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ORIGINAL PAPER

# Total curvature and $L^2$ harmonic 1-forms on complete submanifolds in space forms

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**Abstract** Let $M^n$ be an $n$-dimensional complete noncompact oriented submanifold with finite total curvature, i.e., $\int_M(|A|^2 - n|H|^2)^{\frac{n}{2}} < \infty$, in an $(n+p)$-dimensional simply connected space form $N^{n+p}(c)$ of constant curvature $c$, where $|H|$ and $|A|^2$ are the mean curvature and the squared length of the second fundamental form of $M$, respectively. We prove that if $M$ satisfies one of the following: (i) $n \geq 3, c = 0$ and $\int_M |H|^n < \infty$; (ii) $n \geq 5, c = -1$ and $|H| < 1 - \frac{2}{\sqrt{n}}$; (iii) $n \geq 3$, $c = 1$ and $|H|$ is bounded, then the dimension of the space of $L^2$ harmonic 1-forms on $M$ is finite. Moreover, in the case of (i) or (ii), $M$ must have finitely many ends.

**Keywords** Submanifold · Total curvature · $L^2$ harmonic forms · Mean curvature · Ends

**Mathematics Subject Classification (2000)** 53C40 · 53C21

## 1 Introduction

In [1], Cao-Shen-Zhu proved that a complete immersed stable minimal hypersurface $M^n$ of $R^{n+1}$ with $n \geq 3$ must have only one end. Its strategy was to utilize a result of Schoen-Yau

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asserting that a complete stable minimal hypersurface of $R^{n+1}$ can not admit a non-constant harmonic function with finite Dirichlet integral [11]. Assuming that $M^n$ has more than one end, they constructed a non-constant harmonic function with finite Dirichlet integral in [1]. According to the work of Li-Tam [9], Li-Wang modified this proof to show that each end of a complete immersed minimal submanifold must be non-parabolic in [10]. Due to this connection with harmonic functions, this allows one to estimate the number of ends of the above hypersurface by estimating the dimension of the space of bounded harmonic functions with finite Dirichlet integral. They proved that if $M$ has finite index, then the dimension of the space of $L^2$ harmonic 1-forms $M$ is finite, and $M$ must have finitely many ends [10].

Let $M^n$ be an $n$-dimensional complete oriented submanifold isometrically immersed in an $(n+p)$-dimensional complete simply connected Riemannian manifold $N^{n+p}$. Fix a point $x \in M$ we choose a local orthonormal frame $\{e_1, e_2, \ldots, e_{n+p}\}$ such that $\{e_1, e_2, \ldots, e_n\}$ are tangent fields. For each $\alpha$, $n+1 \le \alpha \le n+p$, define a linear map $A_\alpha : T_xM \to T_xM$ by

$$\langle A_\alpha X, Y\rangle = \langle \widetilde{\nabla}_X Y, e_\alpha\rangle,$$

where $X, Y$ are tangent fields and $\widetilde{\nabla}$ denotes the Riemannian connection on $N^{n+p}$. We denote by $H$ the mean curvature vector of $M$, i.e.,

$$H = \frac{1}{n}\sum_{\alpha=n+1}^{n+p} (Tr A_\alpha) e_\alpha.$$

For each $\alpha$, $n+1 \le \alpha \le n+p$, define a linear map $\phi_\alpha : T_xM \to T_xM$ by

$$\langle \phi_\alpha X, Y\rangle = \langle X, Y\rangle\langle H, e_\alpha\rangle - \langle A_\alpha X, Y\rangle,$$

and a bilinear map $\phi_\alpha : T_xM \times T_xM \to T_xM^\perp$ by

$$\phi(X, Y) = \sum_{\alpha=n+1}^{n+p} \langle \phi_\alpha X, Y\rangle e_\alpha.$$

It is easy to see that the tensor $\phi$ is traceless. Denote by $A$ the second fundamental form of $M$. We have

$$|A|^2 = |\phi|^2 + n|H|^2.$$

If $N^{n+p}$ is a nonpositive curved manifold, then by Hoffman and Spruck's Sobolev inequality [2,6], there exists a positive constant $C_0$ depending only on the dimension $n$ such that

$$\left(\int_M |f|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \le C_0 \int_M (|\nabla f| + n|H||f|), \quad \forall f \in C_0^1(M). \tag{1}$$

We say that $M$ has finite total curvature if

$$\int_M |\phi|^n < \infty.$$

Let $H^1(L^2(M))$ denote the space of $L^2$ harmonic 1-forms on $M$ and $\Delta$ denote the Laplacian on $M$.

When $M^n (n \ge 3)$ is an oriented stable complete minimal hypersurface in $R^{n+1}$, Shen and Zhu [13] showed that if

$$\int_M |A|^n < \infty,$$

then $M$ must be a hyperplane. When $M^n (n \geq 3)$ is a complete oriented minimal hypersurface in $R^{n+1}$, Yun proved that if

$$\int_M |A|^n < \left(\frac{n-2}{2(n-1)C_0}\right)^n,$$

then there are no $L^2$ harmonic 1-forms on $M$, and $M$ has only one end in [16]. The authors generalized Yun's result to the case where $M$ is a complete noncompact oriented submanifold in space forms [5]. In this paper, using arguments due to Li-Wang and Yun, we study a complete noncompact oriented submanifold with finite total curvature in space forms. Throughout this article, we always assume that $M$ is a complete, non-compact, connected Riemannian manifold without boundary. In this case, we will simply say that $M$ is a complete manifold.

Our main results in this paper are stated as follows.

**Theorem 1.1** *Let $M^n (n \geq 3)$ be an oriented complete submanifold with finite total curvature in $R^{n+p}$. If*

$$\int_M |H|^n < \infty,$$

*then $dim H^1(L^2(M)) < \infty$. In particular, $M$ must have finitely many ends.*

From the main theorem in [12], we see that if $M^n (n \geq 3)$ is an oriented complete submanifold with parallel mean curvature and finite total curvature in $R^{n+p}$, then $M$ must be minimal. By Theorem 1.1, we have

**Corollary 1.2** *Let $M^n (n \geq 3)$ be an oriented complete submanifold with parallel mean curvature and finite total curvature in $R^{n+p}$. Then $dim H^1(L^2(M)) < \infty$. In particular, $M$ must have finitely many ends.*

**Theorem 1.3** *Let $M^n (n \geq 5)$ be an oriented complete submanifold with finite total curvature in the hyperbolic space $H^{n+p}$. If*

$$|H| < 1 - \frac{2}{\sqrt{n}},$$

*then $dim H^1(L^2(M)) < \infty$, and $M$ must have finitely many ends.*

**Theorem 1.4** *Let $M^n (n \geq 3)$ be an oriented complete submanifold with bounded mean curvature $|H|$ and finite total curvature in a unit sphere $S^{n+p}$. Then $dim H^1(L^2(M)) < \infty$, and the number of non-parabolic ends of $M$ is at most finitely many.*

## 2 Preliminary

Let $M^n (n \geq 3)$ be an oriented complete immersed submanifold with mean curvature $|H|$ in a complete simply connected $N^{n+p}$ of nonpositive sectional curvature.

If $|H| \leq \alpha < \infty$, putting $f = g^{\frac{2(n-1)}{n-2}}$ with $g \in C_0^1(M)$ in (1), we get

$$\left(\int_M |g|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq 8\frac{(n-1)^2}{(n-2)^2}C_0^2\left(\int_M |\nabla g|^2 + n^2\alpha^2\int_M g^2\right). \tag{2}$$

If $|H| \in L^n(M)$, then there exsits a compact subset $D \subset M$ satisfying

$$\|H\|_{L^n(M\setminus D)} < \frac{1}{2nC_0}.$$

Thus

$$nC_0\int_{M\backslash D}|H||f|\le nC_0\left(\int_{M\backslash D}|H|^n\right)^{\frac{1}{n}}\left(\int_{M\backslash D}|f|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}$$
$$\le\frac{1}{2}\left(\int_{M\backslash D}|f|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}.$$

Substituting the above inequality into (1), we have

$$\left(\int_{M\backslash D}|f|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}\le 2C_0\int_{M\backslash D}|\nabla f|,\quad \forall f\in C_0^1(M\backslash D). \tag{3}$$

Putting $f=g^{\frac{2(n-1)}{n-2}}$ with $g\in C_0^1(M\backslash D)$ in (3), we obtain

$$\left(\int_{M\backslash D}|g|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\le 16\frac{(n-1)^2}{(n-2)^2}C_0^2\int_{M\backslash D}|\nabla g|^2. \tag{4}$$

In this paper, we will investigate the number of ends of submanifolds. Now we state some definitions and theorems.

**Definition 2.1** Let $D\subset M$ be a compact subset of $M$. An end $E$ of $M$ with respect to $D$ is a connected unbounded component of $M\backslash D$. When we say that $E$ is an end, it is implicitly assumed that $E$ is an end with respect to some compact subset $D\subset M$.

**Definition 2.2** ([8]) A manifold is said to be parabolic if it does not admit a positive Green's function. Conversely, a nonparabolic manifold is one which admits a positive Green's function. An end $E$ of a manifold is said to be nonparabolic if it admits a positive Green's function with Neumann boundary condition on $\partial E$. Otherwise, it is said to be parabolic.

**Theorem 2.3** ([10]) *Let $M$ be a complete manifold. Let $\mathcal{H}_D^0(M)$ denote the space of bounded harmonic functions with finite Dirichlet integral. Then the number of non-parabolic ends of $M$ is at most the dimension of $\mathcal{H}_D^0(M)$.*

**Theorem 2.4** ([10]) *Let $E$ be an end of a complete manifold. Suppose for some $\nu\ge 1$, $E$ satisfies a Sobolev type inequality of the form*

$$\left(\int_E|f|^{2\nu}\right)^{\frac{1}{\nu}}\le C\int_E|\nabla f|^2,\quad \forall f\in C_0^1(E).$$

*then $E$ must either have finite volume or be non-parabolic.*

Let $|H|\in L^n(M)$. By the definition of end, every end $E$ of $M$ is contained in $M\backslash D$, then from (3), it follows that

$$\left(\int_E|f|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}\le 2C_0\int_E|\nabla f|,\quad \forall f\in C_0^1(E). \tag{5}$$

For every geodesic ball $B_p(r)$ of $E$ and sufficiently small $\epsilon>0$, we consider the Lipschitz function

$$f_\epsilon(x)=\begin{cases}1, & x\in B_p(r),\\ 1-\frac{1}{\epsilon}d(x,\partial B_p(r)), & x\in E\backslash B_p(r), d(x,\partial B_r(p))<\epsilon,\\ 0, & \textit{otherwise}.\end{cases}$$

By using the regularization argument, we have

$$\left(\int_E |f_\epsilon|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq 2C_0 \int_E |\nabla f_\epsilon|.$$

This implies that

$$\left(vol B_p(r)\right)^{\frac{n-1}{n}} \leq 2C_0 vol \partial B_p(r).$$

Integrating the above, we get

$$vol B_p(r) \geq C_0' r^n.$$

Hence the volume of every end $E$ of $M$ is infinite, and the volume of $M$ is also infinite. By Theorem 2.4 and (4), every end of $M$ is non-parabolic, and $M$ is non-parabolic too. Furthermore, according to Theorem 2.3, the number of its ends is no more than $dim\mathcal{H}_D^0(M)$.

## 3 Proof of the theorems

For each $\omega \in H^1(L^2(M))$, it is the well-known Bochner formula that

$$\Delta|\omega|^2 = 2(|\nabla\omega|^2 + Ric(\omega, \omega)). \tag{6}$$

On the other hand, we have

$$\Delta|\omega|^2 = 2(|\omega|\Delta|\omega| + |\nabla|\omega||^2). \tag{7}$$

From (6), (7) and the generalized Kato's inequality $\frac{n}{n-1}|\nabla|\omega||^2 \leq |\nabla\omega|^2$ in [15], we obtain

$$|\omega|\Delta|\omega| \geq Ric(\omega, \omega) + \frac{1}{n-1}|\nabla|\omega||^2. \tag{8}$$

In [14], Shiohama and Xu proved that the following estimate holds for Ricci curvature of a submanifold $M$ in the simply connected space form $N^{n+p}(c)$ with constant sectional curvature $c$ :

$$Ric \geq \frac{n-1}{n}\left(nc + 2n|H|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\sqrt{|A|^2 - n|H|^2} - |A|^2\right).$$

Applying the above inequality to the traceless second fundamental form $|\phi|$ and using the identity $|A|^2 = |\phi|^2 + n|H|^2$, we get the following inequality:

$$Ric \geq (n-1)c + (n-1)|H|^2 - \frac{(n-2)\sqrt{n(n-1)}|\phi||H|}{n} - \frac{(n-1)|\phi|^2}{n}. \tag{9}$$

Substituting (8) into (9), we obtain

$$\begin{aligned}|\omega|\Delta|\omega| \geq& \frac{1}{n-1}|\nabla|\omega||^2 + (n-1)c|\omega|^2\\ &-\left[\frac{(n-2)\sqrt{n(n-1)}|\phi||H|}{n} + \frac{(n-1)|\phi|^2}{n} - (n-1)|H|^2\right]|\omega|^2.\end{aligned} \tag{10}$$

*Proof of Theorem 1.1* By the assumption $\int_M |H|^n < \infty$, we choose $B_p(r)$ such that the inequality (4) holds on $M\backslash B_p(r)$. Let $\omega \in H^1(L^2(M))$ and $\eta \in C_0^1(M\backslash B_p(r))$. Multiplying (10) by $\eta^2$ and integrating by parts over $M\backslash B_p(r)$, we get

$$
\begin{aligned}
0 \leq& \int_{M\backslash B_p(r)} \left(\eta^2|\omega|\Delta|\omega| - \frac{1}{n-1}\eta^2|\nabla|\omega||^2\right)\\
&+ S\int_{M\backslash B_p(r)} \eta^2\left(\frac{(n-2)\sqrt{n(n-1)}|\phi||H|}{n} + \frac{(n-1)|\phi|^2}{n} - (n-1)|H|^2\right)|\omega|^2\\
=& -2\int_{M\backslash B_p(r)} \eta\langle\nabla\eta, \nabla|\omega|\rangle|\omega| - \frac{n}{n-1}\int_{M\backslash B_p(r)} \eta^2|\nabla|\omega||^2 + \frac{n-1}{n}\int_{M\backslash B_p(r)} \eta^2|\phi|^2|\omega|^2\\
&+ \frac{(n-2)\sqrt{n(n-1)}}{n}\int_{M\backslash B_p(r)} |\phi||H|\eta^2|\omega|^2 - (n-1)\int_{M\backslash B_p(r)} |H|^2\eta^2|\omega|^2\\
\leq& -2\int_{M\backslash B_p(r)} \eta\langle\nabla\eta, \nabla|\omega|\rangle|\omega|\frac{n}{n-1}\int_{M\backslash B_p(r)} \eta^2|\nabla|\omega||^2 + \frac{n}{4}\int_{M\backslash B_p(r)} \eta^2|\phi|^2|\omega|^2.
\end{aligned}
\tag{11}
$$

On the other hand, it follows from (4) and Hölder inequality that

$$
\begin{aligned}
\int_{M\backslash B_p(r)} \eta^2|\phi|^2|\omega|^2 &\leq \left(\int_{M\backslash B_p(r)} |\phi|^n\right)^{\frac{2}{n}}\left(\int_{M\backslash B_p(r)} (\eta|\omega|)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\\
&\leq 16\frac{(n-1)^2}{(n-2)^2}\phi_0 C_0^2\int_{M\backslash B_p(r)} (|\nabla(\eta|\omega|)|^2\\
&= 16\frac{(n-1)^2}{(n-2)^2}\phi_0 C_0^2\int_{M\backslash B_p(r)} \left(|\omega|^2|\nabla\eta|^2 + \eta^2|\nabla|\omega||^2\right)\\
&\quad + 16\frac{(n-1)^2}{(n-2)^2}\phi_0 C_0^2\int_{M\backslash B_p(r)} 2\eta\langle\nabla\eta, \nabla|\omega|\rangle|\omega|,
\end{aligned}
\tag{12}
$$

where $\phi_0 = \left(\int_{M\backslash B_p(r)} |\phi|^n\right)^{\frac{2}{n}}$. Substituting (12) into (11), we have

$$
\begin{aligned}
0 \leq& 2\left(4n\frac{(n-1)^2}{(n-2)^2}\phi_0 C_0^2 - 1\right)\int_{M\backslash B_p(r)} \eta\langle\nabla\eta, \nabla|\omega|\rangle|\omega|\\
&+ \left(4n\frac{(n-1)^2}{(n-2)^2}\phi_0 C_0^2 - \frac{n}{n-1}\right)\int_{M\backslash B_p(r)} \eta^2|\nabla|\omega||^2\\
&+ 4n\frac{(n-1)^2}{(n-2)^2}\phi_0 C_0^2\int_{M\backslash B_p(r)} |\omega|^2|\nabla\eta|^2.
\end{aligned}
\tag{13}
$$

Using the Schwarz inequality, we get

$$
2\left|\int_{M\backslash B_p(r)} \eta\langle\nabla\eta, \nabla|\omega|\rangle|\omega|\right| \leq \epsilon\int_{M\backslash B_p(r)} \eta^2|\nabla|\omega||^2 + \frac{1}{\epsilon}\int_{M\backslash B_p(r)} |\omega|^2|\nabla\eta|^2. \tag{14}
$$

From (13) and (14), we obtain

$$\left[\frac{1}{\epsilon}\left|1-\frac{4n(n-1)^2\phi_0 C_0^2}{(n-2)^2}\right|+\frac{4n(n-1)^2\phi_0 C_0^2}{(n-2)^2}\right]\int_{M\backslash B_p(r)}|\omega|^2|\nabla\eta|^2$$
$$\geq\left[\left(\frac{n}{n-1}-\frac{4n(n-1)^2\phi_0 C_0^2}{(n-2)^2}\right)-\left|1-\frac{4n(n-1)^2\phi_0 C_0^2}{(n-2)^2}\right|\epsilon\right]\int_{M\backslash B_p(r)}\eta^2|\nabla|\omega||^2. \tag{15}$$

We note that the condition $\int_M |\phi|^n < \infty$ implies that there is a decreasing positive function $\epsilon(r)$ satisfying $\lim_{r\to+\infty}\epsilon(r)=0$ such that

$$\int_{M\backslash B_p(r)}|\phi|^n<\epsilon(r)$$

for $r$ large enough. Thus we can choose $r=r_0>0$ and $\epsilon=\epsilon_0>0$ such that (15) is written as follows:

$$\int_{M\backslash B_p(r_0)}\eta^2|\nabla|\omega||^2\leq C_1\int_{M\backslash B_p(r_0)}|\omega|^2|\nabla\eta|^2, \tag{16}$$

where positive constant $C_1$ depends only on $n$. Putting $h=|\omega|$ in (16), we have

$$\int_{M\backslash B_p(r_0)}\eta^2|\nabla h|^2\leq C_1\int_{M\backslash B_p(r_0)}|\nabla\eta|^2h^2. \tag{17}$$

Applying (4) to $\eta h$, we get

$$\left(\int_{M\backslash B_p(r_0)}(\eta h)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\leq 16\frac{(n-1)^2}{(n-2)^2}C_0^2\int_{M\backslash B_p(r_0)}|\nabla\eta h|^2$$
$$\leq 32\frac{(n-1)^2}{(n-2)^2}C_0^2\int_{M\backslash B_p(r_0)}(\eta^2|\nabla h|^2+|\nabla\eta|^2h^2).$$

Combining with (17), we obtain

$$\left(\int_{M\backslash B_p(r_0)}(\eta h)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\leq C_2\int_{M\backslash B_p(r_0)}|\nabla\eta|^2h^2, \tag{18}$$

where positive constant $C_2$ depends only on $n$. For $r>r_0+1$, let us choose the function

$$\eta(x)=\begin{cases}0, & x\in B_p(r_0),\\ d(x,p)-r_0, & x\in B_p(r_0+1)\backslash B_p(r_0),\\ 1, & x\in B_p(r)\backslash B_p(r_0+1),\\ \frac{2r-d(x,p)}{r}, & x\in B_p(2r)\backslash B_p(r),\\ 0, & x\in M\backslash B_p(2r).\end{cases}$$

By using the regularization argument, applying $\eta$ to (18), we have

$$\left(\int_{B_p(r)\backslash B_p(r_0+1)}h^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\leq C_2\int_{B_p(r_0+1)\backslash B_p(r_0)}h^2+C_2r^{-2}\int_{B_p(2r)\backslash B_p(r)}h^2.$$

Since $h \in L^2(M)$, letting $r \to \infty$, the second term in the above tends to 0. Hence we have

$$\left(\int_{M\setminus B_p(r_0+1)} h^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq C_2 \int_{B_p(r_0+1)\setminus B_p(r_0)} h^2. \tag{19}$$

By Hölder inequality, we get

$$\int_{B_p(r_0+2)\setminus B_p(r_0+1)} h^2 \leq vol^{\frac{2}{n}}(B_p(r_0+2))\left(\int_{B_p(r_0+2)\setminus B_p(r_0+1)} h^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}.$$

Together with (19), we conclude that there exists a positive constant $C_3$ depending only on $n$ and $vol(B_p(r_0+2))$ such that

$$\int_{B_p(r_0+2)} h^2 \leq C_3 \int_{B_p(r_0+1)} h^2. \tag{20}$$

Let $\alpha = |\frac{(n-2)\sqrt{n(n-1)}|\phi||H|}{n} + \frac{(n-1)|\phi|^2}{n} - (n-1)|H|^2|$. From (10), we see that

$$h\Delta h \geq -\alpha h^2 + \frac{1}{n-1}|\nabla h|^2. \tag{21}$$

For $\eta \in C_0^1(B_x(1))$, multiplying (21) by $\eta^2 h^{p-2}$ and integrate by parts over $B_x(1)$, we get

$$\left(p-1-\epsilon+\frac{1}{n-1}\right)\int_{B_x(1)} \eta^2 h^{p-2}|\nabla h|^2 \leq \int_{B_x(1)} \left(\alpha\eta^2 + \frac{1}{\epsilon}|\nabla\eta|^2\right)h^p, \tag{22}$$

for any real number $\epsilon > 0$ and $p \geq 2$. It is easy to see that

$$\int_{B_x(1)} |\nabla\left(\eta h^{\frac{p}{2}}\right)|^2 \leq \frac{p}{2}\left(\frac{p}{2}+\epsilon\right)\int_{B_x(1)} \eta^2 h^{p-2}|\nabla h|^2 + \left(1+\frac{p}{2\epsilon}\right)\int_{B_x(1)} |\nabla\eta|^2 h^p. \tag{23}$$

Substituting (22) into (23) and taking $\epsilon = \frac{1}{2}$, we obtain

$$\int_{B_x(1)} |\nabla\left(\eta h^{\frac{p}{2}}\right)|^2 \leq pC_4 \int_{B_x(1)} (\eta^2 + |\nabla\eta|^2)h^p, \tag{24}$$

where $C_4$ is a positive constant depending only on $n$ and $\sup_{B_x(1)} \alpha$.

On the other hand, applying (1) to $f^2$ with $f \in C_0^1(B_x(1))$, we get

$$\begin{aligned}
\left(\int_{B_x(1)} f^{\frac{2n}{n-1}}\right)^{\frac{n-1}{n}} &\leq C_0 \int_{B_x(1)} (2|f||\nabla f| + |H|f^2) \\
&\leq 2C_0\left(\int_{B_x(1)} f^2\right)^{\frac{1}{2}}\left(\int_{B_x(1)} |\nabla f|^2\right)^{\frac{1}{2}} + C_0 \sup_{B_x(1)}|H| \int_{B_x(1)} f^2 \\
&= C_0\left(\int_{B_x(1)} f^2\right)^{\frac{1}{2}}\left(2\left(\int_{B_x(1)} |\nabla f|^2\right)^{\frac{1}{2}} + \sup_{B_x(1)}|H|\left(\int_{B_x(1)} f^2\right)^{\frac{1}{2}}\right) \\
&\leq 2C_0 vol^{\frac{1}{2n}}(B_x(1))\left(\int_{B_x(1)} f^{\frac{2n}{n-1}}\right)^{\frac{n-1}{2n}}\left(\int_{B_x(1)} |\nabla f|^2\right)^{\frac{1}{2}} \\
&\quad + C_0 vol^{\frac{1}{2n}}(B_x(1)) \sup_{B_x(1)}|H|\left(\int_{B_x(1)} f^{\frac{2n}{n-1}}\right)^{\frac{n-1}{2n}}\left(\int_{B_x(1)} f^2\right)^{\frac{1}{2}}.
\end{aligned}$$

From the above inequality, we have

$$\left(\int_{B_x(1)} f^{\frac{2n}{n-1}}\right)^{\frac{n-1}{n}} \le C_5 \int_{B_x(1)} (|\nabla f|^2 + |f|^2), \tag{25}$$

where positive constant $C_5$ depends only on $n$, $vol(B_x(1))$ and $\sup_{B_x(1)} |H|$. Applying (25) to $\eta h^{\frac{p}{2}}$, we get

$$\left(\int_{B_x(1)} (\eta^2 h^p)^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \le C_5 \int_{B_x(1)} \left(|\nabla \left(\eta h^{\frac{p}{2}}\right)|^2 + \eta^2 h^p\right). \tag{26}$$

Substituting (24) into (26), we obtain

$$\left(\int_{B_x(1)} (\eta^2 h^p)^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \le pC_6 \int_{B_x(1)} (|\nabla \eta|^2 + \eta^2) h^p, \tag{27}$$

where positive constant $C_6$ depends only on $n$, $vol(B_x(1))$, $\sup_{B_x(1)} \alpha$ and $\sup_{B_x(1)} |H|$. Let $p_k = \frac{2n^k}{(n-1)^k}$ and $\rho_k = \frac{1}{2} + \frac{1}{2^{k+1}}$, for $k = 0, 1, 2, \ldots$. We choose $\eta_k \in C_0^\infty(B_x(1))$ such that $\eta_k = 1$ on $B_x(\rho_{k+1})$ and $|\nabla \eta_k| \le 2^{k+3}$. Replacing $p$ and $\eta$ in (27) by $p_k$ and $\eta_k$, respectively, we then obtain

$$\left(\int_{B_x(\rho_{k+1})} h^{p_{k+1}}\right)^{\frac{1}{p_{k+1}}} \le \left(2C_6 p_k 4^{k+3}\right)^{\frac{1}{p_k}} \left(\int_{B_x(\rho_k)} h^{p_k}\right)^{\frac{1}{p_k}}. \tag{28}$$

Applying the Moser iteration to the function $h$ via (28), we conclude that

$$h^2(x) \le C_7 \int_{B_x(1)} h^2, \tag{29}$$

where $C_7$ is a positive constant depending only on $n$, $vol(B_x(1))$, $\sup_{B_x(1)} \alpha$ and $\sup_{B_x(1)} |H|$. In particular, if $x \in B_p(r_0 + 1)$ has the property that

$$h^2(x) = \sup_{B_p(r_0+1)} h^2,$$

it follows from (29) that

$$\sup_{B_p(r_0+1)} h^2 \le C_7 \int_{B_p(r_0+2)} h^2.$$

This together with (20) implies that there exists a positive constant $C_8$ depending only on $n$, $vol(B_p(r_0 + 2))$, $\sup_{B_p(r_0+2)} \alpha$ and $\sup_{B_p(r_0+2)} |H|$, such that

$$\sup_{B_p(r_0+1)} h^2 \le C_8 \int_{B_p(r_0+1)} h^2. \tag{30}$$

We are now in a position to prove that $H^1(L^2(M))$ is of finite dimensional. It suffices to show that any finite dimensional subspace $\mathscr{S}$ of $H^1(L^2(M))$ must have its dimension bounded by a fixed constant. Let $l$ be the dimension of $\mathscr{S}$. Let us consider the bilinear form defined on $\mathscr{S}$ given by

$$\int_{B_p(r_0+1)} \langle \omega, \theta \rangle.$$

Note that if

$$\int_{B_p(r_0+1)} |\omega|^2 = 0$$

for some $\omega \in \mathscr{S}$, then by the unique continuation property $\omega$ identically equals to 0. This implies that the bilinear form is an inner product defined on $\mathscr{S}$.

According to Lemma 11 of [7], there exists an $\omega \in \mathscr{S}$ such that

$$l \int_{B_p(r_0+1)} |\omega|^2 \le vol(B_p(r_0+1))(\min\{n,l\}) \sup_{B_p(r_0+1)} |\omega|^2.$$

Combining with (30) we conclude that

$$l \le C_9$$

with positive constant $C_9$ depending only on $n, vol(B_p(r_0+2)), \sup_{B_p(r_0+2)} \alpha$ and $\sup_{B_p(r_0+2)} |H|$. Hence $dim H^1(L^2(M)) < \infty$. Since the number of its ends is no more than $dim\mathcal{H}^0_D(M)$ and $dim\mathcal{H}^0_D(M) \le dim H^1(L^2(M)) + 1$ in Sect. 2, $M$ must have finitely many ends. □

**Lemma 3.1** *([4]) Let $M^n$ be an oriented complete submanifold in $H^{n+p}$ with bounded mean curvature $|H|$. If $|H| \le \alpha$ for some constant $0 \le \alpha < 1$, then*

$$\lambda_1(M) \ge \frac{(n-1)^2(1-\alpha)^2}{4}.$$

**Theorem 3.2** *Let $M^n (n \ge 5)$ be an oriented complete minimal submanifold with finite total curvature in $H^{n+p}$. Then $dim H^1(L^2(M)) < \infty$, and $M$ must have finitely many ends.*

*Proof of Theorem 1.1* Let $\omega \in H^1(L^2(M))$ and $\eta \in C^1_0(M\backslash B_p(r))$. Multiplying (10) by $\eta^2$ and integrating by parts over $M\backslash B_p(r)$, we get the following inequality

$$\begin{aligned} 0 \le -2\int_{M\backslash B_p(r)} \eta\langle\nabla\eta, \nabla|\omega|\rangle|\omega| - \frac{n}{n-1}\int_M \eta^2|\nabla|\omega||^2 \\ + \frac{n-1}{n}\int_{M\backslash B_p(r)} \eta^2|A|^2|\omega|^2 + (n-1)\int_{M\backslash B_p(r)} \eta^2|\omega|^2. \end{aligned} \tag{31}$$

By Lemma 3.1, we have

$$\begin{aligned} (n-1)\int_{M\backslash B_p(r)} \eta^2|\omega|^2 &\le \frac{4}{n-1}\int_{M\backslash B_p(r)} |\nabla(\eta|\omega|)|^2 \\ &\le \frac{4}{n-1}\int_{M\backslash B_p(r)} \left(|\omega|^2|\nabla\eta|^2 + \eta^2|\nabla|\omega||^2\right) \\ &\quad + \frac{4}{n-1}\int_{M\backslash B_p(r)} 2\eta\langle\nabla\eta, \nabla|\omega|\rangle|\omega|. \end{aligned}$$

Substituting the above inequality into (31), using the same argument as the proof of (15), we have

$$\begin{aligned} &\left(\left(\tfrac{n-4}{n-1} - 4\tfrac{(n-1)^3}{n(n-2)^2}A_0C_0^2\right) - \left|\tfrac{n-5}{n-1} - 4\tfrac{(n-1)^3}{n(n-2)^2}A_0C_0^2\right|\epsilon\right) \int_{M\backslash B_p(r)} \eta^2|\nabla|\omega||^2 \\ &\le \left(\tfrac{1}{\epsilon}\left|\tfrac{n-5}{n-1} - 4\tfrac{(n-1)^3}{n(n-2)^2}A_0C_0^2\right| + 4\tfrac{(n-1)^3}{n(n-2)^2}A_0C_0^2 + \tfrac{4}{n-1}\right) \int_{M\backslash B_p(r)} |\omega|^2|\nabla\eta|^2, \end{aligned}$$

where $A_0 = \int_{M \setminus B_p(r)} |A|^n$. Similar to the proof of (20), we get

$$\int_{B_p(r_0+2)} h^2 \leq C_{10} \int_{B_p(r_0+1)} h^2, \tag{32}$$

where $C_{10}$ is a positive constant depending only on $n$ and $vol(B_p(r_0+2))$.

Similar to the proof of (24), we obtain

$$\int_{B_x(1)} |\nabla \left(\eta h^{\frac{p}{2}}\right)|^2 \leq pC_{11} \int_{B_x(1)} (\eta^2 + |\nabla \eta|^2) h^p, \tag{33}$$

where positive constant $C_{11}$ depends only on $n$ and $\sup_{B_x(1)} |A|^2$. Applying (2) to $\eta h^{\frac{p}{2}}$, we get

$$\left(\int_{B_x(1)} (\eta^2 h^p)^{\frac{n}{n-2}}\right)^{\frac{n-2}{n}} \leq 4\frac{(n-1)^2}{(n-2)^2} C_0^2 \int_{B_x(1)} |\nabla(\eta h^{\frac{p}{2}})|^2. \tag{34}$$

Substituting (33) into (34), we obtain

$$\left(\int_{B_x(1)} (\eta^2 h^p)^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq pC_{12} \int_{B_x(1)} (|\nabla \eta|^2 + \eta^2) h^p, \tag{35}$$

where $C_{12}$ is a positive constant depending only on $n$ and $\sup_{B_x(1)} |A|^2$. Let $p_k = \frac{2n^k}{(n-2)^k}$ and $\rho_k = \frac{1}{2} + \frac{1}{2^{k+1}}$, for $k = 0, 1, 2, \ldots$. We choose $\eta_k \in C_0^\infty(B_x(1))$ such that $\eta_k = 1$ on $B_x(\rho_{k+1})$ and $|\nabla \eta_k| \leq 2^{k+3}$. Replacing $p$ and $\eta$ in (35) by $p_k$ and $\eta_k$, respectively, we then obtain

$$\left(\int_{B_x(\rho_{k+1})} h^{p_{k+1}}\right)^{\frac{1}{p_{k+1}}} \leq \left(2C_{12} p_k 4^{k+3}\right)^{\frac{1}{p_k}} \left(\int_{B_x(\rho_k)} h^{p_k}\right)^{\frac{1}{p_k}}. \tag{36}$$

Applying the Moser iteration to the function $h$ via (36), we conclude that

$$h^2(x) \leq C_{13} \int_{B_x(1)} h^2,$$

where positive constant $C_{13}$ depends only on $n$ and $\sup_{B_x(1)} |A|^2$. By the same argument as the proof of Theorem 1.1, we show that $dim H^1(L^2(M)) < \infty$ and $M$ must have finitely many ends. □

*Proof of Theorem 1.3* Using an analogous argument of the proof of Theorem 3.2, we obtain $dim H^1(L^2(M)) < \infty$. It is easy to see that $\lambda_1(M) > 0$. By Corollary 2.1 in [3], the volume of every end of $M$ is infinite. According to Theorems 2.3 and 2.4, $M$ must have finitely many ends. □

*Proof of Theorem 1.4* We choose a real number $\alpha$ such that $|H| \leq \alpha$. For isometric immersion $M^n \to S^{n+p} \hookrightarrow R^{n+p+1}$, we have

$$\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq 8\frac{(n-1)^2}{(n-2)^2} C_0^2 \left(\int_M |\nabla f|^2 + n^2(\alpha^2+1) \int_M |f|^2\right), \forall f \in C_0^1(M).$$

By using the same argument as the proof of Theorem 3.2, we complete the proof of Theorem 1.4. □

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