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Flats and submersions in non-negative curvature

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Abstract We find constraints on the extent to which O'Neill's horizontal curvature equation can be used to create positive curvature on the base space of a Riemannian submersion. In particular, we study when K. Tapp's theorem on Riemannian submersions of compact Lie groups with bi-invariant metrics generalizes to arbitrary manifolds of non-negative curvature.

Keywords Riemannian submersions · Non-negative sectional curvature · Flat submanifolds · Isometric group actions

Mathematics Subject Classification (2000) 53C20

Until very recently all examples of compact, positively curved manifolds were constructed as the image of a Riemannian submersion of a Lie group with a bi-invariant metric [5,14,18]. Earlier constructions of positive curvature in [1–3], and [6–8] combined the fact that Lie groups with bi-invariant metrics are non-negatively curved with the so called Horizontal Curvature Equation,

$$\sec_B(x, y) = \sec_M(\tilde{x}, \tilde{y}) + 3\,|A_{\tilde{x}}\tilde{y}|^2$$

[9,17]. Here $\pi : M \to B$ is a Riemannian submersion, $\{x, y\}$ is an orthonormal basis for a plane in a tangent space to B, $\{\tilde{x}, \tilde{y}\}$ is a horizontal lift of $\{x, y\}$, and A is the "integrability tensor" for the horizontal distribution—that is,

$$A_{\tilde{x}}\tilde{y} \equiv \frac{1}{2}[\tilde{X}, \tilde{Y}]^{\mathrm{vert}}$$

where $\tilde{X}$ and $\tilde{Y}$ are arbitrary extensions of $\tilde{x}$ and $\tilde{y}$ to horizontal vector fields.

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Since the Horizontal Curvature Equation decomposes $\sec_B(x, y)$ into the sum of two non-negative quantities, we see immediately that Riemannian submersions preserve non-negative curvature. In addition, if *either* term on the right is positive, then $\sec_B(x, y) > 0$. Naively, one might expect positively curved examples to be constructed by exploiting the full power of the Horizontal Curvature Equation; however, a survey of the examples shows that this has never been done. In the context in which the examples in [1–3,6–8], and [22] were constructed, it is impossible for a Riemannian submersion to create positive curvature via the A-tensor alone. In fact, in [21] Tapp shows

Theorem 1 (Tapp) *Let $\pi : G \to B$ be a Riemannian submersion of a compact Lie group with a bi-invariant metric. Then*

1. *Every zero-curvature plane of B exponentiates to a flat (meaning a totally geodesic immersion of $\mathbb{R}^2$ with a flat metric), and*
2. *Every horizontal zero-curvature plane of G projects to a zero-curvature plane of B.*

In the case of bi-quotients of Lie groups, this is a consequence of an equation in [10]. This was first observed explicitly in [24].

Examples 2 and 3 (below) show that the theorem fails if the Lie group G is replaced by an arbitrary, compact, non-negatively curved Riemannian manifold M. The inhomogeneous metrics of these examples have zero-planes whose exponentials are locally, but not globally, flat.

Recall, if σ is a zero-curvature plane in a Lie group G with bi-invariant metric, then $\exp(\sigma)$ is a (globally) flat submanifold of G. So it is natural to ask about the extent to which Tapp's theorem holds if σ is assumed to be a horizontal zero-curvature plane whose exponential image is a flat submanifold of M. More formally, we pose:

Problem 1 If $\pi : M \to B$ is a Riemannian submersion of a compact, non-negatively curved manifold M and σ is a horizontal zero-curvature plane in M such that $\exp(\sigma)$ is a flat submanifold, does it follow that $\pi_*(\sigma)$ is a zero-curvature plane in B?

We emphasize that the given flat is not assumed to be globally horizontal.

The following easy consequence of Lemma 1.5 in [20] shows that an affirmative answer to our problem implies that both M and B have a lot of additional structure.

Theorem 2 *Let $\pi : M \to B$ be a Riemannian submersion of complete, non-negatively curved manifolds. Let σ be a zero-curvature plane in B and $\tilde{\sigma}$ a horizontal lift of σ so that $\exp(\tilde{\sigma})$ is a flat in M. Then*

1. *The plane σ exponentiates to a flat in B, and*
2. *Every horizontal lift of σ exponentiates to a horizontal flat in M.*

In Theorem 2, we do not require that M is compact; on the other hand, without compactness, the answer to Problem 1 is "no", even when M is a Lie group.

Example 1 Let $(\mathbb{R}^2, \bar{g})$ be the Cheeger deformation of $\mathbb{R}^2$ obtained from the standard S^1 action on $\mathbb{R}^2$. Let s and g be the usual metrics on S^1 and $\mathbb{R}^2$, respectively. Recall that $\bar{g}$ is defined so that the quotient map,

$$Q : (S^1 \times \mathbb{R}^2, s + g) \to (\mathbb{R}^2, \bar{g})$$

given by $Q(z, q) = \bar{z}q$ is a Riemannian submersion. This new metric is positively curved and is a paraboloid asymptotic to a cylinder of radius 1. All horizontal planes have zero curvature, but each projects to a positively curved plane. So positive curvature is created via the A-tensor alone.

Example 2 (Fish Bowl) Let $\psi : [0, \pi] \longrightarrow \mathbb{R}$ be a smooth, concave-down function that satisfies

$$\psi(t) = \begin{cases} t & \text{for } t \in \left[0, \frac{\pi}{4}\right] \\ \pi - t & \text{for } t \in \left[\frac{3\pi}{4}, \pi\right] \end{cases}$$

Consider the warped product metric

$$g_\psi = dt^2 + \psi^2 d\theta^2$$

on $S^2 = [0, \pi] \times_\psi S^1$. As before, S^1 acts isometrically on $\left(S^2, g_\psi\right)$, so we get a Riemannian submersion

$$\left(S^2, g_\psi\right) \times S^1 \longrightarrow \left(S^2, \bar{g}_\psi\right),$$

where $\bar{g}_\psi$ is the metric induced by the submersion. Notice that $\left(S^2, g_\psi\right) \times S^1$ is flat in a neighborhood of the set $\{0, \pi\} \times S^1$, but, as in Example 1, $\left(S^2, \bar{g}_\psi\right)$ is positively curved in the image of this neighborhood. If, in addition,

$$\psi''|_{\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)} < 0,$$

then $\left(S^2, \bar{g}_\psi\right)$ is positively curved. This shows that even in the compact case, the A-tensor can be responsible for creating positive curvature and that conclusion 2 of Tapp's theorem fails for arbitrary Riemannian submersions of compact, non-negatively curved manifolds.

Example 3 To see how conclusion 1 of Tapp's theorem can fail to hold, choose ψ in the previous example to be constant in a neighborhood of $\pi/2$. This makes $\left(S^2, g_\psi\right)$ isometric to a flat cylinder near a neighborhood of the equator. In the Cheeger deformed metric, the image of this region is a smaller flat cylinder. Since the base, $\left(S^2, \bar{g}_\psi\right)$, is not flat, we have zero–curvature planes near the equator that do not exponentiate to flats.

If we assume the fibers of the submersion are totally geodesic, then, even in the non-compact case, the conclusion of Tapp's theorem holds.

Theorem 3 *Let $\pi : M \to B$ be a Riemannian submersion of complete, non-negatively curved manifolds with totally geodesic fibers. Let $\tilde{\sigma}$ be a horizontal zero-curvature plane in M such that* $\exp(\tilde{\sigma})$ *is a flat. Then*

1. *$\tilde{\sigma}$ projects to a zero-curvature plane σ in B that exponentiates to a flat submanifold of B, and*
2. *Every horizontal lift of σ exponentiates to a horizontal flat in M.*

We also give an affirmative answer to Problem 1 in the special case when the submersion is induced by an isometric group action with only principal orbits.

Theorem 4 *Let a compact Lie group G act by isometries on a compact, non-negatively curved manifold M. Suppose all of the orbits are principal, and let $\pi : M \to M/G$ be the induced Riemannian submersion.*

Suppose $\tilde{\sigma}$ is a horizontal zero-curvature plane in M such that $\exp_p(\tilde{\sigma})$ *is a flat. Then*

1. *$\tilde{\sigma}$ projects to a zero-curvature plane σ in M/G that exponentiates to a flat submanifold of M/G, and*
2. *Every horizontal lift of σ exponentiates to a horizontal flat in M.*

Example 1 shows that this result does not hold if we remove the hypothesis that M is compact. On the other hand, appropriate associated bundles also inherit this property.

Corollary 1 *Let G be a compact Lie group, P be compact, and $\pi_P : P \to B \equiv P/G$ a principal G–bundle with non-negatively curved G–invariant metric. Let F be a non-negatively curved manifold that carries an isometric G–action and $\pi : E := P \times_G F \to B$ the corresponding associated bundle with fiber F. Give E and B the corresponding non-negatively curved metrics so that π and $Q : P \times F \to P \times_G F = E$ become Riemannian submersions.*
If $\tilde{\sigma}$ is a π–horizontal zero-curvature plane in E such that $\exp_p(\tilde{\sigma})$ is a flat, then

1. *$\tilde{\sigma}$ projects to a zero-curvature plane σ in B that exponentiates to a flat submanifold of B, and*
2. *Every horizontal lift of σ exponentiates to a horizontal flat in E.*

There is an abstract application of Theorem 2 in [19]. It allows for a simplification of one of the axioms for the Orthogonal Partial Conformal Change. There are also quite a few concrete examples of our results in the literature that are not examples of Theorem 1.

Example 4 Grove and Ziller have shown how to lift the product metric on $S^2 \times S^2$ and Cheeger's metric on $\mathbb{C}P^2\# - \mathbb{C}P^2$ to various principal $SO(k)$ bundles and hence to all of the associated bundles [15]. According to Lemma 4 (below) the flat tori in $S^2 \times S^2$ lift to flats in all of these non-negatively curved bundles. Similarly, the flat Klein bottles in Cheeger's $\mathbb{C}P^2\# - \mathbb{C}P^2$ must also lift to flats in all of the non-negatively curved bundles of [15]. It follows from the construction of the metric that the principal bundles all have totally geodesic fibers. Therefore the principal bundles give examples of Theorems 2, 3, and 4. The associated bundles give examples of Theorem 2 and Corollary 1.

To prove Theorems 3 and 4 we establish a main lemma on holonomy fields, whose defintion we recall from [11].

Definition 1 Given a Riemannian submersion $\pi : M \to B$ let A and T be the corresponding fundamental tensors as defined in [17]. A Jacobi field J along a horizontal geodesic $c : I \to M$ is said to be a holonomy field if $J(0)$ is vertical and satisfies

$$J'(0) = A_{\dot{c}(0)} J(0) + T_{J(0)} \dot{c}(0). \tag{0.1}$$

Main Lemma *Let $\pi : M \to B$ be a Riemannian submersion of complete, non-negatively curved manifolds so that each holonomy field is bounded. Let $\tilde{\sigma}$ be a horizontal zero-curvature plane in M such that $\exp(\tilde{\sigma})$ is a flat. Then*

1. *$\tilde{\sigma}$ projects to a zero-curvature plane σ in B that exponentiates to a flat submanifold of B, and*
2. *Every horizontal lift of σ exponentiates to a horizontal flat in M.*

The Main Lemma is a consequence of Lemmas 3 and 4 (below). These along with Theorems 2 and 3 are proven in Sect. 1. In Sect. 2, we prove Theorem 4 by showing that such submersions satisfy the hypotheses of the main lemma. Corollary 1 is also proven in Sect. 2.

1 Jacobi fields along geodesics contained in flats

The symmetries of the curvature tensor imply that the map $X \longmapsto R(X, W)W$ is self-adjoint. This combined with the spectral theorem yields the following result, which appears implictly in [18].

Proposition 1 *Let* span$\{X, W\}$ *be a zero-curvature plane in a nonnegatively curved manifold, then*

$$R(X, W)W = R(W, X)X = 0.$$

In a compact Lie group G with bi-invariant metric, solutions to the Jacobi equation along a geodesic $\gamma(t)$ have the form

$$J(t) = E_0 + tF_0 + \sum_{i=0}^{l}\left(\cos(\sqrt{k_i}t)E_i + \sin(\sqrt{k_i}r)F_i\right),$$

where E_i and F_i are parallel along γ (see [16]). We generalize this decomposition in the following way:

Lemma 1 *Suppose γ is a geodesic in a complete, non-negatively curved manifold M, and suppose J_0 is a normal, parallel, Jacobi field along γ, then any normal Jacobi field J along γ can be written as*

$$J(t) = (a + bt)J_0(t) + W(t), \tag{1.1}$$

where $a, b \in \mathbb{R}$ and W and W' are perpendicular to J_0.

Proof Extend J_0 to an orthonormal basis $\{J_0, E_2, ..., E_{n-1}\}$ of normal, parallel fields along γ. Since $J_0(t)$ and $\gamma'(t)$ span a zero-curvature plane and M is non-negatively curved, $R(J_0, \gamma')\gamma' = 0$, by Proposition 1. Therefore, if we write

$$J(t) = f(t)J_0(t) + \sum_{i=2}^{n-1} f_i(t)E_i(t),$$

we have

$$\begin{aligned} J''(t) &= -R(J(t), \gamma'(t))\gamma'(t) \\ &= -\sum_{i=2}^{n-1} f_i(t)R(E_i, \gamma'(t))\gamma'(t) \end{aligned}$$

and

$$\langle R(E_i, \gamma')\gamma', J_0\rangle = \langle R(J_0, \gamma')\gamma', E_i\rangle = 0$$

by a symmetry of the curvature tensor. Thus $J'' \perp J_0$. Since $\{J_0, E_2, ..., E_{n-1}\}$ is parallel and orthogonal, we also have

$$J''(t) = f''(t)J_0(t) + \sum_{i=2}^{n-1} f_i''(t)E_i(t).$$

Combining this with $J'' \perp J_0$, we see that $f'' = 0$ as claimed.

Since $W' = \sum_{i=2}^{n-1} f_i'(t)E_i(t)$, we also have $W' \perp J_0$. □

Given a Riemannian submersion $\pi : M \to B$, let $\mathcal{V}$ and $\mathcal{H}$ be the vertical and horizontal distributions. As holonomy fields are the variational fields arising from horizontal lifts of geodesics in B, they never vanish, they remain vertical, and they satisfy (0.1) for all time. In fact, we can find a collection $\{J_i(t)\}$ of such fields that span $\mathcal{V}$ along c. This description of $\mathcal{V}$ allows one to determine precisely when a field along a curve in M has values in $\mathcal{H}$. In particular, we have the following, as observed by Tapp when M is a Lie group.

Lemma 2 *Suppose $\pi : M \to B$ is a Riemannian submersion of a complete, non-negatively curved manifold M, γ is a horizontal geodesic in M, and J_0 is a parallel Jacobi field along γ such that $J_0(0)$ is horizontal. If all holonomy fields V along γ have bounded length, then J_0 is everywhere horizontal.*

Proof Let V be a holonomy field. Since V is always vertical, the decomposition in Lemma 1 simplifies to

$$V(t) = btJ_0(t) + W(t).$$

Since V has bounded length, $b = 0$ and therefore $V(t) = W(t)$, which is perpendicular to J_0. As the collection of all holonomy fields spans the vertical distribution along γ, the result follows. □

Part 1 of the main lemma is a consequence of the next result.

Lemma 3 *Suppose $\pi : M \to B$ is a Riemannian submersion of a complete, non-negatively curved manifold M, and all holonomy fields of π have bounded length. Suppose $\tilde{\sigma}$ is a horizontal zero-curvature plane and* $\exp(\tilde{\sigma})$ *is a totally geodesic flat.*

Then $\sigma := d\pi(\tilde{\sigma})$ has a zero-curvature and $\exp(\sigma)$ *is a totally geodesic flat submanifold of B.*

Proof Let $\{X, Y\}$ be any orthonormal pair in $\tilde{\sigma}$. Let γ be the geodesic: $t \longmapsto \exp(tX)$, and let J be the parallel Jacobi field along γ with $J(0) = Y$. Then by the previous Lemma, $J(t)$ is horizontal for all t. Hence $\exp(\tilde{\sigma})$ is everywhere horizontal, and, by assumption, a totally geodesic flat.

It follows from the Horizontal Curvature Equation that $\pi(\exp(\tilde{\sigma}))$ is also flat, and from the formula for covariant derivatives of horizontal fields it follows that $\pi(\exp(\tilde{\sigma}))$ is totally geodesic. Since horizontal geodesics project to geodesics, $\pi(\exp(\tilde{\sigma})) = \exp(d\pi(\tilde{\sigma})) = \exp(\sigma)$. So $\exp(\sigma)$ is a totally geodesic flat submanifold of B. □

The following lemma is probably a well known application of the Horizontal Curvature Equation. We include it as it establishes part 2 of our main lemma and is also used in the proof of Theorem 2.

Lemma 4 *Let $\pi : M \to B$ be a Riemannian submersion of a complete, non-negatively curved manifold M. Let σ be a tangent plane to B so that* $\exp(\sigma)$ *is a totally geodesic flat.*

Then for any horizontal lift $\tilde{\sigma}$ of σ, $\exp(\tilde{\sigma})$ *is a totally geodesic flat that is everywhere horizontal.*

Proof The Horizontal Curvature Equation implies that any horizontal lift $\hat{\tau}$ of a plane τ tangent to $\exp(\sigma)$ satisfies

$$\sec_M(\hat{\tau}) = 0 \text{ and } A(\hat{\tau}) = 0.$$

In particular, the collection of all such $\hat{\tau}$s gives us an integrable 2-dimensional distribution that is horizontal. The vanishing A-tensor combined with our hypothesis that $\exp(\sigma)$ is totally geodesic gives us that all the integral submanifolds of this distribution are also totally geodesic. If $\tilde{\sigma}$ is a horizontal lift of σ, then it follows that $\exp(\tilde{\sigma})$ is tangent to this distribution and hence is a totally geodesic flat that is everywhere horizontal.

We now proceed with proofs of Theorems 3 and 2. □

Proof of Theorem 3 When the fibers of a Riemannian submersion are totally geodesic, the T-tensor for the submersion vanishes. If V is a holonomy field along a horizontal geodesic γ, by (0.1) we have

$$\langle V(t), V(t)\rangle' = 2\langle V(t), V'(t)\rangle = 2\langle V(t), T_{V(t)}\gamma'(t)\rangle = 0,$$

so V has constant norm. An application of the main lemma completes the proof. □

In contrast to our other results the proof of Theorem 2 does not use the main lemma. Instead we exploit the infinitesimal geometry of the submersion.

Proof of Theorem 2 Let σ be a zero-curvature plane in B and $\tilde{\sigma}$ a horizontal lift of σ so that $\exp(\tilde{\sigma})$ is contained in a flat of M. Let γ be a geodesic in $\exp(\tilde{\sigma})$ and J_0 be a parallel Jacobi field along γ such that

$$\tilde{\sigma} = \text{span}\left\{\gamma'(0), J_0(0)\right\}.$$

Now $A_{\gamma'(0)}J_0(0) = 0$ because $\sec_M(\tilde{\sigma}) = \sec_B(\sigma) = 0$; so for any holonomy field V, we have

$$\begin{aligned}\langle J_0(t), V'(t)\rangle\big|_{t=0} &= \langle J_0(t), A_{\gamma'(t)}V(t)\rangle\big|_{t=0}, \text{ since } J_0(0) \text{ is horizontal}\\ &= -\langle A_{\gamma'(t)}J_0(t), V(t)\rangle\big|_{t=0}\\ &= 0.\end{aligned}$$

On the other hand, differentiating the right hand side of $V(t) = btJ_0(t) + W(t)$, we find

$$\begin{aligned}\langle J_0(t), V'(t)\rangle\big|_{t=0} &= \langle J_0(t), bJ_0(t)\rangle|_{t=0} + \langle J_0(t), W'(t)\rangle\big|_{t=0}\\ &= b\,|J_0(0)|^2.\end{aligned}$$

Therefore $b = 0$ and $V = W$, and it follows that $N := \exp(\tilde{\sigma})$ is everywhere horizontal. Thus its projection, $\exp(\sigma)$, is a totally geodesic flat in B.

By Lemma 4, every horizontal lift of σ exponentiates to a horizontal flat in M. □

2 The holonomy of π

In this section we prove Theorem 4 by showing that such submersions have bounded holomomy fields and hence satisfy the hypotheses of the main lemma. At the end of the section we prove Corollary 1.

Given a point $b \in B$, we define the *holonomy group* $\text{hol}(b)$ to be the group of all diffeomorphisms of the fiber $\pi^{-1}(b)$ that occur as holonomy diffeomorphisms $h_c : \pi^{-1}(b) \to \pi^{-1}(b)$ obtained by lifting piecewise smooth loops c at b. If M is compact, the T-tensor is globally bounded in norm. It follows that each holonomy diffeomorphism h_c is Lipschitz with Lipschitz constant dependent only on the length of c (see [12], Lemma 4.2). Since this Lipschitz constant can actually depend on the length of c, this is generally not enough to conclude that the holonomy fields are uniformly bounded (see [21], Example 6.1).

On the other hand, if B is compact and $\text{hol}(b)$ is a compact, finite-dimensional Lie group, then there is a uniform Lipschitz constant for all of $\text{hol}(b)$. Thus the holonomy fields are uniformly bounded ([21], Proposition 6.2). So to prove Theorem 4, it suffices to show that $\text{hol}(b)$ is a compact, finite-dimensional Lie group.

Proof of Theorem 4 Set $B = M/G$, and for $p \in M$, let G_p denote the isotropy subgroup of G. Note that the map $f : G/G_p \to M$ given by $f(gG_p) = g(p)$ is an imbedding onto the orbit $G(p)$ of p. Now take any piecewise smooth curve $c : [0, 1] \to B$. The holonomy diffeomorphism

$$h_c : \pi^{-1}(c(0)) \to \pi^{-1}(c(1))$$

is defined by

$$h_c(p) = \bar{c}(1),$$

where $\bar{c}$ is the unique horizontal lift of c starting at p. By assumption, G acts isometrically on M, so $g\bar{c}$ is also horizontal. Since $(g\bar{c})(1) = g(\bar{c}(1))$, we have that

$$h_c(gp) = gh_c(p).$$

In other words, h_c is G-equivariant.

By the above, $\text{hol}(b)$ is a subgroup of the collection $\text{Diff}_G(\pi^{-1}(b))$ of all G-equivariant diffeomorphisms of the fiber $\pi^{-1}(b)$. Take any $p \in \pi^{-1}(b)$. Set $H \equiv G_p$, and identify $\pi^{-1}(b)$ with G/H. Then $\text{Diff}_G(G/H)$ is isomorphic to the Lie group $N(H)/H$, where $N(H)$ is the normalizer of H (see [11], Lemma 2.3.3).

In [23], Wilking associates to a given metric foliation $\mathcal{F}$ the so-called *dual foliation* $\mathcal{F}^\#$. The dual leaf through a point $p \in M$ is defined as all points $q \in M$ such that there is a piecewise smooth, horizontal curve from p to q. Let $L_p^\#$ be the dual leaf through p.

We shall see that for any $p \in M$, $\text{hol}(b)$ is homeomorphic to $L_p^\# \cap \pi^{-1}(b)$.

We have the continuous map

$$\text{ev}_p : \text{hol}(b) \to L_p^\# \cap \pi^{-1}(b)$$

defined by

$$\text{ev}_p : h_c \mapsto h_c(p).$$

To construct the inverse, let q be in $L_p^\# \cap \pi^{-1}(b)$. There is a piecewise smooth, horizontal curve $\bar{c}$ from p to q. Now $\pi \circ \bar{c}$ is a piecewise smooth loop at b and

$$h_{\pi \circ \bar{c}}(p) = q.$$

We therefore propose to define ev_p^{-1} by

$$\text{ev}_p^{-1} : q \longmapsto h_{\pi \circ \bar{c}}.$$

To see that ev_p^{-1} is well-defined, suppose $\tilde{c}$ is another piecewise smooth, horizontal curve from p to q. By construction, we have $h_{\pi \circ \bar{c}}(p) = h_{\pi \circ \tilde{c}}(p)$. Since all holonomy diffeomorphisms are G-equivariant and G acts transitively on $\pi^{-1}(b)$, it follows that

$$h_{\pi \circ \bar{c}} = h_{\pi \circ \tilde{c}}.$$

Now take a sequence of points $q_i \in L^\# \cap \pi^{-1}(b)$ converging to $q_0 \in L^\# \cap \pi^{-1}(b)$. There are horizontal curves $\bar{c}_i$ from p to q_i such that $h_{\pi \circ \bar{c}_i}(p) = q_i$. Again by G-equivariance and the transitive action of G, these holonomy diffeomorphisms are completely determined by their behavior at a point. Thus $h_{\pi \circ \bar{c}_i} \to h_{\pi \circ \bar{c}_0}$, and so ev_p^{-1} is continuous. Therefore $\text{hol}(b)$ is homeomorphic to $L^\# \cap \pi^{-1}(b)$.

Since $\mathcal{F}$ is given by the orbit decomposition of an isometric group action, the dual foliation has complete leaves ([23], Theorem 3(a)). In particular, this says $L^\# \cap \pi^{-1}(b) \cong \text{hol}(b)$ is a closed subset of the compact space $\pi^{-1}(b)$ and hence is also compact. It follows that hol(b) is closed in the Lie group $\mathrm{Diff}_G(G/H) \cong N(H)/H$, so is a Lie subgroup of $\mathrm{Diff}_G(G/H)$. Thus hol(b) is a compact, finite-dimensional Lie group. □

Remark 1 In general, hol(b) need not even be a Lie group, let alone a compact Lie group [21]. However, it is shown in [13] that when the fibers come from principal G-actions, hol(b) is always a Lie group.

Recall (see [11], p. 92) that if P is the total space of the principal G-bundle $\pi_P : P \to B := P/G$ and F is a manifold that carries a G-action, then G acts freely on the product $P \times F$. In particular, if P and F have G-invariant metrics of non-negative curvature, G acts by isometries on the product $P \times F$. As a result, the total space $E = P \times_G F := (P \times F)/G$ of the associated bundle inherits a metric of non-negative curvature such that the quotient map $Q : P \times F \to P \times_G F$ is a Riemannian submersion [4]. Similarly, B inherits a metric of non-negative curvature such that $\pi_P : P \to B$ is a Riemannian submersion. If $\pi_1 : P \times F \to P$ is projection onto the first factor, the diagram

$$\begin{array}{ccc} P \times F & \xrightarrow{Q} & E \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ P & \xrightarrow[\pi_P]{} & B \end{array}$$

commutes and so $\pi : E \to B$ is also a Riemannian submersion.

Proof of Corollary 1: Consider the composition

$$\pi_P \circ \pi_1 : P \times F \longrightarrow B.$$

The holonomy fields for $\pi_P \circ \pi_1$ are the products of holonomy fields for $\pi_P : P \to B$ and π_1. The former are bounded by the proof of Theorem 4, the latter are bounded because the fibers of π_1 are totally geodesic.

Now suppose that $\tilde{\sigma}$ is a horizontal zero-curvature plane for $\pi : E \longrightarrow B$ such that $\exp_p(\tilde{\sigma})$ is a flat. Apply Lemma 4 to $Q : P \times F \to E$ to conclude that any horizontal lift $\tilde{\sigma}_{P\times F}$ of $\tilde{\sigma}$ exponentiates to a (Q–horizontal) flat. Since the holonomy fields of $\pi_P \circ \pi_1 = \pi \circ Q$ are bounded, we can apply Lemma 3 to conclude that $\sigma := d(\pi \circ Q)(\tilde{\sigma}_{P\times F}) = d\pi(\tilde{\sigma})$ is a zero plane that exponentiates to a flat. Applying Lemma 4 to $\pi : E \to B$ we conclude that every horizontal lift of σ is a horizontal flat. □

Remark 2 Combining the Main Lemma with the concept of projectable Jacobi fields from [11] one gets a shorter (but more learned) proof of the Corollary.

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