}((M, \gamma'), L) \longrightarrow \mathcal{Q}((M, \gamma'), L) \tag{3.20}$$

given by

$$(V, \gamma) \mapsto [(\theta_\gamma, \det(\gamma))]. \tag{3.21}$$

Let us now see that we also have a map the other way around and which is inverse of the above one. As we are assuming $d_U - d > g - 1$, we have $\chi(M^{-2}L^{-2}U^2) > 0$, hence $H^0(X, M^{-2}L^{-2}U^2) \neq 0$ and then

$$\mathcal{Q}((M, \gamma'), L) \neq \emptyset.$$

Take $[(q, \eta)] \in \mathcal{Q}((M, \gamma'), L)$ and choose a representative (q, η). We construct a pair (V, γ) as follows.

Consider the following complexes

$$C_1^\bullet : L^{-1}M \xrightarrow{\cong} L^{-1}M^{-1}U(-D)$$
$$C_2^\bullet : L^{-1}M \xrightarrow{c} L^{-1}M^{-1}U$$

and

$$C_3^\bullet : 0 \longrightarrow L^{-1}M^{-1}U|_D$$

where

$$c(\psi) = \gamma'\psi. \tag{3.22}$$

We have the short exact sequence

$$0 \longrightarrow C_1^\bullet \longrightarrow C_2^\bullet \xrightarrow{r(D)} C_3^\bullet \longrightarrow 0$$

given by the commutative diagram of sheaves of holomorphic sections

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
L^{-1}M & \xrightarrow{\cong} & L^{-1}M^{-1}U(-D) \\
{\scriptstyle =}\downarrow & & \downarrow{\scriptstyle c} \\
L^{-1}M & \xrightarrow{c} & L^{-1}M^{-1}U \\
\downarrow & & \downarrow{\scriptstyle r(D)} \\
0 & \xrightarrow{0} & L^{-1}M^{-1}U|_D \\
\downarrow & & \downarrow \\
0 & & 0.
\end{array}
$$

From this we obtain a long exact sequence in hypercohomology of the complexes

$$0 \longrightarrow \mathbb{H}^0(X, C_1^\bullet) \longrightarrow \mathbb{H}^0(X, C_2^\bullet) \xrightarrow{r(D)} \mathbb{H}^0(X, C_3^\bullet) \longrightarrow \mathbb{H}^1(X, C_1^\bullet) \longrightarrow \mathbb{H}^1(X, C_2^\bullet)$$
$$\xrightarrow{r(D)} \mathbb{H}^1(X, C_3^\bullet) \longrightarrow \mathbb{H}^2(X, C_1^\bullet) \longrightarrow \mathbb{H}^2(X, C_2^\bullet) \xrightarrow{r(D)} \mathbb{H}^2(X, C_3^\bullet) \longrightarrow 0 \quad (3.23)$$

from which it follows that $r(D)$ yields a natural isomorphism

$$\mathbb{H}^1(X, C_2^\bullet) \overset{r(D)}{\underset{\cong}{}} \mathbb{H}^1(X, C_3^\bullet) \cong H^0(D, L^{-1}M^{-1}U). \tag{3.24}$$

Consider the element $(0, q) \in \mathbb{H}^1(X, C_3^\bullet)$ and the corresponding class $r(D)^{-1}(0, q) \in \mathbb{H}^1(X, C_2^\bullet)$. With respect to some open covering $(U_a)_a$ of X, choose a representative

$$(\lambda_{ab}, \gamma_a'')$$

of the class $r(D)^{-1}(0, q) \in \mathbb{H}^1(X, C_2^\bullet)$. Recall then that $\gamma'\lambda_{ab} = \gamma_b'' - \gamma_a''$.

Let V be the vector bundle defined by taking on each open U_a the direct sum

$$M|_{U_a} \oplus L|_{U_a} \tag{3.25}$$

and gluing over U_{ab} through the map

$$f_{ab} = \begin{pmatrix} 1_M & \lambda_{ab} \\ 0 & 1_L \end{pmatrix}. \tag{3.26}$$

Also over each open U_a, consider the section of $H^0(U_a, S^2(M \oplus L)^* \otimes U)$ given, with respect to the decomposition (3.25), by

$$\gamma_a = \begin{pmatrix} \gamma' & \gamma_a'' \\ \gamma_a''' \otimes 1_U & (\gamma_a''' \otimes 1_U)\gamma'^{-1}\gamma_a'' + \eta\gamma'^{-1} \end{pmatrix}. \tag{3.27}$$

Observe that, since $r(D)(U_a)(\gamma_a'') = q|_{D \cap U_a}$ and $q^2 + \eta|_D = 0$, then

$$(\gamma_a''' \otimes 1_U)\gamma'^{-1}\gamma_a'' + \eta\gamma'^{-1}$$

is defined over D. One has $\gamma_b = f_{ab}^t \gamma_a f_{ab}$, so the collection of symmetric maps $(\gamma_a)_a$ yields a global symmetric map $\gamma : V \to V^* \otimes U$.

So, from $((M, \gamma'), L) \in \mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times \mathrm{Jac}^{\alpha_k}(X)$ and (q, η) both non-zero and such that $q^2 + \eta|_D = 0$, we have built a U-quadratic pair (V, γ) over X such that $\det(\gamma) = \eta$, $\theta_\gamma = q$, which lies in $\mathcal{S}^1_{\alpha_k^-}(2, d)$ and which is mapped onto $((M, \gamma'), L)$ by the map p.

Suppose now that we had a different choice of the representative of the class $[(q, \eta)] \in \mathcal{Q}((M, \gamma'), L)$, say $(\beta q, \beta^2 \eta)$ with $\beta \in \mathbb{C}^*$. From (3.24), this pair defines a new class in $\mathbb{H}^1(X, C_2^\bullet)$ whose representative is $(\beta\lambda_{ab}, \beta\gamma_{ab}'')$. The vector bundle $\tilde{V}$ constructed again from (3.25) and gluing by

$$\begin{pmatrix} 1_M & \beta\lambda_{ab} \\ 0 & 1_L \end{pmatrix} = \tilde{f}_{ab}$$

is isomorphic to V through the isomorphism $g : \tilde{V} \to V$ defined locally by

$$g_a = \begin{pmatrix} 1_M & 0 \\ 0 & \beta \end{pmatrix}$$

because $f_{ab} g_a = g_b \tilde{f}_{ab}$. Moreover, we consider the section of $H^0(U_a, S^2(M \oplus L)^* \otimes U)$ given by

$$\begin{pmatrix} \gamma' & \beta\gamma_a'' \\ \beta\gamma_a''' \otimes 1_U & \beta^2((\gamma_a''' \otimes 1_U)\gamma'^{-1}\gamma_a'' + \eta\gamma'^{-1}) \end{pmatrix} = \tilde{\gamma}_a$$

and we have again $\tilde{\gamma}_b = \tilde{f}_{ab}^t \tilde{\gamma}_a \tilde{f}_{ab}$, so we have the pair $(\tilde{V}, \tilde{\gamma}) \in p^{-1}((M, \gamma'), L)$. Since $g_a^t \gamma_a g_a = \tilde{\gamma}_a$ the isomorphism g is indeed an isomorphism between $(\tilde{V}, \tilde{\gamma})$ and (V, γ).

In other words, we have a map

$$\mathcal{Q}((M, \gamma'), L) \longrightarrow p^{-1}((M, \gamma'), L)$$

defined by

$$[(q, \eta)] \mapsto \text{ isomorphism class of } (V, \gamma) \text{ defined by (3.25), (3.26) and (3.27).}$$

Clearly, this map is inverse of that defined in (3.20) and (3.21) and this gives an isomorphism $\mathcal{Q}((M, \gamma'), L) \cong p^{-1}((M, \gamma'), L)$. We have then seen that p is surjective with fibre isomorphic to $\mathcal{Q}((M, \gamma'), L)$. It remains to check that p is a morphism.

Let $p_X : \mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times X \to X$ be the projection. From Remark 2.4 and Theorem I of [15], there is a universal $p_X^* U$-quadratic pair $(\mathcal{L}_1, \boldsymbol{\gamma}')$ over $\mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times X$. Consider also the Poincaré line bundle $\mathcal{L}_2$, of degree α_k, over $\mathrm{Jac}^{\alpha_k}(X) \times X$. Let

$$\mathrm{pr}_{13} : \mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times \mathrm{Jac}^{\alpha_k}(X) \times X \longrightarrow \mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times X$$

$$\mathrm{pr}_{23} : \mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times \mathrm{Jac}^{\alpha_k}(X) \times X \longrightarrow \mathrm{Jac}^{\alpha_k}(X) \times X$$

and

$$\mathrm{pr}_{12} : \mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times \mathrm{Jac}^{\alpha_k}(X) \times X \longrightarrow \mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times \mathrm{Jac}^{\alpha_k}(X)$$

be the projections. Consider the following sheaves over $\mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times \mathrm{Jac}^{\alpha_k}(X)$:

$$R^0 \mathrm{pr}_{12*}(\mathrm{pr}_{13}^* \mathcal{L}_1^{-1} \mathrm{pr}_{23}^* \mathcal{L}_2^{-1} p_X^* U|_\Delta)$$

where $\Delta \subset \mathrm{Sym}^{d_U - 2d + 2\alpha_k}(X) \times X$ is the universal divisor, and

$$R^0 \mathrm{pr}_{12*}(\mathrm{pr}_{13}^* \mathcal{L}_1^{-2} \mathrm{pr}_{23}^* \mathcal{L}_2^{-2} p_X^* U^2).$$

Since we are assuming $d_U - d > g - 1$, these spaces have constant dimension, hence are locally free. We consider the subsheaf $\widetilde{\mathcal{C}}((\mathcal{L}_1, \boldsymbol{\gamma}'), \mathcal{L}_2)$ (of sets) of

$$R^0 \mathrm{pr}_{12*}(\mathrm{pr}_{13}^* \mathcal{L}_1^{-1} \mathrm{pr}_{23}^* \mathcal{L}_2^{-1} p_X^* U|_\Delta) \smallsetminus \{0\} \times R^0 \mathrm{pr}_{12*}(\mathrm{pr}_{13}^* \mathcal{L}_1^{-2} \mathrm{pr}_{23}^* \mathcal{L}_2^{-2} p_X^* U^2) \smallsetminus \{0\}$$

consisting of pairs of non-zero sections $(\mathbf{q}, \boldsymbol{\eta})$ satisfying the equation

$$\mathbf{q}^2 + \boldsymbol{\eta}|_\Delta = 0.$$

If $\widetilde{\mathcal{Q}}((\mathcal{L}_1, \boldsymbol{\gamma}'), \mathcal{L}_2)$ denotes the sheaf obtained from $\widetilde{\mathcal{C}}((\mathcal{L}_1, \boldsymbol{\gamma}'), \mathcal{L}_2)$ by identifying sections of the form $(\mathbf{q}, \boldsymbol{\eta})$ and $(\lambda \mathbf{q}, \lambda^2 \boldsymbol{\eta})$ for some $\lambda \in \mathbb{C}^*$, then this is a locally trivial fibration over $\mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times \mathrm{Jac}^{\alpha_k}(X)$ such that its fibre over $((M, \gamma'), L)$ is $\mathcal{Q}((M, \gamma'), L)$, as defined in (3.19). As in the proof of Proposition 3.10 (see also Proposition 3.4 of [30]), we have the following commutative diagram:

$$
\begin{array}{ccc}
\widetilde{\mathcal{Q}}((\mathcal{L}_1, \boldsymbol{\gamma}'), \mathcal{L}_2) & \xrightarrow{\;\cong\;} & \mathcal{S}_{\alpha_k^-}(2, d) \\
& \searrow & \downarrow{\scriptstyle p} \\
& & \mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times \mathrm{Jac}^{\alpha_k}(X).
\end{array}
$$

So p is a morphism and the result follows. □

One consequence of Corollary 3.19 and of the previous proposition is that $\mathcal{S}_{\alpha_k^-}(2, d) = \mathcal{S}_{\alpha_k^-}^0(2, d) \sqcup \mathcal{S}_{\alpha_k^-}^1(2, d)$ is compact. It follows from Proposition 3.7 (see also Remark 3.8) is a disconnected subvariety of $\mathcal{N}_{\alpha_k^-}(2, d)$. Hence we can compute its dimension.

From Corollaries 2.8 and 3.19, we have

$$\dim \mathcal{S}_{\alpha_k^-}^0(2, d) = 2d_U - 2d. \tag{3.28}$$

On the other hand, for $\mathcal{S}_{\alpha_k^-}^1(2, d)$, we have:

Corollary 3.23 *If $d_U - d > g - 1$, then $\dim \mathcal{S}_{\alpha_k^-}^1(2, d) = 3d_U - 4d + 2\alpha_k$.*

Proof Since $d_U - d > g - 1$, Proposition 3.22 holds. From Corollary 2.8 we have

$$\dim(\mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times \mathrm{Jac}^{\alpha_k}(X)) = 2\alpha_k - 2d + d_U + g. \tag{3.29}$$

Given $((M, \gamma'), L) \in \mathcal{N}_{d-\alpha_k^+}(1, d - \alpha_k) \times \mathrm{Jac}^{\alpha_k}(X)$, we now compute

$$\dim \mathcal{Q}((M, \gamma'), L) = \dim \mathcal{C}((M, \gamma'), L) - 1.$$

If $F : H^0(D, M^{-1}L^{-1}U) \smallsetminus \{0\} \times H^0(X, M^{-2}L^{-2}U^2) \smallsetminus \{0\} \to H^0(D, M^{-2}L^{-2}U^2)$ is given by

$$F(q, \eta) = q^2 + \eta|_D$$

then $\mathcal{C}((M, \gamma'), L) = F^{-1}(0)$. Linearising the map F at a point (q, η), we are lead to $F_{(q,\eta)*} : H^0(D, M^{-1}L^{-1}U) \times H^0(X, M^{-2}L^{-2}U^2) \to H^0(D, M^{-2}L^{-2}U^2)$ with

$$F_{(q,\eta)*}(\dot{q}, \dot{\eta}) = 2q\dot{q} + \dot{\eta}|_D.$$

Choose (q, η) such that F_* is surjective (for instance, $(1, \eta)$, where $q = 1$ means that $q(p) = 1$ for each $p \in \mathrm{Supp}(D)$). Then $\dim \mathcal{C}((M, \gamma'), L)$ is

$$\dim \ker(F_{(1,\eta)*}) = \dim H^0(X, M^{-2}L^{-2}U^2) = 2(d_U - d) + 1 - g$$

because $d_U - d > g - 1$ implies $\dim H^1(X, M^{-2}L^{-2}U^2) = 0$. Hence

$$\dim \mathcal{Q}((M, \gamma'), L) = 2(d_U - d) - g. \tag{3.30}$$

The result now follows from (3.29) and (3.30).   □

From this, from (3.28) and from $\alpha_k \geqslant \alpha_m = d - [d_U/2]$ follows that

$$\dim \mathcal{S}^1_{\alpha_k^-}(2, d) \geqslant \dim \mathcal{S}^0_{\alpha_k^-}(2, d)$$

with equality if and only if $\alpha_k = \alpha_m$ and d_U even. Hence from (3.18) and the previous corollary, we conclude the following.

Corollary 3.24 *Let $\alpha \neq \alpha_M$ and d such that $d_U - d > g - 1$. Then each connected component of $\mathcal{S}_{\alpha_k^-}(2, d)$ has dimension less or equal than $3d_U - 4d + 2\alpha_k$.*

4 The space $\mathcal{N}_{\alpha_m^-}(2, d)$

Having examined the differences which occur on the moduli spaces when we cross a critical value of α, we now address the problem of studying the number of connected components of one of them. This will be done in this section and the moduli space which will be analysed is $\mathcal{N}_{\alpha_m^-}(2, d)$, the one for which the parameter α is less than the minimum critical value $\alpha_m = d - [d_U/2]$. In Sect. 5 we join the results of this and the previous sections to achieve the goal of computing the number of connected components of $\mathcal{N}_\alpha(2, d)$, for any $\alpha \leqslant d/2$.

The method we shall employ to give the desired description of $\mathcal{N}_{\alpha_m^-}(2, d)$ is the theory of spectral curves together with an analogue of the Hitchin map which is slightly outlined in the next sections.

4.1 The spectral curve

We shall give a rough description of the spectral curve of X corresponding to a line bundle L and a section of L^2. Then we shall see how to associate a spectral curve to a quadratic pair (V, γ), with γ generically non-degenerate. The classical references for this theory, particularly its relations with Higgs bundles, are [1,18].

Let then L be a holomorphic line bundle over X with $\deg(L) > 0$. We begin by reviewing the construction of the spectral curve $X_{s,L}$ associated to a section $s \in H^0(X, L^2)$. Consider the complex surface T given by the total space of the line bundle L, and let $\pi : T \to X$ be the projection. The pullback π^*L of L to its total space has a tautological section

$$\lambda \in H^0(T, \pi^*L)$$

defined by $\lambda(x) = x$.

Definition 4.1 Let $s \in H^0(X, L^2)$. The *spectral curve* $X_{s,L}$ associated to s is the zero scheme in the surface T of the section

$$\lambda^2 + \pi^*s \in H^0(T, \pi^*L^2).$$

Remark 4.2 In the present case, the spectral curve $X_{s,L}$ is always reduced, but it may be singular and reducible. In fact, it is smooth if and only if s only has simple zeros and it is irreducible if and only if s is not the square of a section of L.

Remark 4.3 The above definition of spectral curve is a very particular case of a general definition. In fact, one can define a spectral curve associated to an element of the sum $\bigoplus_{k=1}^n H^0(X, L^k)$. See [1,18].

4.2 An analogue of the Hitchin map and its generic fibre

Consider a U-quadratic pair $(V, \gamma) \in \mathcal{N}_{\alpha_m^-}(2, d)$. By Proposition 3.1, $\det(\gamma)$ is a non-zero holomorphic section of $\Lambda^2 V^{-2} U^2$ and one can consider its divisor $\mathrm{div}(\det(\gamma)) \in \mathrm{Sym}^{2d_U - 2d}(X)$. Let $\mathcal{P}_X$ be the 2^{2g}-cover of $\mathrm{Sym}^{2d_U - 2d}(X)$ which fits in the commutative diagram

$$\begin{array}{ccc}
\mathcal{P}_X & \longrightarrow & \mathrm{Jac}^{d_U - d}(X) \\
\downarrow & & \downarrow{\scriptstyle L \mapsto L^2} \\
\mathrm{Sym}^{2d_U - 2d}(X) & \underset{D \mapsto \mathcal{O}(D)}{\longrightarrow} & \mathrm{Jac}^{2d_U - 2d}(X).
\end{array} \qquad (4.1)$$

In other words,

$$\mathcal{P}_X = \mathrm{Sym}^{2d_U - 2d}(X) \times_{\mathrm{Jac}^{2d_U - 2d}(X)} \mathrm{Jac}^{d_U - d}(X)$$

i.e., it is the fibred product of $\mathrm{Sym}^{2d_U - 2d}(X)$ and $\mathrm{Jac}^{d_U - d}(X)$ over $\mathrm{Jac}^{2d_U - 2d}(X)$, and its elements are pairs $(D, L) \in \mathrm{Sym}^{2d_U - 2d}(X) \times \mathrm{Jac}^{d_U - d}(X)$ such that

$$\mathcal{O}(D) \cong L^2.$$

In order to describe $\mathcal{N}_{\alpha_m^-}(2, d)$, we shall use the following map, which is analogue to the so-called Hitchin map defined for the first time by Hitchin in [18]. Consider then the map

$$\begin{array}{rcl}
h : \mathcal{N}_{\alpha_m^-}(2, d) & \longrightarrow & \mathcal{P}_X \\
(V, \gamma) & \longmapsto & (\mathrm{div}(\det(\gamma)), \Lambda^2 V^{-1} U).
\end{array} \qquad (4.2)$$

Definition 4.4 An *L-twisted Higgs pair* of type (n, d) over X is a pair (V, φ), where V is a holomorphic vector bundle over X, with $\text{rk}(V) = n$ and $\deg(V) = d$, and φ is a global holomorphic section of $\text{End}(V) \otimes L$, called the *Higgs field*.

Two *L*-twisted Higgs pairs (V, φ) and (V', φ') are *isomorphic* if there is a holomorphic isomorphism $f : V \to V'$ such that $\varphi' f = (f \otimes 1_L)\varphi$.

Definition 4.5 Let (V, γ) be a U-quadratic pair of type $(2, d)$ over X and let $\xi = \Lambda^2 V^{-1} U$. The ξ-*twisted Higgs pair* (V, φ) *associated to* (V, γ) is the one induced from (V, γ) and from the isomorphism

$$g : V \otimes \xi \xrightarrow{\cong} V^* \otimes U \qquad (4.3)$$

given by

$$g(v \otimes \phi \otimes u) = \phi(v \wedge -) \otimes u,$$

where $v \otimes \phi \otimes u \in V \otimes \xi = V \otimes \Lambda^2 V^{-1} U$. In other words, $\varphi = g^{-1}\gamma$.

Lemma 4.6 *Suppose that V and V' are rank 2 holomorphic vector bundles with the same determinant. Let $\xi = \Lambda^2 V^{-1} U$. Let (V, γ) and (V', γ') be two U-quadratic pairs, and (V, φ) and (V', φ') be, respectively, the associated ξ-twisted Higgs, as in Definition 4.5.*

(1) *If (V, φ) is isomorphic to (V', φ') as ξ-twisted Higgs pairs, then (V, γ) is isomorphic to (V', γ') as U-quadratic pairs.*

(2) *If (V, γ) is isomorphic to (V', γ') as U-quadratic pairs, then there is some $\lambda \in \mathbb{C}^*$ such that (V, φ) is isomorphic to $(V', \lambda\varphi')$ as ξ-twisted Higgs pairs.*

Proof Let $f : V \to V'$ be an isomorphism between (V, φ) and (V', φ'), that is,

$$\varphi' f = (f \otimes 1_\xi)\varphi. \qquad (4.4)$$

Since $\Lambda^2 V = \Lambda^2 V'$, then $\det(f) = \lambda \in \mathbb{C}^*$.

Let $g : V \otimes \xi \to V^* \otimes U$ be the isomorphism (4.3), and define $g' : V' \otimes \xi \to V'^* \otimes U$ similarly. Now, we have that

$$\begin{aligned}
(f^t \otimes 1_U)g'(f \otimes 1_\xi)(v \otimes \phi \otimes u) &= (f^t \otimes 1_U)g'(f(v) \otimes \phi \otimes u) \\
&= (f^t \otimes 1_U)(\phi(f(v) \wedge -) \otimes u) \\
&= \phi(f(v) \wedge f(-)) \otimes u \\
&= (\phi \det(f))(v \wedge -) \otimes u \\
&= \lambda\phi(v \wedge -) \otimes u
\end{aligned}$$

so we conclude, from the definition of g in (4.3), that

$$(f^t \otimes 1_U)g'(f \otimes 1_\xi) = \lambda g.$$

From this, from (4.4) and noticing that $g\varphi = \gamma$ and $g'\varphi' = \gamma'$, we conclude that

$$(f^t \otimes 1_U)\gamma' f = \lambda\gamma.$$

Thus $\sqrt{\lambda^{-1}} f$ is an isomorphism between (V, γ) and (V', γ') and this settles the first item.

For the second item, if $f : V \to V'$ is an isomorphism between (V, γ) and (V', γ') then $(f^t \otimes 1_U)\gamma' f = \gamma$. It follows, as above, that $\varphi' f = \lambda^{-1}(f \otimes 1_\xi)\varphi$ where $\mathbb{C}^* \ni \lambda = \det(f)$. So

$$(\lambda\varphi')f = (f \otimes 1_\xi)\varphi$$

and f is an isomorphism between (V, φ) and $(V', \lambda\varphi')$. $\square$

Definition 4.7 A ξ-twisted Higgs pair (V, φ) of type $(2, d)$ is *semistable* if $\deg(L) \leqslant d/2$ for any line subbundle $L \subset V$ such that $\varphi(L) \subset L\xi$.

Proposition 4.8 *Let (V, γ) be a U-quadratic pair of type $(2, d)$ and $\xi = \Lambda^2 V^{-1} U$. Let (V, φ) be the corresponding ξ-twisted Higgs pair, in the sense of Definition 4.5. Then (V, γ) is α_m^--semistable if and only if (V, φ) is semistable.*

Proof Assume that (V, φ) is semistable and let $L \subset V$. As (V, γ) is α_m^--semistable, then Proposition 3.1 says that $\gamma(L) \neq 0$. Suppose that $\gamma(L) \subset L^\perp U$. It is easy to see that

$$\gamma(L) \subset L^\perp U \Longleftrightarrow \varphi(L) \subset L\xi, \tag{4.5}$$

and since (V, φ) is semistable, it follows that $\deg(L) \leqslant d/2$.

Finally, suppose that $\gamma(L) \not\subset L^\perp U$, and $\deg(L) > d - \alpha_m^-$. Then L is a destabilizing subbundle for α_m^-. So by Proposition 3.2, (V, γ) is α-unstable for every $\alpha < \alpha_m^-$ and, from above, the destabilizing subbundle must also be of type **(C)**. We see that for any $\alpha < \alpha_m$, there is $L' \subset V$ such that $\deg(L') > d - \alpha$. Letting $\alpha \to -\infty$ this contradicts the fact that the degrees of subbundles of V are bounded above (see Corollary 10.9 of [21]). We conclude that (V, γ) is α_m^--semistable.

The proof of the other direction is straightforward, using (4.5).   □

If $\xi = \Lambda^2 V^{-1} U$ and (V, φ) is a ξ-twisted Higgs pair, consider the sections defined by the coefficients of the characteristic polynomial of φ:

$$(-\operatorname{tr}(\varphi), \det(\varphi)) \in H^0(X, \xi) \oplus H^0(X, \xi^2).$$

We have $\det(\varphi) = \det(\gamma)$ and, as γ is symmetric, φ has trace zero. Hence one can view $h(V, \gamma) = (\operatorname{div}(\det(\gamma)), \Lambda^2 V^{-1} U)$ in (4.2) as given by ξ and by the divisor of the section given by the characteristic polynomial of φ. The spectral curve $X_{s,\xi}$ associated to ξ and to the section $s = \det(\gamma) \in H^0(X, \xi^2)$ is the curve inside the total space T of ξ defined by the equation

$$\lambda^2 + \pi^* \det(\gamma) = 0.$$

Now, let (D, ξ) be any pair in $\mathcal{P}_X$, defined in (4.1). We want to describe the fibre of h over (D, ξ), i.e., the space of isomorphism classes of α_m^--semistable U-quadratic pairs (V, γ) with $\operatorname{div}(\det(\gamma)) = D$ and $\Lambda^2 V$ isomorphic to $U\xi^{-1}$.

From (D, ξ) we have a section $s \in H^0(X, \mathcal{O}(D)) = H^0(X, \xi^2)$, defined up to a non-zero scalar, and one can construct the spectral curve associated to this section s. We denote this spectral curve by

$$X_{D,\xi}$$

(in Remark 4.10 below we give an explanation of this notation).

Given line bundles L and Λ, let

$$\mathcal{M}_L^\Lambda$$

denote the moduli space of L-twisted Higgs pairs of rank two, with fixed determinant Λ and with traceless Higgs field. In [17], we carry out a study of the singular fibre of the Hitchin map $\mathcal{H}$ defined in $\mathcal{M}_L^\Lambda$ for any L with positive degree and any Λ:

$$\begin{aligned} \mathcal{H} : \quad \mathcal{M}_L^\Lambda &\longrightarrow H^0(X, L^2) \\ (V, \varphi) &\longmapsto \det(\varphi). \end{aligned}$$

Moreover we have the following proposition, which is immediate from Lemma 4.6 and Proposition 4.8:

Proposition 4.9 *Let $(D, \xi) \in \mathcal{P}_X$. Then $h^{-1}(D, \xi) \in \mathcal{N}_{\alpha_m^-}(2, d)$ is isomorphic to $\mathcal{H}^{-1}(s) \in \mathcal{M}_\xi^{U\xi^{-1}}$, where $s \in H^0(X, \xi^2)$ is such that $\mathrm{div}(s) = D$.*

Remark 4.10 Recall that we made a choice of a section s associated to the divisor D and this choice induces a choice of the corresponding spectral curve, as explained in Sect. 4.1. However, the fibre of h does not depend of this choice, due to Lemma 4.6. In fact, if we had a different choice λs, for some $\lambda \in \mathbb{C}^*$, then we would be working on the spectral curve $X_{\lambda s, \xi} : x^2 + \pi^* \lambda s = 0$ and we would be working with ξ-Higgs pairs of the form $(V, \sqrt{\lambda}\varphi)$, where (V, φ) is a ξ-Higgs pair coming from $X_{s,\xi}$. But, although these two ξ-Higgs pairs are not isomorphic, the corresponding U-quadratic pairs (V, γ) and $(V, \sqrt{\lambda}\gamma)$ are isomorphic. This yields an isomorphism between the fibres of h using $X_{s,\xi}$ and $X_{\lambda s, \xi}$. This is the reason why we denote "the" spectral curve associated to (D, ξ) by $X_{D,\xi}$.

The study of $\mathcal{H}^{-1}(s)$ in [17] is done by considering the cases where $X_{s,\xi}$ is smooth, singular and irreducible, and singular and reducible. The smooth case is the generic one, and it is well known that the fibre $\mathcal{H}^{-1}(s)$ in that case is a torsor for the Prym variety of the spectral curve (cf. [1,18]). The case of singular and irreducible spectral curve is carried out by a careful study of the compactification of the Jacobian of the singular spectral curve associated to s, using the relation between this Jacobian with that of its desingularization. Finally, the study of $\mathcal{H}^{-1}(s)$ when $X_{s,\xi}$ is reducible is done by a direct analysis of the eigenbundles of φ.

Theorem 8.1 in [17] and Proposition 4.9 imply the following:

Theorem 4.11 *Let $(D, \xi) \in \mathcal{P}_X$. Then the fibre of $h : \mathcal{N}_{\alpha_m^-}(2, d) \to \mathcal{P}_X$ over (D, ξ) is connected and has dimension $d_U - d + g - 1$.*

5 Components of $\mathcal{N}_\alpha(2, d)$

From Theorem 4.11 and from the fact that $\mathcal{P}_X$ is connected and $\dim \mathcal{P}_X = 2d_U - 2d$, one concludes the following:

Theorem 5.1 *For every $d < d_U$, the space $\mathcal{N}_{\alpha_m^-}(2, d)$ is connected and has dimension $3(d_U - d) + g - 1$.*

Hence the dimension of $\mathcal{N}_{\alpha_m^-}(2, d)$ is the expected dimension given in (2.6).

Before stating our main result, we need one final lemma. In the following all spaces are assumed to be second countable and Hausdorff (and thus metrizable). Thus compactness is equivalent to sequential compactness.

Lemma 5.2 *Let $N^\pm$ be compact spaces and let $S^\pm \subset N^\pm$ be proper closed subspaces. Assume that $\overline{(N^\pm \smallsetminus S^\pm)} = N^\pm$ and that there is a homeomorphism $N^+ \smallsetminus S^+ \cong N^- \smallsetminus S^-$. If N^- and S^+ are connected, then so is N^+.*

Proof Let $U^\pm = N^\pm \smallsetminus S^\pm$. Then $U^\pm$ are non-compact.

Suppose now that N^+ is not connected. Then, since the closure of a connected set is connected, U^+ is not connected. Let $N^+ = N_1^+ \cup N_2^+$ be a decomposition into disjoint

non-empty closed subsets. Then $U_1^+ = U^+ \cap N_1^+$ and $U_2^+ = U^+ \cap N_2^+$ are disjoint non-empty open subsets of $U^+ \cong U^-$. By the connectedness of N^-, the intersections $\overline{U_i^+} \cap S^-$ are non-empty, where we are considering closures in N^-. As above, this implies that U_i^+ is non-compact for $i = 1, 2$. Considering now the closures in N^+, we have $\overline{U_i^+} = N_i^+$ and it follows that $N_i^+ \cap S^+$ is non-empty for $i = 1, 2$. This shows that S^+ is disconnected, a contradiction.   □

Now we reach our main result about the moduli of quadratic pairs.

Theorem 5.3 *Let d be such that $d_U - d > g - 1$. For every $\alpha \leqslant d/2$, the moduli space $\mathcal{N}_\alpha(2, d)$ is connected.*

Proof By Theorem 5.1, $\mathcal{N}_\alpha(2, d)$ is connected, for every $\alpha < \alpha_m$. We will see the flip loci described in Sect. 3.2 have sufficient high codimension so that they do not affect the number of components of adjacent moduli spaces.

Again by Theorem 5.1, $\mathcal{N}_{\alpha_m^-}(2, d)$ has dimension $3(d_U - d) + g - 1$.

From Corollary 3.11 we have

$$\dim \mathcal{S}_{\alpha_m^+}(2, d) = d_U - d + 2g - 2$$

hence, as $d_U - d > g - 1$, we have

$$\dim \mathcal{S}_{\alpha_m^+}(2, d) < \dim \mathcal{N}_{\alpha_m^-}(2, d). \tag{5.1}$$

On the other hand, from Corollary 3.24 every point in $\mathcal{S}_{\alpha_m^-}(2, d)$ is contained in a component whose dimension is less or equal than

$$3d_U - 4d + 2\alpha_m = 3d_U - 2[d_U/2] - 2d$$

hence,

$$\dim \mathcal{S}_{\alpha_m^-}(2, d) < \dim \mathcal{N}_{\alpha_m^-}(2, d). \tag{5.2}$$

Using (3.1), we conclude that $\dim \mathcal{N}_{\alpha_m^+}(2, d) = \dim \mathcal{N}_{\alpha_m^-}(2, d) = 3(d_U - d) + g - 1$.

Now, observe that (5.1) and (5.2) are valid for all critical value $\alpha_k < \alpha_M$ and not just α_m. Hence we conclude that, for all $\alpha < \alpha_M$,

$$\dim \mathcal{N}_\alpha(2, d) = 3(d_U - d) + g - 1. \tag{5.3}$$

So, for all $\alpha_k < \alpha_M = d/2$,

$$\mathrm{codim}\, \mathcal{S}_{\alpha_k^+}(2, d) = 2(d_U - d) - g + 1 > g - 1 \geqslant 1 \tag{5.4}$$

and, from Corollary 3.24, every point in $\mathcal{S}_{\alpha_k^-}(2, d)$ is contained in a component whose codimension is greater or equal than

$$d + g - 1 - 2\alpha_k > g - 1 \geqslant 1. \tag{5.5}$$

Recall that the flip loci measure the difference between two moduli spaces whose parameter lie on opposite sides of a critical value. From Theorem 5.1, (5.4) and (5.5), we see that the spaces $\mathcal{N}_{\alpha_k^\pm}(2, d)$ and $\mathcal{S}_{\alpha_k^\pm}$ satisfy the conditions of Lemma 5.2. From this it follows that $\mathcal{N}_\alpha(2, d)$ is connected for every generic α.

If $\alpha_k \neq \alpha_M$ is a critical value, we have two obvious continuous maps

$$\pi_\pm : \mathcal{N}_{\alpha_k^\pm}(2, d) \longrightarrow \mathcal{N}_{\alpha_k}(2, d).$$

From the definition of the flip loci

$$\mathcal{N}_{\alpha_k}(2, d) = \pi_-(\mathcal{N}_{\alpha_k^-}(2, d)) \cup \pi_+(\mathcal{N}_{\alpha_k^+}(2, d)).$$

From above, $\pi_-(\mathcal{N}_{\alpha_k^-}(2, d)) \cap \pi_+(\mathcal{N}_{\alpha_k^+}(2, d))$ is non-empty and the images of $\pi_\pm$ are connected. The conclusion is that $\mathcal{N}_{\alpha_k}(2, d)$ is also connected. $\qquad\square$

6 An application to surface group representations

6.1 Higgs bundles

Let $H \subset G$ be a maximal compact subgroup, and let $H^{\mathbb{C}}$ be the complexification of H. The Cartan decomposition, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, of $\mathfrak{g}$, yields a decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$ of the corresponding complexified Lie algebra. Then $\mathfrak{m}^{\mathbb{C}}$ is a representation of $H^{\mathbb{C}}$ via the *isotropy representation*

$$\iota : H^{\mathbb{C}} \longrightarrow \mathrm{Aut}(\mathfrak{m}^{\mathbb{C}}) \tag{6.1}$$

obtained by restricting the adjoint representation of $G^{\mathbb{C}}$ on $\mathfrak{g}^{\mathbb{C}}$. If $E_{H^{\mathbb{C}}}$ is a principal $H^{\mathbb{C}}$-bundle over X, we denote by $E_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}) = E \times_{H^{\mathbb{C}}} \mathfrak{m}^{\mathbb{C}}$ the vector bundle, with fibre $\mathfrak{m}^{\mathbb{C}}$, associated to the isotropy representation. Let $K = T^*X^{1,0}$ be the canonical line bundle of X.

Definition 6.1 A G-*Higgs bundle* over a compact Riemann surface X is a pair $(E_{H^{\mathbb{C}}}, \varphi)$ where $E_{H^{\mathbb{C}}}$ is a principal holomorphic $H^{\mathbb{C}}$-bundle over X and φ is a global holomorphic section of $E_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}) \otimes K$, called the *Higgs field*.

A G-Higgs bundle $(E_{H^{\mathbb{C}}}, \varphi)$ is topologically classified by the topological invariant of the corresponding $H^{\mathbb{C}}$-bundle $E_{H^{\mathbb{C}}}$ and, as the maximal compact subgroup of $H^{\mathbb{C}}$ is H, the topological classification of G-Higgs bundles is the same as the one of H-principal bundles. Thus, whenever G is connected, the topological class of a G-Higgs bundle is given by an element in $H^2(X, \pi_1 H) \cong \pi_1 H$.

In [11], a general notion of (semi,poly)stability of G-Higgs bundles was developed, allowing for proving a Hitchin–Kobayashi correspondence between polystable G-Higgs bundles and solutions to certain gauge theoretic equations known as Hitchin's equations. On the other hand, Schmitt [26–28] introduced stability conditions for *decorated bundles* and used these in his general Geometric Invariant Theory construction of moduli spaces. In particular, Schmitt's constructions give moduli of G-Higgs bundles for the groups considered in this paper, and his stability conditions coincide with the ones relevant for the Hitchin–Kobayashi correspondence. It should be noted that the stability conditions depend on a parameter $\alpha \in \sqrt{-1}\mathfrak{h} \cap \mathfrak{z}$, where $\mathfrak{z}$ is the centre of $\mathfrak{h}^{\mathbb{C}}$. We denote by

$$\mathcal{M}_d(X, G)$$

the moduli space of semistable (for the parameter value $\alpha = 0$) G-Higgs bundles with topological invariant $d \in \pi_1 H$. As usual, the moduli space $\mathcal{M}_d(X, G)$ can also be viewed as parametrizing isomorphism classes of polystable G-Higgs bundles.

6.2 Higgs bundles for the adjoint form of the symplectic group

Let $\mathrm{Sp}(2n, \mathbb{R})$ be the real symplectic group of linear automorphisms of $\mathbb{R}^{2n}$ which preserve the standard symplectic form. The centre of $\mathrm{Sp}(2n, \mathbb{R})$ is $\mathbb{Z}/2$ and we denote by

$\mathrm{PSp}(2n, \mathbb{R}) = \mathrm{Sp}(2n, \mathbb{R})/(\mathbb{Z}/2)$ the projectivization of $\mathrm{Sp}(2n, \mathbb{R})$. A maximal compact subgroup of $\mathrm{PSp}(2n, \mathbb{R})$ is isomorphic to $\mathrm{U}(n)/(\mathbb{Z}/2)$, so the Cartan decomposition for $\mathfrak{psp}(2n, \mathbb{C}) = \mathfrak{sp}(2n, \mathbb{C})$ is given by $\mathfrak{sp}(2n, \mathbb{C}) = \mathfrak{gl}(n, \mathbb{C}) \oplus \mathfrak{m}^{\mathbb{C}}$ where

$$\mathfrak{m}^{\mathbb{C}} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \mid B, C \in \mathfrak{gl}(n, \mathbb{C}),\ B^T = B,\ C^T = C \right\} \cong S^2\mathbb{C}^n \oplus S^2(\mathbb{C}^n)^*. \quad (6.2)$$

Hence a $\mathrm{PSp}(2n, \mathbb{R})$-*Higgs bundle* over a compact Riemann surface X is a pair (E, φ), where E is a holomorphic principal $\mathrm{GL}(n, \mathbb{C})/(\mathbb{Z}/2)$-principal bundle and φ is a holomorphic global section of the vector bundle $E \times_{\mathrm{GL}(n,\mathbb{C})/(\mathbb{Z}/2)} (S^2\mathbb{C}^n \oplus S^2(\mathbb{C}^n)^*) \otimes K$.

We want to work with holomorphic vector bundles, so we shall use a very similar procedure to the one taken in [24] for $G = \mathrm{PGL}(n, \mathbb{R})$. Consider the group $\mathrm{Sp}(2n, \mathbb{R}) \times \mathrm{U}(1)$, the normal subgroup $\{(I_n, 1), (-I_n, -1)\} \cong \mathbb{Z}/2 \lhd \mathrm{GL}(n, \mathbb{R}) \times \mathrm{U}(1)$ and the corresponding quotient group

$$\mathrm{Sp}(2n, \mathbb{R}) \times_{\mathbb{Z}/2} \mathrm{U}(1) = (\mathrm{Sp}(2n, \mathbb{R}) \times \mathrm{U}(1))/(\mathbb{Z}/2).$$

Notation 6.2 We shall write

$$\mathrm{ESp}(2n, \mathbb{R}) = \mathrm{Sp}(2n, \mathbb{R}) \times_{\mathbb{Z}/2} \mathrm{U}(1),$$
$$\mathrm{EU}(n) = \mathrm{U}(n) \times_{\mathbb{Z}/2} \mathrm{U}(1),$$
$$\mathrm{EGL}(n, \mathbb{C}) = \mathrm{GL}(n, \mathbb{C}) \times_{\mathbb{Z}/2} \mathbb{C}^*.$$

The "E" stands for enhanced or extended.

The complexification of the maximal compact subgroup $\overline{H} = \mathrm{EU}(n) \subset \mathrm{ESp}(2n, \mathbb{R})$ is $\overline{H}^{\mathbb{C}} = \mathrm{EGL}(n, \mathbb{C})$. Also, $\overline{\mathfrak{g}}^{\mathbb{C}} = \overline{\mathfrak{h}}^{\mathbb{C}} \oplus \overline{\mathfrak{m}}^{\mathbb{C}}$ where $\overline{\mathfrak{g}}^{\mathbb{C}} = \mathfrak{sp}(2n, \mathbb{C}) \oplus \mathbb{C}$, $\overline{\mathfrak{h}}^{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C}) \oplus \mathbb{C}$ and $\overline{\mathfrak{m}}^{\mathbb{C}} = \mathfrak{m}^{\mathbb{C}} \oplus \{0\} \cong \mathfrak{m}^{\mathbb{C}}$, where $\mathfrak{m}^{\mathbb{C}}$ is given by (6.2), so

$$\overline{\mathfrak{m}}^{\mathbb{C}} \cong \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \mid B, C \in \mathfrak{gl}(n, \mathbb{C}),\ B^T = B,\ C^T = C \right\}. \quad (6.3)$$

Definition 6.3 An $\mathrm{ESp}(2n, \mathbb{R})$-*Higgs bundle* over X is a pair (E, φ), where E is a holomorphic principal $\mathrm{EGL}(n, \mathbb{C})$-bundle and $\varphi \in H^0(X, E \times_{\mathrm{EGL}(n,\mathbb{C})} \overline{\mathfrak{m}}^{\mathbb{C}} \otimes K)$, where $\overline{\mathfrak{m}}^{\mathbb{C}}$ is given by (6.3).

Consider the actions of $\mathrm{EGL}(n, \mathbb{C})$ on $\mathbb{C}^n$ and on $\mathbb{C}$ induced, respectively, by the group homomorphisms

$$\mathrm{EGL}(n, \mathbb{C}) \longrightarrow \mathrm{GL}(n, \mathbb{C}), \qquad [(w, \lambda)] \mapsto \lambda w \quad (6.4)$$

and

$$\mathrm{EGL}(n, \mathbb{C}) \longrightarrow \mathbb{C}^*, \qquad [(w, \lambda)] \mapsto \lambda^2. \quad (6.5)$$

Note that together these two actions define an isomorphism

$$\mathrm{EGL}(n, \mathbb{C}) \overset{\cong}{\to} \mathrm{GL}(n, \mathbb{C}) \times \mathbb{C}^*$$
$$[(w, \lambda)] \mapsto (\lambda w, \lambda^2). \quad (6.6)$$

We have the following description of an $\mathrm{ESp}(2n, \mathbb{R})$-Higgs bundle in terms of vector bundles:

Proposition 6.4 *Let $(\overline{E}, \overline{\varphi})$ be an ESp$(2n, \mathbb{R})$-Higgs bundle on X. Through the actions (6.4) and (6.5) of EGL$(n, \mathbb{C})$ on $\mathbb{C}^n$ and on $\mathbb{C}$, associated to $(\overline{E}, \overline{\varphi})$ there is a quadruple (V, L, β, γ), where V is a rank n holomorphic vector bundle, L is a holomorphic line bundle and $(\beta, \gamma) \in H^0(X, (S^2 V \otimes L^{-1} \oplus S^2 V^* \otimes L) \otimes K)$.*

Moreover, two ESp$(2n, \mathbb{R})$-Higgs bundles $(\overline{E}_\nu, \overline{\varphi}_\nu)$, $\nu = 1, 2$, are isomorphic if and only if and only if the corresponding quadruples $(V_\nu, L_\nu, \beta_\nu, \gamma_\nu)$ are isomorphic, i.e., there are isomorphisms $V_1 \cong V_2$ and $L_1 \cong L_2$ intertwining (β_1, γ_1) and (β_2, γ_2).

Proof From the actions (6.4) and (6.5) we define, respectively, the vector bundle $V = \overline{E} \times_{\mathrm{EGL}(n, \mathbb{C})} \mathbb{C}^n$ and the line bundle $L = \overline{E} \times_{\mathrm{EGL}(n, \mathbb{C})} \mathbb{C}$.

Consider the representations $\sigma : \mathrm{EGL}(n, \mathbb{C}) \to \mathrm{GL}(S^2 \mathbb{C}^n)$ and $\sigma^* : \mathrm{EGL}(n, \mathbb{C}) \to \mathrm{GL}(S^2 (\mathbb{C}^n)^*)$ given by

$$\sigma([w, \lambda])(B) = w B w^T$$

and

$$\sigma^*([w, \lambda])(C) = (w^T)^{-1} C w^{-1}.$$

If $\overline{\iota} : \mathrm{EGL}(n, \mathbb{C}) \to \mathrm{GL}(\overline{\mathfrak{m}}^{\mathbb{C}})$ is the isotropy representation of EGL$(n, \mathbb{C})$ on $\overline{\mathfrak{m}}^{\mathbb{C}}$, then it is clear that $\overline{\iota}([(w, \lambda)])(A) = \iota([w])(A)$, where ι is the isotropy representation of $\mathrm{GL}(n, \mathbb{C})/(\mathbb{Z}/2)$ in $\mathfrak{m}^{\mathbb{C}}$. It is easy to see that

$$\overline{\iota} = \sigma \oplus \sigma^*$$

hence, taking into account the actions (6.4) and (6.5), from σ we obtain the vector bundle $S^2 V \otimes L^{-1}$ and from σ^* the vector bundle $S^2 V^* \otimes L$. The Higgs field $\overline{\varphi} \in H^0(X, \overline{E} \times_{\overline{H}^{\mathbb{C}}} \overline{\mathfrak{m}}^{\mathbb{C}} \otimes K)$ is therefore given, in terms of V and L by two sections:

$$\beta \in H^0(X, S^2 V \otimes L^{-1} K) \quad \text{and} \quad \gamma \in H^0(X, S^2 V^* \otimes L K).$$

The final statement about isomorphism of quadruples follows from the isomorphism (6.6). □

We shall slightly abuse notation and also call a quadruple (V, L, β, γ) as introduced in the preceding proposition an ESp$(2n, \mathbb{R})$-Higgs bundles.

Remark 6.5 An ESp$(2n, \mathbb{R})$-Higgs bundle (V, L, β, γ) with $L = \mathcal{O}$ is the same thing as an Sp$(2n, \mathbb{R})$-Higgs bundle (V, β, γ)(cf. [11]).

Projection on the first factor gives a homomorphism

$$\mathrm{ESp}(2n, \mathbb{R}) \longrightarrow \mathrm{PSp}(2n, \mathbb{R})$$

and so, to any ESp$(2n, \mathbb{R})$-Higgs bundle, we can naturally associate a PSp$(2n, \mathbb{R})$-Higgs bundle. Note that this association is given by extension of structure group in the principal bundles via the map $\mathrm{EGL}(n, \mathbb{C}) \xrightarrow{p} \mathrm{GL}(n, \mathbb{C})/(\mathbb{Z}/2)$, where $p([(w, \lambda)]) = [w]$ and that the Higgs fields β and γ are unchanged since the map p intertwines the identity map between the respective isotropy representations (6.3) and (6.2).

The following result is very similar to Proposition 5.4 of [24], so we omit the proof.

Proposition 6.6 *Every PSp$(2n, \mathbb{R})$-Higgs bundle (E, φ) on X lifts to an ESp$(2n, \mathbb{R})$-Higgs bundle $(\overline{E}, \varphi)$.*

Proposition 6.7 *Two* $\mathrm{ESp}(2n, \mathbb{R})$*-Higgs bundle* $(V_\nu, L_\nu, \beta_\nu, \gamma_\nu)$, $\nu = 1, 2$ *give rise to isomorphic* $\mathrm{PSp}(2n, \mathbb{R})$*-Higgs bundles if and only if there is a line bundle* M *on* X *such that the* $\mathrm{ESp}(2n, \mathbb{R})$*-Higgs bundles* $(V_1, L_1, \beta_1, \gamma_1)$ *and* $(V_2 \otimes M, L_2 \otimes M^2, \beta_2, \gamma_2)$ *are isomorphic.*

Proof The correspondence with isomorphism of the underlying bundles is immediate from their definition. The complete statement including the Higgs fields follows because the Higgs fields are unchanged under the correspondence between $\mathrm{ESp}(2n, \mathbb{R})$ and $\mathrm{PSp}(2n, \mathbb{R})$-Higgs bundles. □

In view of Propositions 6.6 and 6.7 we can now work interchangeably with either isomorphism classes of $\mathrm{PSp}(2n, \mathbb{R})$-Higgs bundles or with equivalence classes of $\mathrm{ESp}(2n, \mathbb{R})$-Higgs bundles under the equivalence relation introduced in the latter Proposition. Thus we have the following immediate corollaries (analogous, respectively, to Proposition 5.3 and Corollary 5.1 of [24]).

Corollary 6.8 *Given a* $\mathrm{PSp}(2n, \mathbb{R})$*-Higgs bundle* (E, φ)*, it is possible to choose a lift of* (E, φ) *to an* $\mathrm{ESp}(2n, \mathbb{R})$*-Higgs bundle* (V, L, β, γ) *such that* L *is trivial or* $\deg(L) = 1$.

Corollary 6.9 *Let* (E, φ) *be a* $\mathrm{PSp}(2n, \mathbb{R})$*-Higgs bundle and* (V, L, β, γ) *be an* $\mathrm{ESp}(2n, \mathbb{R})$*-Higgs bundle which is a lift of* (E, φ). *Then* (E, φ) *lifts to an* $\mathrm{Sp}(2n, \mathbb{R})$*-Higgs bundle if and only if* $\deg(L)$ *is even.*

Next we give the topological classification of $\mathrm{PSp}(2n, \mathbb{R})$ and $\mathrm{ESp}(2n, \mathbb{R})$ bundles. Restriction of the isomorphism (6.6) gives an isomorphism

$$\epsilon : \mathrm{EU}(n) \xrightarrow{\cong} \mathrm{U}(n) \times \mathrm{U}(1)$$
$$[(w, \lambda)] \mapsto (\lambda w, \lambda^2). \tag{6.7}$$

Hence (using the standard identification $\pi_1 \mathrm{U}(n) \cong \mathbb{Z}$)

$$\pi_1 \mathrm{EU}(n) \cong \mathbb{Z} \times \mathbb{Z}. \tag{6.8}$$

This means that $\mathrm{ESp}(2n, \mathbb{R})$-Higgs bundles are classified by a pair of integers and, thinking of an $\mathrm{ESp}(2n, \mathbb{R})$-Higgs bundle as a quadruple (V, L, β, γ), we see from Proposition 6.4 that this pair can be identified with

$$(\deg(V), \deg(L)) \in \mathbb{Z} \times \mathbb{Z}.$$

Using the identification (6.7), the natural projection $\mathrm{EU}(n, \mathbb{R}) \to \mathrm{U}(n)/(\mathbb{Z}/2)$ takes the form

$$\mathrm{U}(n) \times \mathrm{U}(1) \xrightarrow{\epsilon} \mathrm{EU}(n) \to \mathrm{U}(n)/(\mathbb{Z}/2),$$
$$(g, \mu) \mapsto \left[\frac{g}{\sqrt{\mu}}\right].$$

Thus we have a short exact sequence

$$1 \to \mathrm{U}(1) \to \mathrm{U}(n) \times \mathrm{U}(1) \to \mathrm{U}(n)/(\mathbb{Z}/2) \to 1.$$

Again using the standard identification $\pi_1 \mathrm{U}(n) \cong \mathbb{Z}$, the associated homotopy sequence gives

$$1 \to \mathbb{Z} \xrightarrow{(n\cdot, 2\cdot)} \mathbb{Z} \times \mathbb{Z} \to \pi_1\big(\mathrm{U}(n)/(\mathbb{Z}/2)\big) \to 1.$$

It follows that,

$$\pi_1\big(U(n)/(\mathbb{Z}/2)\big) \cong \begin{cases} \mathbb{Z} \times \mathbb{Z}/2 & \text{for } n \text{ even,} \\ \mathbb{Z} & \text{for } n \text{ odd.} \end{cases} \qquad (6.9)$$

In particular, the composition $U(n) \hookrightarrow EU(n) \xrightarrow{\epsilon} U(n) \times U(1) \to U(n)/(\mathbb{Z}/2)$ induces an isomorphism of $\pi_1 U(n)$ onto the $\mathbb{Z}$-factor in $\pi_1\big(U(n)/(\mathbb{Z}/2)\big)$. For n even and (V, L) with $(\deg(V), \deg(L)) = (d_1, d_2)$, the projection $\pi_1 EU(n) \to \pi_1(U(n)/(\mathbb{Z}/2))$ can then easily be calculated to be

$$(d_1, d_2) \mapsto (d_1 - nd_2/2, d_2 \mod 2), \qquad (6.10)$$

in terms of the identifications (6.8) and (6.9). Thus, from the point of view of $PSp(2n, \mathbb{R})$-Higgs bundles it is more natural to work with the following invariants when n is even:

$$(a, b)(V, L, \beta, \gamma) = (\deg(V) - n\deg(L)/2, \deg(L)). \qquad (6.11)$$

In terms of these

$$\deg(V) = a + nb/2 \quad \text{and} \quad \deg(L) = b.$$

Notice that for a line bundle F we have

$$(a, b)(V, L, \beta, \gamma) = (a, b)(V \otimes F, LF^2, \beta, \gamma)$$

which is consistent with Proposition 6.7.

Remark 6.10 From either point of view, we see that the obstruction to lifting a $PSp(2n, \mathbb{R})$-Higgs bundle to an $Sp(2n, \mathbb{R})$-Higgs bundle is given by the invariant $d_2 = b$ (cf. Remark 6.5 and Corollary 6.9).

Remark 6.11 We remark that $PSp(4, \mathbb{R})$ is isomorphic to the group $SO_0(2, 3)$ (i.e., the connected component of the identity of $SO(2, 3)$). As explained in [7], an $SO_0(2, 3)$-Higgs bundle is given by the the data $(W, Q_W, F, \beta, \gamma)$ where F is a line bundle, (W, Q_W) is a rank 3 vector bundle equipped with a non-degenerate F^2-valued quadratic form, β is a section of $\mathrm{Hom}(W, F) \otimes K$ and γ a section of $\mathrm{Hom}(W, F^{-1}) \otimes K$. The objects are classified by two invariants, namely the degree of F (which is actually the Toledo invariant: see (6.12) below) and the second Stiefel-Whitney class $w_2(W, Q_W) \in \mathbb{Z}/2$.

Generalizing the construction of an $SO_0(2, 3)$-Higgs bundle from an $Sp(4, \mathbb{R})$-Higgs bundle given in [7, Section 3.3], we can obtain an $SO_0(2, 3)$-Higgs bundle from an $ESp(4, \mathbb{R})$-Higgs bundle, as follows: if (V, L, β, γ) is an $ESp(4, \mathbb{R})$-Higgs bundle, define

$$W = S^2 V \otimes \Lambda^2 V^{-1}$$

and

$$F = \Lambda^2 V \otimes L^{-1}.$$

If $Q(x \otimes y, x' \otimes y') = (x \wedge x') \otimes (y \wedge y')$, then Q is an $F^2 L^2$-valued quadratic form on $S^2 V$, hence W has the induced non-degenerate quadratic form Q_W. Moreover, γ is a section of $\mathrm{Hom}(W, F^{-1}) \otimes K$ and, since $W \cong W^*$, we can view β as a section of $\mathrm{Hom}(W, F) \otimes K$. Hence we have obtained the $SO_0(2, 3)$-Higgs bundle $(W, Q_W, F, \beta, \gamma)$. It is easily checked that the invariants (a, b) of the $ESp(4, \mathbb{R})$-Higgs bundle (V, L, β, γ) are given by

$$\begin{aligned} a &= \deg(L), \\ b &= \deg(L) + w_2(W, Q_W) \mod 2. \end{aligned} \qquad (6.12)$$

6.3 Stability, moduli spaces and the non-abelian Hodge theorem

In [11], a general notion of (semi,poly)stability for G-Higgs bundles was introduced and a Hitchin–Kobayashi correspondence was established showing that polystability of a G-Higgs bundles is equivalent to the existence of a solution to certain gauge theoretic equations, known as the Hitchin equations. The general definition of stability is fairly involved but in many examples it can be significantly simplified. In the case of $G = \mathrm{ESp}(2n, \mathbb{R})$ a simplification can be carried out in a manner entirely analogous to the case of $G = \mathrm{Sp}(2n, \mathbb{R})$ studied in [11, Section 4] and the stability condition then takes the following form.

Proposition 6.12 *An* $\mathrm{ESp}(2n, \mathbb{R})$-*Higgs bundle* (V, L, β, γ) *is*

- *semistable if and only if for any filtration of holomorphic subbundles* $0 \subset V_1 \subset V_2 \subset V$ *such that*

$$(\beta, \gamma) \in H^0(X, (S^2 V_2 + V_1 \otimes_S V) \otimes L^{-1}K \oplus (S^2 V_1^\perp + V_2^\perp \otimes_S V^*) \otimes LK),$$

we have

$$\deg(V_1) + \deg(V_2) \leqslant \deg(V).$$

- *stable if and only if for any filtration of holomorphic subbundles* $0 \subset V_1 \subset V_2 \subset V$ *such that*

$$(\beta, \gamma) \in H^0(X, (S^2 V_2 + V_1 \otimes_S V) \otimes L^{-1}K \oplus (S^2 V_1^\perp + V_2^\perp \otimes_S V^*) \otimes LK),$$

the following holds: if at least one of the subbundles V_1 *or* V_2 *is proper, then*

$$\deg(V_1) + \deg(V_2) < \deg(V)$$

and in any other case,

$$\deg(V_1) + \deg(V_2) \leqslant \deg(V).$$

Remark 6.13 (1)   The general notion of semistability of G-Higgs bundles depends on a parameter $\alpha \in \sqrt{-1}\mathfrak{h} \cap \mathfrak{z}$ where $\mathfrak{z}$ is the centre of $\mathfrak{h}^{\mathbb{C}}$. For $G = \mathrm{ESp}(2n, \mathbb{R})$, we have $\sqrt{-1}\mathfrak{h} \cap \mathfrak{z} = \sqrt{-1}(\mathfrak{u}(1) \oplus \mathfrak{u}(1)) = \mathbb{R} \times \mathbb{R}$, so there is a (α_1, α_2)-semistability condition. This is very similar to the case of $G = \mathrm{Sp}(2n, \mathbb{R})$. However, if (V, L, β, γ) is an $\mathrm{ESp}(2n, \mathbb{R})$-Higgs bundle, it can be seen that if $\alpha_2 \neq \deg(L)$, then it is (α_1, α_2)-unstable. Therefore we are considering $\alpha_2 = \deg(L)$ fixed. Also, in the preceding theorem, we have restricted ourselves to the case of $\alpha_1 = 0$ semistability, because it is for this value of the parameter that the fundamental correspondence between moduli spaces G-Higgs bundles and G-character varieties (Theorem 6.15 below) holds.

(2)   The notion of polystablity of $\mathrm{ESp}(2n, \mathbb{R})$-Higgs bundles is analogous to the one in Proposition 4.16 of [11], with the obvious modifications.

With regard to the relation between the stability conditions for $\mathrm{PSp}(2n, \mathbb{R})$-Higgs bundles and $\mathrm{ESp}(2n, \mathbb{R})$-Higgs bundles we have the following result.

Proposition 6.14 *An* $\mathrm{ESp}(2n, \mathbb{R})$-*Higgs bundle* (V, L, β, γ) *is polystable if and only if the associated* $\mathrm{PSp}(2n, \mathbb{R})$-*Higgs bundle is polystable.*

Proof This can be checked by specializing and comparing directly the general polystability conditions given in [11]. An alternative proof can be given by invoking the Hitchin Kobayashi correspondence proved in that paper, since the existence of solutions to the Hitchin equations on an $\mathrm{ESp}(2n, \mathbb{R})$-Higgs bundle is clearly equivalent to the existence of solutions on the corresponding $\mathrm{PSp}(2n, \mathbb{R})$-Higgs bundle.   □

Next we recall the non-abelian Hodge theory correspondence. Let G be a connected semi-simple real Lie group with maximal compact subgroup $H \subseteq G$. By a *representation* of $\pi_1 X$ in G we mean a homomorphism $\rho \colon \pi_1 X \to G$. A representation ρ is *reductive* if its composition with the adjoint representation of G on $\mathfrak{g}$ is a completely reducible representation. The *character variety* for representations of $\pi_1 X$ in G is

$$\mathcal{R}(\pi_1 X, G) = \mathrm{Hom}^{\mathrm{red}}(\pi_1 X, G)/G,$$

where G acts by overall conjugation on homomorphisms. Any representation ρ has a topological invariant $c(\rho) \in \pi_1 H$ defined as the topological class of the associated flat bundle. Let

$$\mathcal{R}_d(\pi_1 X, G) \subset \mathcal{R}(\pi_1 X, G)$$

be the subspace of equivalence classes of representations whose topological invariant is $c(\rho) = d$.

The non-abelian Hodge Theorem ([9–11, 19, 29]) now states the following.

Theorem 6.15 *There is a homeomorphism*

$$\mathcal{M}_d(X, G) \cong \mathcal{R}_d(\pi_1 X, G).$$

6.4 Bounds on invariants

From now on we restrict to the case of n being even. Let

$$\mathcal{M}_{d_1, d_2} = \mathcal{M}_{d_1, d_2}(X, \mathrm{ESp}(2n, \mathbb{R}))$$

the moduli space of polystable $\mathrm{ESp}(2n, \mathbb{R})$-Higgs bundles (V, L, β, γ) with $\deg(V) = d_1$ and $\deg(L) = d_2$. Let also

$$\widehat{\mathcal{M}}_{a,b} = \mathcal{M}_{a,b}(X, \mathrm{PSp}(2n, \mathbb{R})),$$

the moduli space of polystable $\mathrm{PSp}(2n, \mathbb{R})$-Higgs bundles with topological invariants $(a, b) \in \mathbb{Z} \times \mathbb{Z}/2$.

For a semistable $\mathrm{ESp}(2n, \mathbb{R})$-Higgs bundle (V, L, β, γ) with $\deg(V) = d_1$ and $\deg(L) = d_2$, we have a Milnor-Wood inequality (a Higgs bundle proof of this inequality can be easily given, cf. [5] for the case $G = \mathrm{U}(p, q)$ which implies the result in the current setting):

$$n(1 - g) + d_2 \leqslant d_1 \leqslant n(g - 1) + d_2. \tag{6.13}$$

This is equivalent to

$$|a| \leqslant n(g - 1)$$

where $a = d_1 - n d_2/2 \in \mathbb{Z}$ is the invariant introduced in (6.11). In the context of surface group representations the invariant a is the *Toledo invariant*.

Furthermore, for $(d_1, d_2) \in \mathbb{Z} \times \mathbb{Z}$ such that (6.13) holds, we have the isomorphism

$$\mathcal{M}_{d_1, d_2} \cong \mathcal{M}_{n d_2 - d_1, d_2}$$

given by $(V, L, \beta, \gamma) \mapsto (V^* \otimes L, L, \gamma^t \otimes 1_K, \beta^t \otimes 1_K)$ and the induced isomorphism

$$\widehat{\mathcal{M}}_{a,b} \cong \widehat{\mathcal{M}}_{-a,b}.$$

We can, therefore, assume that

$$0 \leqslant a \leqslant n(g - 1). \tag{6.14}$$

Let now L_0 be a fixed line bundle of degree 1 over X. Denote by

$$\mathcal{M}_{d,L_0} \subset \mathcal{M}_{d,1}$$

be the subspace of $\mathrm{ESp}(4, \mathbb{R})$-Higgs bundles (V, L, β, γ) with $L = L_0$. Similarly, let

$$\mathcal{M}_{d,\mathcal{O}} \subset \mathcal{M}_{d,0}$$

be the subspace of $\mathrm{ESp}(4, \mathbb{R})$-Higgs bundles (V, L, β, γ) with L isomorphic to the trivial line bundle $\mathcal{O}$.

From Corollary 6.8 and (6.10) the following is clear:

Proposition 6.16 *Let d and d' be two integers satisfying $0 \leqslant d \leqslant n(g-1)$ and $1 \leqslant d' \leqslant n(g-1) + 1$. Let $[(V, L, \beta, \gamma)]$ denote the class of the corresponding $\mathrm{PSp}(4, \mathbb{R})$-Higgs bundle under the equivalence relation given in Proposition 6.7. Then the projection $(V, L, \beta, \gamma) \mapsto [(V, L, \beta, \gamma)]$ yields a continuous surjective map*

$$\mathcal{M}_{d,\mathcal{O}} \sqcup \mathcal{M}_{d',L_0} \longrightarrow \widehat{\mathcal{M}}_{d,0} \sqcup \widehat{\mathcal{M}}_{d'-1,1}$$

preserving the decompositions.

6.5 Relation with quadratic pairs and connectedness theorems

For the remainder of the paper we specialize to the case $n = 2$, i.e., $G = \mathrm{ESp}(4, \mathbb{R})$ or $G = \mathrm{PSp}(4, \mathbb{R})$. Our goal is to count the number of connected components of $\widehat{\mathcal{M}}_{a,b}$ for $(a, b) \in \mathbb{Z} \times \mathbb{Z}/2$ such that $0 < |a| < 2g - 2$. The situation for $|a| = 0$ and $|a| = 2g - 2$ is somewhat special and, at any rate, in these cases the count follows from the results of [16] and [7]. Note also that the count for $b = 0$ (corresponding to $\mathrm{PSp}(4, \mathbb{R})$-Higgs bundles which lift to $\mathrm{Sp}(4, \mathbb{R})$-Higgs bundles) follows from the results of García-Prada and Mundet [14].

We will analyze the spaces $\mathcal{M}_{d,0}$ and $\mathcal{M}_{d,L_0}$ and from that draw our conclusions about $\widehat{\mathcal{M}}_{a,b}$, using Proposition 6.16. Let us deal first with $\mathcal{M}_{d,L_0}$, with

$$1 < d < 2g - 1.$$

We introduce the following *Hitchin functional*. It is defined as

$$f : \mathcal{M}_{d,L_0} \longrightarrow \mathbb{R}$$
$$(V, L_0, \beta, \gamma) \longmapsto \|\beta\|_{L^2}^2 + \|\gamma\|_{L^2}^2$$

Remark 6.17 The definition of the Hitchin functional uses a *harmonic metric* on V coming from the Hitchin-Kobayashi correspondence—see [19].

Using the Hitchin equations and Uhlenbeck's weak compactness theorem it follows that the Hitchin functional is proper (see Hitchin [19, Proposition 7.1]). The following is an elementary consequence of properness of the non-negative function f (cf. [19] or Proposition 4.3 of [5]).

Proposition 6.18 *The space $\mathcal{M}_{d,L_0}$ is connected if the subspace of local minima of the Hitchin proper function is connected.*

Using very similar methods to the ones of [12,14,16] for $G = \mathrm{Sp}(2n, \mathbb{R})$, one can prove the following result:

Proposition 6.19 *Let (V, L_0, β, γ) represent a point in $\mathcal{M}_{d,L_0}$, with $1 < d < 2g - 1$. Then (V, L_0, β, γ) is a minimum of f if and only if $\beta = 0$.*

We have the following immediate corollary.

Proposition 6.20 *For any integer* $1 < d < 2g - 1$, *the subvariety of local minima of* f *is the moduli space*

$$\mathcal{N}_{d,L_0}$$

of semistable $\mathrm{ESp}(4, \mathbb{R})$*-Higgs bundles* $(V, L_0, 0, \gamma)$ *such that* V *is a rank* 2 *holomorphic vector bundle of degree* d *and* $\gamma \in H^0(X, S^2V^* \otimes L_0K)$.

Now, we connect this with the study of quadratic pairs made in the first part of the paper. Let

$$\mathcal{N}_0(2, d)$$

be the moduli space of 0-semistable L_0K-quadratic pairs of type $(2, d)$.

Proposition 6.21 *The spaces* $\mathcal{N}_{d,L_0}$ *and* $\mathcal{N}_0(2, d)$ *are isomorphic.*

Proof In view of Proposition 6.20 the result follows by comparing the notions of 0-(semi,poly)stability given in Proposition 2.9 for quadratic pairs, and from the notion of (semi,poly)stability for $\mathrm{ESp}(4, \mathbb{R})$-Higgs bundles in Theorem 6.12. By considering all possible filtrations $0 \subset V_1 \subset V_2 \subset V$ of the rank 2 bundle V in Theorem 6.12, one easily checks that these notions coincide. □

Proposition 6.22 *For each integer* d *such that* $3 - 2g < d < 2g - 1$ *and* $d \neq 1$, *the space* $\mathcal{M}_{d,L_0}$ *is connected.*

Proof Recall that we can assume $1 < d < 2g - 1$. By Theorem 5.3 one has that $\mathcal{N}_0(2, d)$ is connected for every $1 < d < g$, hence, by Proposition 6.21, the same is valid for $\mathcal{N}_{d,L_0}$ for such d.

If $g \leqslant d < 2g - 1$, then $\mathcal{N}_0(2, d)$ corresponds to the case $\mathcal{N}_{\alpha_m^-}(2, d)$, because in this case the formula for α_m given in Notation 3.4, yields $\alpha_m = d - g + 1 > 0$. Hence, from Theorem 5.1, $\mathcal{N}_0(2, d)$ is connected, so Proposition 6.21, says that $\mathcal{N}_{d,L_0}$ is connected as well.

Now the result follows from Proposition 6.18. □

The connectedness of $\mathcal{M}_{d,\mathcal{O}}$ with $0 < d < 2g - 2$ was proved by García-Prada and Mundet (alternatively the argument used above to prove Proposition 6.22 could be applied to give a proof):

Proposition 6.23 ([14, Theorem 5]) *For each integer* d *such that* $0 < |d| < 2g - 2$, $\mathcal{M}_{d,\mathcal{O}}$ *is connected.*

We are now ready to state the theorem on the connectedness of the moduli space of $\mathrm{PSp}(4, \mathbb{R})$-Higgs bundles, with fixed topological classes.

Theorem 6.24 *For each* $(a, b) \in \mathbb{Z} \times \mathbb{Z}/2$ *such that* $0 < |a| < 2g - 2$, *the space* $\widehat{\mathcal{M}}_{a,b}$ *is connected.*

Proof Follows from Propositions 6.22, 6.23 and 6.16. □

Using the non-abelian Hodge theory correspondence of Theorem 6.15, we can rephrase our Theorem 6.24 as follows:

Theorem 6.25 *For each* $(a, b) \in \mathbb{Z} \times \mathbb{Z}/2$ *such that* $0 < |a| < 2g - 2$, *the space* $\mathcal{R}_{a,b}(\pi_1 X, \mathrm{PSp}(4, \mathbb{R}))$ *is connected.*

Recalling the correspondence of Remark 6.11, we can alternatively consider the character variety

$$\mathcal{R}_{a,w}(\pi_1 X, \mathrm{SO}_0(2, 3))$$

of representations of $\pi_1 X$ in $\mathrm{SO}_0(2, 3)$ with invariants $(a, w) \in \mathbb{Z} \times \mathbb{Z}/2$. We then have an identification

$$\mathcal{R}_{a,w}(\pi_1 X, \mathrm{SO}_0(2, 3)) = \mathcal{R}_{a,b}(\pi_1 X, \mathrm{PSp}(4, \mathbb{R}))$$

where the invariants are related by $(a, b) = (a, a + w \mod 2)$ (see (6.12)). (There is of course an analogous identification of the corresponding Higgs bundle spaces.) We thus have the following equivalent formulation of Theorem 6.25:

Theorem 6.26 *For each* $(a, w) \in \mathbb{Z} \times \mathbb{Z}/2$ *such that* $0 < |a| < 2g - 2$, *the space* $\mathcal{R}_{a,w}(\mathrm{SO}_0(2, 3))$ *is connected.*

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