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# Conformal paracontact curvature and the local flatness theorem

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**Abstract** A curvature-type tensor invariant called para contact (pc) conformal curvature is defined on a paracontact manifold. It is shown that a paracontact manifold is locally paracontact conformal to the hyperbolic Heisenberg group or to a hyperquadric of neutral signature iff the pc conformal curvature vanishes. In the three dimensional case the corresponding result is achieved through employing a certain symmetric (0,2) tensor. The well known result of Cartan–Chern–Moser giving necessary and sufficient condition a CR-structure to be CR equivalent to a hyperquadric in $\mathbb{C}^{n+1}$ is presented in-line with the paracontact case. An explicit formula for the regular part of a solution to the sub-ultrahyperbolic Yamabe equation on the hyperbolic Heisenberg group is shown.

**Keywords** Paracontact · CR structures · Pseudo conformal flat · Paracontact conformally invariant curvature · Cartan–Chern–Moser theorem

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## 1 Introduction

A paracontact structure on a real (2n+1)-dimensional manifold $M$ is a codimension one distribution $\mathbb{H}$ and a paracomplex structure $I$ on $\mathbb{H}$, i.e. $I^2 = id$ and the $\pm$ eigen-distributions $\mathbb{H}^{\pm}$ have equal dimension. Locally, $\mathbb{H}$ is the kernel of a 1-form $\eta$, $\mathbb{H} = Ker\,\eta$. A paracontact Hermitian structure is a paracontact structure with the additional assumption that $\eta$ is a para Hermitian contact form in the sense that we have a non-degenerate pseudo-Riemannian metric $g$, which is defined on $\mathbb{H}$, and compatible with $\eta$ and $I$, $d\eta(X,Y) = 2g(IX,Y)$, $g(IX,IY) = -g(X,Y)$, $X,Y \in \mathbb{H}$. The signature of $g$ on $\mathbb{H}$ is necessarily of (signature) type (n,n). A para contact structure is said to be integrable if the para complex structure $I$ on $\mathbb{H}$ is formally integrable, i.e., $[\mathbb{H}^{\pm}, \mathbb{H}^{\pm}] \subset \mathbb{H}^{\pm}$. A para-contact manifold with an integrable para-contact structure is called a *para CR-manifold*

The 1-form $\eta$ is determined up to a conformal factor and hence $\mathbb{H}$ become equipped with a conformal class [g] of neutral Riemannian metrics of signature (n,n). Transformations preserving a given para contact hermitian structure $\eta$, i.e. $\bar{\eta} = \mu\eta$ for a non-vanishing smooth function $\mu$ are called *para contact conformal (pc conformal for short) transformations* [19].

A basic example is provided by a para-Sasakian manifold, which can be defined as a $(2n+1)$-dimensional Riemannian manifold whose metric cone is a para-Kähler manifold [1]. It was shown in [19] that the torsion endomorphism of the canonical connection is the obstruction for a given integrable para contact hermitian structure to be locally para-Sasakian, up to a multiplication with a constant factor.

Any non-degenerate hypersurface in $(R^{2n+2}, \mathbb{I}, g)$ considered with the standard flat para-hermitian structure inherits an integrable para-contact hermitian structure. We consider the (2n+1)-dimensional Heisenberg group with a left-invariant para-contact hermitian structure $\eta$ and call it *hyperbolic Heisenberg group*, denoted by $(\boldsymbol{G}\,(\mathbb{P}), \eta)$. We show that in dimension greater than three the hyperbolic Heisenberg group is the unique example of an integrable para-contact hermitian structure with flat canonical connection. In the three dimensional case the same statement holds under the additional assumption of vanishing of the torsion tensor. As a manifold $\boldsymbol{G}\,(\mathbb{P}) = \mathbb{R}^{2n} \times \mathbb{R}$ with the group law given by $(p'', t'') = (p', t') \circ (p, t) = (p' + p, t' + t - \sum_{k=1}^{n}(u'_k v_k - v'_k u_k)$, where $p', p \in \mathbb{R}^{2n}$ with the standard coordinates $(u_1, v_1, \ldots, u_n, v_n)$ and $t', t \in \mathbb{R}$. Define the 'standard' para-contact structure by the left-invariant para-contact form

$$\tilde{\Theta} = -\frac{1}{2}\,dt - \sum_{k=1}^{n}(u_k dv_k - v_k du_k).$$

In this paper we find a tensor invariant characterizing locally the integrable para-contact hermitian structures which are para-contact conformally equivalent to the flat structure on $\boldsymbol{G}\,(\mathbb{P})$. To any integrable para-contact hermitian structure we associate a curvature-type tensor $W^{pc}$ defined in terms of the curvature and torsion of the canonical connection by (5.12), whose form is similar to the Weyl conformal curvature in Riemannian geometry (see e.g. [6]) and to the Chern-Moser curvature in CR geometry [3]. We call $W^{pc}$ *para-contact conformal curvature or pc conformal curvature*. When $M$ is three dimensional, we define in (5.13) a symmetric (0,2) tensor $F$ on $\mathbb{H}$, which plays a role similar to the Schouten tensor in a 3-dimensional locally conformally flat Riemannian manifold.

The main purpose of this article is to prove the following two results.

**Theorem 1.1** *The pc conformal curvature $W^{pc}$ of an integrable para-contact hermitian manifold is invariant under para-contact conformal transformations.*

**Theorem 1.2** *Let $(M, \eta)$ be a $2n + 1$ dimensional integrable para-contact hermitian manifold.*

(i) *If $n > 1$ then $(M, \eta)$ is locally para-contact conformal to the standard flat para-contact hermitian structure on the hyperbolic Heisenberg group $\boldsymbol{G}(\mathbb{P})$ if and only if the para-contact conformal curvature vanishes, $W^{pc} = 0$.*
(ii) *If $n = 1$ then $W^{pc}$ vanishes identically and $(M, \eta)$ is locally para-contact conformal to the standard flat para-contact hermitian structure on the 3-dimensional hyperbolic Heisenberg group $\boldsymbol{G}(\mathbb{P})$ if and only if the symmetric tensor $F$ vanishes, $F = 0$.*

We define a Cayley transform which establishes a conformal para-contact equivalence between the standard para-Sasaki structure on the hyperboloid, cf. 4.2,

$$\begin{aligned} HS^{2n+1} = \big\{&(x_1, y_1, \dots, x_{n+1}, y_{n+1}) : x_1^2 + \cdots \\ &+x_{n+1}^2 - y_1^2 - \cdots - y_{n+1}^2 = 1\big\} \subset \mathbb{R}^{2n+2} \end{aligned} \tag{1.1}$$

and the standard para-contact hermitian structure on $\boldsymbol{G}(\mathbb{P})$. As a consequence of Theorem 1.2 and the fact that the Cayley transform is a para-contact conformal equivalence between the hyperboloid and the group $\boldsymbol{G}(\mathbb{P})$, we obtain

**Corollary 1.3** *Let $(M, \eta)$ be a $2n + 1$ dimensional integrable para-contact hermitian manifold. $(M, \eta)$ is locally para-contact conformal to the hyperboloid $HS^{2n+1}$ if and only if conditions i) or ii) of Theorem 1.2 hold.*

Our investigations are close to the classical approach used by Weyl (see e.g. [6]) and follow the steps of [8], compare with [3] where the Cartan method of equivalence is used. Recall, that in the CR case the vanishing of the Chern-Moser tensor is a necessary and sufficient condition for a non-degenerate CR manifold M of dimension $2n + 1, n > 1$, to be locally equivalent to a real hyperquadric in $\mathbb{C}^{n+1}$ of the same signature as M. When $M$ is three dimensional, the same conclusion can be reached using the Cartan invariant [2]. Both results can be obtained following the steps of the proof of Theorem 1.2. In particular, we express the flatness condition for an abstract three-dimensional pseudohermitian structure in terms of the covariant derivatives of the pseudohermitian scalar and torsion of the Webster connection.

Let us note that, as observed, the hyperboloid $HS^{2n+1}$ is always para-contact conformally flat, while the hyperboloid $HS^{4n+1}$ considered as an embedded CR submanifold of $\mathbb{C}^{2n+1}$ is a degenerate CR manifold.

In the last section we consider the CR-Yamabe equation on a CR manifold of neutral signature. This leads to the non-linear sub ultra-hyperbolic Eq. (8.1), which coincides with the Yamabe equation for the considered para CR manifolds. Using this relation we show an explicit formula for the regular part of solutions to the Yamabe equation.

The paper uses a Webster-like connection, the canonical connection considered in [19] and the properties of its torsion and curvature described in Sect. 3.

**Convention 1.4** *In the first six sections of the paper we use*

(a) $X, Y, Z...$ *denote horizontal vector fields, i.e.* $X, Y, Z... \in \mathbb{H}$
(b) $\{e_1, \dots, e_n, Ie_1, \dots, Ie_n\}$ *denotes an adapted orthonormal basis of the horizontal space* $\mathbb{H}$.
(c) *The summation convention over repeated vectors from the basis* $\{e_1, \dots, e_{2n}\}$ *will be used. For example, for a* (0,4)*-tensor* $P$ *we have*

$$P(e_b, X, Y, e_b) = \sum_{b=1}^{n} g(e_b, e_b) P(e_b, X, Y, e_b) + \sum_{b=1}^{n} g(Ie_b, Ie_b) P(Ie_b, X, Y, Ie_b).$$

## 2 Integrable para-contact manifolds

A para-contact manifold $(M^{2n+1}, \eta, I, g)$ is a (2n+1)-dimensional smooth manifold equipped with a codimension one distribution $\mathbb{H}$, locally given as the kernel of a 1-form $\eta$, $\mathbb{H} = Ker\, \eta$ and a paracomplex structure $I$ on $\mathbb{H}$. Recall that a paracomplex structure is an endomorphism $I$ satisfying $I^2 = id$ and the $\pm$ eigen-distributions have equal dimension. If in addition there exists a pseudo-Riemannian metric $g$ defined on $\mathbb{H}$ compatible with $\eta$ and $I$ in the sense that

$$g(IX, IY) = -g(X, Y), \quad d\eta(X, Y) = 2g(IX, Y), \quad X, Y \in \mathbb{H}, \tag{2.1}$$

we have para contact hermitian manifold. The signature of $g$ restricted to $\mathbb{H}$ is necessarily neutral of type $(n, n)$.

The para-contact Reeb vector field $\xi$ (of length $-1$) is the dual vector field to $\eta$ via the metric $g$, $g(X, \xi) = \eta(X)$, $\eta(\xi) = -1$ satisfying $d\eta(\xi, .) = 0$. The metric $g$ extends to the metric in the whole manifold by requiring $g(\xi, \xi) = -1$. In addition, the 1-form $\eta$ is a contact form and the fundamental 2-form is defined by

$$2\omega(X, Y) = 2g(IX, Y) = d\eta(X, Y). \tag{2.2}$$

The paracomplex structure $I$ on $\mathbb{H}$ is formally integrable [19] if the $\pm$ eigen-distrubutions $\mathbb{H}^{\pm}$ of $I$ in $\mathbb{H}$ are formally integrable in the sense that $[\mathbb{H}^{\pm}, \mathbb{H}^{\pm}] \in \mathbb{H}^{\pm}$. Using the Nijenhuis tensor $N(X, Y) = [IX, IY] + [X, Y] - I[IX, Y] - I[X, IY]$, the formal integrability of $I$ is equivalent to

$$N(X, Y) = 0 \quad \text{and} \quad [IX, Y] + [X, IY] \in \mathbb{H}. \tag{2.3}$$

A para-contact manifold is called para-sasakian if $N(X, Y) = d\eta(X, Y)\xi$.

### 2.1 The canonical connection

The canonical para-contact connection $\nabla$ on a para-contact hermitian manifold defined in [19] is similar to the Webster connection in the pseudohermitian case. We summarize the properties of $\nabla$ on an integrable para-contact hermitian manifold from [19]).

**Theorem 2.1** [19] *On an integrable para-contact hermitian manifold* $(M, \eta, I, g)$ *there exists a unique linear connection preserving the integrable para-contact hermitian structure, i.e.*

$$\nabla\xi = \nabla I = \nabla\eta = \nabla g = 0 \tag{2.4}$$

*with torsion tensor* $T(A, B) = \nabla_A B - \nabla_B A - [A, B]$ *given by*

$$T(X, Y) = -d\eta(X, Y)\xi = -2\omega(X, Y)\xi, \quad T(\xi, X) \in \mathbb{H}, \tag{2.5}$$

$$g(T(\xi, X), Y) = g(T(\xi, Y), X) = g(T(\xi, IX), IY) = \frac{1}{2}\mathcal{L}_\xi g(X, Y). \tag{2.6}$$

It is shown in [19] that the endomorphism $T(\xi, .)$ is the obstruction an integrable para-contact hermitian manifold to be parasasakian. We denote the symmetric endomorphism $T_\xi : \mathbb{H} \longrightarrow \mathbb{H}$ by $\tau$ and call it *the torsion of the integrable para-contact hermitian manifold.* It follows that the torsion $\tau$ is completely trace-free [19], i.e.

$$\tau(e_a, e_a) = \tau(e_a, Ie_a) = 0. \tag{2.7}$$

## 3 The Bianchi identities

Let $R = [\nabla, \nabla] - \nabla_{[,]}$ be the curvature of the canonical connection $\nabla$. We shall also denote with $R$ the corresponding (0,4) tensor defined with the help of the metric $g$. The Ricci tensor $r$, the Ricci 2-form $\rho$ and the pc-scalar curvature Scal of $\nabla$ are defined, respectively, by

$$\begin{aligned} &r(A, B) = R(e_a, A, B, e_a), \quad \rho(A, B) = \frac{1}{2}R(A, B, e_a, Ie_a), \\ &\text{Scal} = r(e_a, e_a), \quad A, B \in \Gamma(TM). \end{aligned}$$

**Proposition 3.1** *Let* $(M, \eta, I, g)$ *be an integrable para-contact hermitian manifold. Then:*

(i) *The curvature of the canonical connection has the properties:*

$$\begin{aligned} &R(X, Y, IZ, IV) = -R(X, Y, Z, V), \\ &R(X, Y, Z, V) = -R(X, Y, V, Z), \quad R(X, Y, Z, \xi) = 0. \end{aligned} \tag{3.1}$$

$$\begin{aligned} &R(X, Y, Z, V) + R(IX, IY, Z, V) \\ &= 2\,[g(X, Z)\tau(Y, IV) + g(Y, V)\tau(X, IZ) - g(Y, Z)\tau(X, IV) \\ &\quad - g(X, V)\tau(Y, IZ)] \\ &\quad + 2\,[\omega(X, Z)\tau(Y, V) + \omega(Y, V)\tau(X, Z) - \omega(Y, Z)\tau(X, V) \\ &\quad - \omega(X, V)\tau(Y, Z)]; \end{aligned} \tag{3.2}$$

$$R(\xi, X, Y, Z) = (\nabla_Y \tau)(Z, X) - (\nabla_Z \tau)(Y, X). \tag{3.3}$$

(ii) *The horizontal Ricci tensor is symmetric,* $r(X, Y) = r(Y, X)$ *and has the property*

$$r(X, Y) + r(IX, IY) = 4(1 - n)\tau(X, IY). \tag{3.4}$$

(iii) *The horizontal Ricci 2-from satisfies the relations*

$$2\rho(X, IY) = r(X, Y) - r(IX, IY) = R(e_a, Ie_a, X, IY). \tag{3.5}$$

(iv) *The following differential identity holds*

$$2(\nabla_{e_a} r)(e_a, X) = dScal(X). \tag{3.6}$$

*Proof* Equation (2.4) implies immediately (3.1). The first Bianchi identity

$$\sum_{(A,B,C)} \{R(A, B)C - (\nabla_A T)(B, C) - T(T(A, B), C)\} = 0, \quad A, B, C \in \Gamma(TM). \tag{3.7}$$

together with (2.5) and (2.6) yield

$$R(X,Y,Z,V)-R(Z,V,X,Y)=2\omega(X,Z)\tau(Y,V)+2\omega(Y,V)\tau(X,Z)$$
$$-2\omega(Y,Z)\tau(X,V)-2\omega(X,V)\tau(Y,Z). \quad (3.8)$$
$$R(\xi,X,Y,Z)-R(Y,Z,\xi,X)=(\nabla_Y\tau)(Z,X)-(\nabla_Z\tau)(Y,X). \quad (3.9)$$

Combining (3.1) with (3.8) and (3.9) we obtain (3.2) and (3.3).

When we take the trace of (3.8), use (2.6) and (2.7) we find

$$r(Y,Z)-r(Z,Y)=2\omega(e_a,Z)\tau(Y,e_a)+2\omega(Y,e_a)\tau(e_a,Z)$$
$$=-2\tau(IZ,Y)+2\tau(Y,IZ)=0.$$

Furthermore, (3.1) and (3.2) imply

$$r(Y,Z)+r(IY,IZ)=R(e_a,Y,Z,e_a)+R(Ie_a,Y,Z,Ie_a)+4(1-n)\tau(Y,IZ)$$
$$=4(1-n)\tau(Y,IZ).$$

The first Bianchi identity (3.7) together with (2.4) and (2.6) yields

$$2\rho(X,IY)=r(X,Y)-r(IY,IX)+2\tau(X,IY)-2\tau(IY,X)$$
$$=r(X,Y)-r(IX,IY).$$

The second Bianchi identity reads

$$\sum_{(A,B,C)}\{(\nabla_A R)(B,C,D,E)+R(T(A,B),C,D,E)\}=0,$$
$$A,B,C,D\in\Gamma(TM). \quad (3.10)$$

A suitable trace of (3.10) leads to

$$(\nabla_{e_a}R)(X,Y,Z,e_a)-(\nabla_X r)(Y,Z)+(\nabla_Y r)(X,Z)+2R(\xi,Y,Z,IX)$$
$$-2R(\xi,X,Z,IY)+2\omega(X,Y)r(\xi,Z)=0. \quad (3.11)$$

The trace of (3.11) gives $2(\nabla_{e_a}r)(X,e_a)-d\mathrm{Scal}(X)+4r(\xi,IX)-4\rho(\xi,X)=0$ while Eq. (3.3) implies $r(\xi,IX)=(\nabla_{e_a}\tau)(IX,e_a)=\rho(\xi,X)$. Now, the identity (3.6) follows from the last two equalities. □

## 4 Basic examples

Let $\{x_1,y_1,\dots,x_{n+1},y_{n+1}\}$ be the standard coordinate system in $\mathbb{R}^{2n+2}$. The standard parahermitian structure $(\mathbb{I},g)$ is defined by

$$\mathbb{I}\frac{\partial}{\partial x_j}=\frac{\partial}{\partial y_j},\quad \mathbb{I}\frac{\partial}{\partial y_j}=\frac{\partial}{\partial x_j},\quad g\left(\frac{\partial}{\partial x_j},\frac{\partial}{\partial x_k}\right)=-g\left(\frac{\partial}{\partial y_j},\frac{\partial}{\partial y_k}\right)=\delta_{jk},$$
$$g\left(\frac{\partial}{\partial x_j},\frac{\partial}{\partial y_k}\right)=0,$$

where $j,k=1,\dots,n$. Recall that a smooth map $f=(u_1,v_1,\dots,u_n,v_n):\mathbb{R}^{2n+2}\longrightarrow\mathbb{R}^{2n+2}$ preserves the paracomplex structure $\mathbb{I}$ iff it is *paraholomorphic*, i.e., satisfies the (para) Cauchy-Riemann equations, see e.g.[12], $df\circ\mathbb{I}=\mathbb{I}\circ df$, or,

$$\frac{\partial u_k}{\partial x_j}=\frac{\partial v_k}{\partial y_j},\quad \frac{\partial u_j}{\partial y_j}=\frac{\partial v_k}{\partial x_j}. \quad (4.1)$$

Let $(\mathbb{R}^{2n+2}, \mathbb{I}, g)$ be the standard flat parahermitian structure on $\mathbb{R}^{2n+2}$ and $M^{2n+1}$ be a hypersurface with unit normal $N$ such that $T_pM^{2n+1} \oplus N = R^{2n+2}$, $p \in M^{2n+1}$. Consider the vector field $\xi := \mathbb{I}N$, the dual 1-form $\eta(\xi) = -1$ and denote $\mathbb{H} = \xi^{\perp} = Ker\,\eta$. A para CR-structure on $M$ is defined by $(\mathbb{H}, I = \mathbb{I}_{|\mathbb{H}})$. Moreover

$$d\eta(X, Y) = -\eta([X, Y]) = -d\eta(IX, IY)$$

in view of the integrability condition (2.3). If in addition $d\eta_{|\mathbb{H}}$ is non-degenerate then it necessarily has signature $(n, n)$ and $(M, \eta)$ is an integrable para-contact hermitian manifold.

**Proposition 4.1** *Any non-degenerate hypersurface in $(\mathbb{R}^{2n+2}, \mathbb{I}, g)$ admits an integrable para-contact hermitian structure.*

Since the horizontal space $\mathbb{H}$ is invariant under the standard paracomplex structure of $\mathbb{R}^{2n+2}$, a restriction of a paraholomorphic map $f : R^{2n+2} \longrightarrow R^{2n+2}$ on $(M^{2n+1}, \eta)$ induces a para conformal transformation of the embedded paracontact hermitian structure $\bar{\eta} = \mu\eta$ on the hypersurface $M^{2n+1}$.

## 4.1 Hyperbolic Heisenberg group

The hyperbolic Heisenberg group is the example of an integrable para-contact hermitian structure with flat canonical connection. The difference between this group and the standard Heisenberg group is in the metric, while the groups are identical. As a group $\boldsymbol{G}(\mathbb{P}) = \mathbb{R}^{2n} \times \mathbb{R}$ with the group law given by

$$(p'', t'') = (p', t') \circ (p, t) = \left(p' + p, t' + t - \sum_{k=1}^{n} \left(u'_k v_k - v'_k u_k\right)\right).$$

where $p', p \in \mathbb{R}^{2n}$, $t', t \in \mathbb{R}$, $p = (u_1, v_1, \ldots, u_n, v_n)$ and $p' = (u'_1, v'_1, \ldots, u'_n, v'_n)$. A basis of left-invariant vector fields is given by $U_k = \frac{\partial}{\partial u_k} - 2v_k\frac{\partial}{\partial t}$, $V_k = \frac{\partial}{\partial v_k} + 2u_k\frac{\partial}{\partial t}$, $\xi = 2\frac{\partial}{\partial t}$. Define $\tilde{\Theta} = -\frac{1}{2}dt - \sum_{k=1}^{n}(u_k dv_k - v_k du_k)$ with corresponding horizontal bundle $\mathbb{H}$ given by the span of the left-invariant horizontal vector fields $\{U_1, \ldots U_n, V_1 \ldots V_n, \}$. We consider an endomorphism on $\mathbb{H}$ by defining $IU_k = V_k, IV_k = U_k$, hence $I^2 = \mathrm{Id}$ on $\mathbb{H}$ and $I$ is a paracomplex structure on $\mathbb{H}$. The form $\tilde{\Theta}$ and the para-complex structure $I$ (on $\mathbb{H}$) define a para-contact manifold, which will be called the hyperbolic Heisenberg group. Note that by definition $\{U_1, ...U_n, V_1...V_n, \xi\}$ is an orthonormal basis of the tangent space, $g(U_j, U_j) = -g(V_j, V_j) = 1$, $j = 1, ..., n$.

**Theorem 4.2** *Let $(M, \eta, I, g)$ be an integrable para-contact hermitian manifold of dimension $2n + 1$.*

(i) *If $n > 1$ then $(M, \eta, I, g)$ is locally isomorphic to the hyperbolic Heisenberg group exactly when the canonical connection has vanishing horizontal curvature, $R(X, Y, Z, V) = 0$,*
(ii) *If $n = 1$ then $(M, \eta, I, g)$ is locally isomorphic to the 3-dimensional hyperbolic Heisenberg group exactly when the canonical connection has vanishing horizontal curvature and zero torsion.*

*Proof* It is easy to see that the canonical connection on the hyperbolic Heisenberg group is the left-invariant connection on the group which is flat and with zero torsion endomorphism. For the converse, we first show that if $n > 1$ and the horizontal curvature vanishes then the canonical connection is flat and with zero torsion endomorphism, $R = \tau = 0$. Indeed, (3.4) yields $\tau = 0$ and (3.3) shows $R(\xi, X, Y, Z) = 0$.

Let $\{e_1, \dots, e_n, Ie_1, \dots, Ie_n, \xi\}$ be a local basis parallel with respect to $\nabla$. Then (2.5) and (2.6) show that $M$ has the structure of the Lie algebra of the hyperbolic Heisenberg group, which proves the claim. □

4.2 Hyperboloid of neutral signature.

Let $\{x_0, y_0, \dots, x_n, y_n\}$ be the standard coordinate system in $(\mathbb{R}^{2n+2}, \mathbb{I}, g)$. Consider the hypersurface

$$HS^{2n+1} = \left\{(x_0, y_0, \dots, x_n, y_n) \subset \mathbb{R}^{2n+2} \mid x_0^2 + \dots + x_n^2 - y_0^2 \dots - y_n^2 = 1\right\}.$$

$HS^{2n+1}$ carries a natural para-CR structure inherited from its embedding in $(\mathbb{R}^{2n+2}, \mathbb{I}, g)$. The horizontal bundle $\mathbb{H}$ is the maximal subspace of the tangent space of $HS^{2n+1}$ which is invariant under the (restriction of the) action of $\mathbb{I}$. We take

$$\tilde{\eta} = -\sum_{j=0}^{n} \left(x_j dy_j - y_j dx_j\right)$$

noting that here $N = \sum_{j=0}^{n} \left(x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}\right)$ and $\xi = \sum_{j=0}^{n} \left(x_j \frac{\partial}{\partial y_j} + y_j \frac{\partial}{\partial x_j}\right)$. We will also consider $HS^{2n+1}$ as the boundary of the "ball" $B = \{(x_0, y_0, \dots, x_n, y_n) \subset \mathbb{R}^{2n+2} : x_0^2 + \dots + x_n^2 - y_0^2 \dots - y_n^2 < 1\}$.

4.3 The Cayley transform

Let $\Sigma_0 = \{(x_0, y_0, \dots, x_n, y_n) \in HS^{2n+1} : (1+x_0)^2 = y_0^2\}$. The Cayley transform (centered at $\Sigma_0$), is defined as follows

$$\mathcal{C} : HS^{2n+1} \setminus \Sigma_0 \to \boldsymbol{G}(\mathbb{P})$$
$$t = \frac{2y_0}{(1+x_0)^2 - y_0^2}, \qquad u_k = \frac{x_k(1+x_0) - y_k y_0}{(1+x_0)^2 - y_0^2}, \qquad v_k = \frac{y_k(1+x_0) - x_k y_0}{(1+x_0)^2 - y_0^2}. \tag{4.2}$$

A small calculation shows

$$\mathcal{C}^* \tilde{\Theta} = \frac{1}{(1+x_0)^2 - y_0^2} \tilde{\eta}.$$

Furthermore, the para-complex structure is preserved. In order to see the last claim, we can consider $G(\mathbb{P})$ as the boundary of the domain $D = \left\{(u_0, v_0, \dots, u_n, v_n) \subset \mathbb{R}^{2n+2} : u_1^2 + \dots + u_n^2 - v_1^2 \dots - v_n^2 < v_0\right\}$ by identifying the point $(p, t) \in G(\mathbb{P})$ with the point $(t, \Sigma_{k=1}^n \left(u_k^2 - v_k^2\right), u_1, v_1 \dots, u_n, v_n) \in \partial D$ and define the diffeomorphism $\mathcal{C} : B \setminus \Sigma_0 \to D \setminus \Xi_0$, $\Xi_0 = \left\{(1+u_0)^2 - v_0^2 = 0\right\}$,

$$u_0 = \frac{2y_0}{(1+x_0)^2 - y_0^2}, \qquad v_0 = \frac{1 - x_0^2 + y_0^2}{(1+x_0)^2 - y_0^2}$$
$$u_k = \frac{x_k(1+x_0) - y_k y_0}{(1+x_0)^2 - y_0^2}, \qquad v_k = \frac{y_k(1+x_0) - x_k y_0}{(1+x_0)^2 - y_0^2}.$$

A calculation shows that the above map is para-holomorphic, i.e., the coordinate functions satisfy the (para) Cauchy-Riemann equations (4.1). Thus, $\mathcal{C}$ preserves the para-CR structure when considered as a map between the boundaries of $B$ and $D$.

Using hyperbolic rotations, which preserve the para-contact structure, and Cayley maps similar to the above we see that the hyperboloid is locally para-contact conformal to the hyperbolic Heisenberg group.

Finally, it is worth noting that according to Theorem 1.2 the hyperboloid $HS^{4n+1}$is para-contact conformally flat, while regarded as a CR submanifold of $\mathbb{C}^{2n+1}$ it is a degenerate CR manifold.

## 5 Paracontact conformal curvature

In this section we define para-contact conformal invariant and prove Theorem 1.1.

### 5.1 Paracontact conformal transformations

A conformal para-contact transformation (*pc transformartion* ) between two para-contact manifold is a diffeomorphism $\Phi$ which preserves the para-contact structure i.e. $\Phi^*\eta = \mu\eta$, for a nowhere vanishing smooth function $\mu$.

Let $u$ be a smooth nowhere vanishing function on a para-contact manifold $(M, \eta)$. Let $\bar\eta = \frac{1}{2}e^{-2u}\eta$ be a conformal deformation of $\eta$. We will denote the objects related to $\bar\eta$ by over-lining the same object corresponding to $\eta$. Thus, $d\bar\eta = -e^{-2u}du\wedge\eta + \frac{1}{2}e^{-2u}d\eta,\ \bar g = \frac{1}{2}e^{-2u}g$. The new para-contact Reeb vector field $\bar\xi$ is [19]

$$\bar\xi \ = \ 2e^{2u}\,\xi \ + 2e^{2u}\ I\nabla u, \tag{5.1}$$

where $\nabla u$ is the horizontal gradient defined by $g(\nabla u, X) = du(X), \quad X \in \mathbb{H}$. The horizontal sub-Laplacian and the norm of the horizontal gradient are defined respectively by $\triangle u = tr^g_H(\nabla du) = \nabla du(e_a, e_a) = \sum_{s=1}^n(\nabla du(e_s, e_s) - \nabla du(Ie_s, Ie_s)),\ |\nabla u|^2 = du(e_a)^2 = \sum_{s=1}^n(du(e_s)^2 - du(Ie_s)^2)$. The canonical para-contact connections $\nabla$ and $\bar\nabla$ are related by a (1,2) tensor S,

$$\bar\nabla_A B = \nabla_A B + S_A B, \qquad A, B \in \Gamma(TM). \tag{5.2}$$

Suppose the para contact structure is integrable. The conditions (2.5) and $\bar\nabla\bar g = 0$ determine $g(S(X, Y), Z)$ for $X, Y, Z \in \mathbb{H}$ due to the equality

$$\begin{aligned} g(S(X, Y), Z) = &- du(X)g(Y, Z) - du(IX)\omega(Y, Z) - du(Y)g(Z, X)\\ &+ du(IY)\omega(Z, X) + du(Z)g(X, Y) + du(IZ)\omega(X, Y). \end{aligned} \tag{5.3}$$

We obtain after some calculations using (5.1) that

$$\bar\tau(X, Y) - 2e^{2u}\tau(X, Y) - g(S(\bar\xi, X), Y) = -2e^{2u}\nabla du(X, IY) - 4e^{2u}du(X)du(IY). \tag{5.4}$$

From (5.4) and (2.6) we find

$$\begin{aligned} g(S(\bar\xi, X), Y) - g(S(\bar\xi, IX)IY) = 2e^{2u}\,[&\nabla du(X, IY) - \nabla du(IX, Y)\\ &+ 2du(X)du(IY) - 2du(IX)du(Y)]\,. \end{aligned} \tag{5.5}$$

The condition $\bar{\nabla} I = \nabla I = 0$ yield $g(S(\bar{\xi}, X), Y) = -g(S(\bar{\xi}, IX)IY)$. Substitute the latter into (5.4) and (5.5), use (5.1) and (5.3) to get

$$g(S(\xi, X), Y) = \frac{1}{2}\left[\nabla du(X, IY) - \nabla du(IX, Y)\right] \\ - du(X)du(IY) + du(IX)du(Y) + |\nabla u|^2 \omega(X, Y), \tag{5.6}$$

$$\bar{\tau}(X, Y) = e^{2u}\left[2\tau(X, Y) - \nabla du(X, IY) - \nabla du(IX, Y) \\ - 2du(X)du(IY) - 2du(IX)du(Y)\right]. \tag{5.7}$$

In addition, the pc-scalar curvature changes according to the formula [19]

$$\overline{\mathrm{Scal}} = 2e^{2u}\,\mathrm{Scal} - 8n(n+1)e^{2u}|\nabla u|^2 + 8(n+1)e^{2u}\triangle u. \tag{5.8}$$

The identity $d^2 = 0$ yields $\nabla du(X, Y) - \nabla du(Y, X) = -du(T(X, Y))$. Applying (2.5), we can write

$$\nabla du(X, Y) = [\nabla du]_{[sym]}(X, Y) + du(\xi)\omega(X, Y), \tag{5.9}$$

where $[.]_{[sym]}$ denotes the symmetric part of the corresponding (0,2)-tensor.

### 5.2 Paracontact conformal curvature tensor

Let $(M, \eta, I, g)$ be a (2n+1)-dimensional integrable para-contact hermitian manifold. Let us consider the symmetric (0,2) tensor $L$ defined on $\mathbb{H}$ by the equality

$$L(X, Y) = \frac{1}{2(n+2)}\rho(X, IY) - \tau(IX, Y) \\ - \frac{\mathrm{Scal}}{8(n+1)(n+2)}g(X, Y), \qquad X, Y \in \mathbb{H}. \tag{5.10}$$

We define the (0,4) tensor $PW$ on $\mathbb{H}$ by

$$g(PW(X, Y)Z, V) = g(R(X, Y)Z, V) + g(X, Z)L(Y, V) + g(Y, V)L(X, Z) \\ - g(Y, Z)L(X, V) - g(X, V)L(Y, Z) + \omega(X, Z)L(Y, IV) \\ + \omega(Y, V)L(X, IZ) - \omega(Y, Z)L(X, IV) - \omega(X, V)L(Y, IZ) \\ + \omega(X, Y)\left[L(Z, IV) - L(IZ, V)\right] \\ + \omega(Z, V)\left[L(X, IY) - L(IX, Y)\right]. \tag{5.11}$$

**Proposition 5.1** *The tensor $PW$ is completely trace-free, i.e.*

$$r(PW) = \rho(PW) = 0.$$

*Proof* The claim follows after taking the corresponding traces in (5.11) keeping in mind (5.10). □

If we compare (5.11) and (3.3) we obtain the following

**Proposition 5.2** *For $n > 1$ the tensor $PW$ has the properties*

$$PW(X, Y, Z, V) + PW(IX, IY, Z, V) = 0,$$

$$PW(X,Y,Z,V) - PW(IX,IY,Z,V) = R(X,Y,Z,V) - R(IX,IY,Z,V)$$
$$+ \frac{Scal}{2(n+1)(n+2)} [\omega(X,Z)\omega(Y,V) - \omega(Y,Z)\omega(X,V) + 2\omega(X,Y)\omega_s(Z,V)]$$
$$- \frac{Scal}{2(n+1)(n+2)} [g(X,Z)g(Y,V) - g(Y,Z)g(X,V)] + \frac{2}{n+2} [\omega(X,Y)\rho(Z,V)$$
$$+ \omega(Z,V)\rho(X,Y)] + \frac{1}{n+2} [g(X,Z)\rho(Y,IV) - g(Y,Z)\rho(X,IV)$$
$$+ g(Y,V)\rho(X,IZ) - g(X,V)\rho(Y,IZ)] + \frac{1}{n+2} [\omega(X,Z)\rho(Y,V)$$
$$- \omega(Y,Z)\rho(X,V) + \omega(Y,V)\rho(X,Z) - \omega(X,V)\rho(Y,Z)]. \tag{5.12}$$

*For $n = 1$ the tensor $PW$ vanishes identically.*

**Definition 5.3** We denote the tensor $PW(X,Y,Z,V) - PW(IX,IY,Z,V)$ by $2W^{pc}$ and call it *the para-contact conformal curvature*.

If $n = 1$ we define on $\mathbb{H}$ the following symmetric (0,2) tensor $F$ by the equality

$$F(X,Y) = (\nabla d(Scal))(X,IY) + (\nabla d(Scal))(Y,IX) + 16(\nabla^2_{Xe_a}\tau)(Y,e_a)$$
$$+ 16(\nabla^2_{Ye_a}\tau)(X,e_a) - 48(\nabla^2_{e_aIe_a}\tau)(X,IY) + 36Scal\tau(X,Y)$$
$$+ 3g(X,Y)(\nabla d(Scal))(e_aIe_a). \tag{5.13}$$

## 5.3 Proof of Theorem 1.1

First we show

**Theorem 5.4** *The para-contact conformal curvature $W^{pc}$ of an integrable para-contact hermitian manifold is invariant under conformal para-contact transformations, i.e., if*

$$2\bar\eta = e^{-2u}\eta \quad \textit{for any smooth function } u \textit{ then } 2e^{2u}W^{pc}_{\bar\eta} = W^{pc}_{\eta}.$$

*Proof* After a straightforward computation using (5.2), (5.3) and (5.6) we obtain the formula

$$2e^{2u}g(\bar R(X,Y)Z,V) - g(R(X,Y)Z,V) = -g(Z,V)[M(X,Y) - M(Y,X)]$$
$$- g(X,Z)M(Y,V) - g(Y,V)M(X,Z) + g(Y,Z)M(X,V) + g(X,V)M(Y,Z)$$
$$- \omega(X,Z)M(Y,IV) - \omega(Y,V)M(X,IZ) + \omega(Y,Z)M(X,IV) + \omega(X,V)M(Y,IZ)$$
$$- \omega(X,Y)[M(Z,IV) - M(IZ,V)] - \omega(Z,V)[M(X,IY) - M(Y,IX)]. \tag{5.14}$$

where the (0,2) tensor $M$ is given by

$$M(X,Y) = \nabla du(X,Y) + du(X)du(Y) + du(IX)du(IY) - \frac{1}{2}g(X,Y)|\nabla u|^2. \tag{5.15}$$

Let $trM = M(e_a,e_a)$ be the trace of the tensor $M$. Using (5.15) and (5.9) we obtain

$$trM = \triangle u - n|\nabla u|^2, \quad M(X,Y) + M(IX,IY) = M(Y,X) + M(IY,IX), \tag{5.16}$$
$$M(e_a,Ie_a) = -2ndu(\xi), \quad M(e_a,Ie_a)\omega(X,Y) = -n[M(X,Y) - M(Y,X)]. \tag{5.17}$$

Taking the trace in (5.14) and using (5.15), (5.16), and (5.17) we come to

$$\bar r(X,Y) - r(X,Y) = (n+1)M(X,Y) + nM(Y,X) - M(IX,IY)$$
$$- 2M(IY,IX) + trM\, g(X,Y); \tag{5.18}$$
$$e^{-2u}\overline{Scal} - 2\,\mathrm{Scal} = 8(n+1)trM.$$

Proposition 3.1 together with (5.18) and (5.10) imply

$$M_{[sym]}(X,Y)=\overline{L}(X,Y)-L(X,Y). \tag{5.19}$$

Now, from (5.15) and (5.9) we obtain

$$M(X,Y)=M_{[sym]}(X,Y)+du(\xi)\omega(X,Y). \tag{5.20}$$

Substituting (5.19) into (5.20), then inserting the obtained equality in (5.14) and finally using (5.16) allows us to complete the proof of Theorem 5.4. □

At this point a combination of Theorem 5.4 and Proposition 5.2 ends the proof of Theorem 1.1.

## 6 Converse problem. Proof of Theorem 1.2

Suppose $W^{pc}=0$, hence $PW=0$ by Proposition 5.2. We shall show that in this case there exists (locally) a smooth conformal factor $u$ which changes by a pc conformal transformation the integrable para-contact hermitian structure to a torsion-free flat one.

Consider the following system of differential equations with respect to the unknown function $u$

$$\begin{aligned}\nabla du(X,Y)&=-L(X,Y)-du(X)du(Y)-du(IX)du(IY)\\&\quad+\frac{1}{2}g(X,Y)|\nabla u|^2+du(\xi)\omega(X,Y)\end{aligned} \tag{6.1}$$

$$\nabla du(X,\xi)=-\mathbb{B}(X,\xi)-L(X,I\nabla u)+\frac{1}{2}du(IX)|\nabla u|^2-du(X)du(\xi_i) \tag{6.2}$$

$$\nabla du(\xi,\xi)=-\mathbb{B}(\xi,\xi)-\mathbb{B}(I\nabla u,\xi)-\frac{1}{4}|\nabla u|^4-(du(\xi))^2, \tag{6.3}$$

where $\mathbb{B}(X,\xi)$ and $\mathbb{B}(\xi,\xi)$ do not depend on the function $u$ and are determined in (6.7) and (6.22).

In order to prove Theorem 1.2 it is sufficient to show the existence of a local smooth solution to (6.1). Indeed, suppose $u$ is a local smooth solution to (6.1). Then the canonical connection of the para-contact hermitian structure $2\bar{\eta}=e^{-2u}\eta$ has in view of (5.7) zero torsion. Furthermore, the curvature restricted to $\mathbb{H}$ vanishes when $W^{pc}=0$ taking into account Proposition 5.2 and the proof of Theorem 5.4. Therefore, we can apply Theorem 4.2 to conclude the result.

The rest of this section is devoted to showing the existence of a smooth solution to the system (6.1)–(6.3).

The integrability conditions for this overdetermined system are the Ricci identities,

$$\begin{aligned}\nabla du(A,B,C)-\nabla du(B,A,C)&=-R(A,B,C,\nabla u)-\nabla du((T(A,B),C),\\&\qquad A,B,C\in\Gamma(TM).\end{aligned} \tag{6.4}$$

We consider as separate cases the four possibilities for $A$, $B$ and $C$.

**Case 1** $[A,B,C\in\mathbb{H}]$. Invoking (2.5) we see that Eq. (6.4) on $\mathbb{H}$ has the form

$$\nabla du(Z,X,Y)-\nabla du(X,Z,Y)+R(Z,X,Y,\nabla u)-2\omega(Z,X)\nabla du(\xi,Y)=0, \tag{6.5}$$

Take a covariant derivative of (6.1) along $Z \in \mathbb{H}$, substitute in the obtained equality (6.1) and (5.10), anticommute the covariant derivatives, let $W^{pc} = 0$ in (5.11), and finally use (6.2) and (6.1) to see that the integrability condition (6.5) is

$$(\nabla_Z L)(X,Y) - (\nabla_X L)(Z,Y) = \omega(Z,Y)\mathbb{B}(X,\xi) - \omega(X,Y)\mathbb{B}(Z,\xi) + 2\omega(Z,X)\mathbb{B}(Y,\xi). \tag{6.6}$$

The 1-forms $\mathbb{B}(X,\xi)$ can be determined by taking traces in (6.6). Thus, we have

$$(\nabla_{e_a} L)(Ie_a, IX) = -(2n+1)\mathbb{B}(IX,\xi) \text{ and } (\nabla_X tr\, L) - (\nabla_{e_a} L)(e_a, X) = 3\mathbb{B}(IX,\xi). \tag{6.7}$$

Notice that the consistence of the first and second equalities in (6.7) is equivalent to (3.6).

**Lemma 6.1** *Suppose $W^{pc} = 0$ and the dimension is bigger than three. Then (6.6) holds.*

*Proof* Using (2.5), the second Bianchi identity (3.10) gives

$$(\nabla_{e_a} R)(X,Y,Z,e_a) - (\nabla_X r)(Y,Z) + (\nabla_Y r)(X,Z) + 2R(\xi,Y,Z,IX) - 2R(\xi,X,Z,IY) + 2\omega(X,Y)r(\xi,Z) = 0, \tag{6.8}$$

$$(\nabla_X \rho)(Y,Z) + (\nabla_Y \rho)(Z,X) + (\nabla_Z \rho)(X,Y) - 2\omega(X,Y)\rho(\xi,Z) - 2\omega(Y,Z)\rho(\xi,X) - 2\omega(Z,X)\rho(\xi,Y) = 0, \tag{6.9}$$

$$(\nabla_X \rho)(Y,Z) + (\nabla_{e_a} R)(Ie_a,X,Y,Z) + 2(n-1)R(\xi,X,Y,Z) = 0. \tag{6.10}$$

From $W^{pc} = 0$ and (5.10) we can express $r$, $\rho$ and $\tau$ in terms of $L$ and $tr\, L$, obtaing

$$r(X,Y) = (2n+1)L(X,Y) - 3L(IX,IY) + (trL)g(X,Y) \tag{6.11}$$

$$\rho(X,Y) = (n+2)L(X,IY) - (n+2)L(IX,Y) - (trL)\omega(X,Y) \tag{6.12}$$

$$2\tau(IX,Y) = -L(X,Y) - L(IX,IY). \tag{6.13}$$

Inserting (5.11) and (3.3) in (6.8), and then using (6.11), (6.13) we come after some standard calculations to the following identity

$$\begin{aligned}&-3g(Z,X)\mathbb{B}(IY,\xi) + 3g(Z,Y)\mathbb{B}(IX,\xi) - (2n+1)\omega(X,Z)\mathbb{B}(Y,\xi)\\ &+(2n+1)\omega(Y,Z)\mathbb{B}(X,\xi) - 2(2n+1)\omega(X,Y)\mathbb{B}(Z,\xi) + 2n\,[(\nabla_X L)(Y,Z)\\ &-(\nabla_Y L)(X,Z)] \quad + [(\nabla_{IZ} L)(X,IY) - (\nabla_{IZ} L)(IX,Y)] - [(\nabla_{IX} L)(IY,Z)\\ &-(\nabla_{IY} L)(IX,Z)] - 2\,[(\nabla_{IX} L)(Y,IZ) - (\nabla_{IY} L)(X,IZ)]\\ &-3\,[(\nabla_X L)(IY,IZ) - (\nabla_Y L)(IX,IZ)] = 0.\end{aligned} \tag{6.14}$$

A substitution of (5.11) and (3.3) in (6.9) together with (6.12) give

$$\begin{aligned}&(n+2)[(\nabla_X L)(Y,IZ) - (\nabla_Y L)(X,IZ)] - (n+2)[(\nabla_X L)(IY,Z) - (\nabla_Y L)(IX,Z)]\\ &+ (n+2)[(\nabla_Z L)(X,IY) - (\nabla_Z L)(IX,Y)] - 2(n+2)\,[\omega(X,Y)\mathbb{B}(IZ,\xi)\\ &+ \omega(Y,Z)\mathbb{B}(IX,\xi) + \omega(Z,X)\mathbb{B}(IY,\xi)] = 0.\end{aligned} \tag{6.15}$$

Take $IZ$ instead of $Z$ in (6.15), then set $IX$ and $IY$, correspondingly, for $X$ and $Y$ into the obtained result. Taking the sum of thus achieved equalities we derive

$$\begin{aligned}&[(\nabla_X L)(Y,Z) - (\nabla_Y L)(X,Z)] - [(\nabla_X L)(IY,IZ) - (\nabla_Y L)(IX,IZ)] + [(\nabla_{IX} L)\,(IY,Z)\\ &-(\nabla_{IY} L)(IX,Z)] - [(\nabla_{IX} L)(Y,IZ) - (\nabla_{IY} L)(X,IZ)] + 2g(Y,Z)\mathbb{B}(IX,\xi)\\ &- 2g(Z,X)\mathbb{B}(IY,\xi) + 2\omega(Y,Z)\mathbb{B}(X,\xi) - 2\omega(X,Z)\mathbb{B}(Y,\xi) = 0.\end{aligned} \tag{6.16}$$

Insert (5.11), (3.3) in (6.10) using (6.12), (6.13), replace $Y$ and $Z$ respectively with $IY$ and $IZ$ into the obtained equality and then take the sum of both equations to obtain

$$\begin{aligned}&(n-1)\left[(\nabla_{IZ}L)(X,IY)-(\nabla_{IY}L)(X,IZ)\right]\\ &\quad+(n-1)\left[(\nabla_{IZ}L)(IX,Y)-(\nabla_{IY}L)(IX,Z)\right]\\ &\quad+(n-1)\left[(\nabla_{Z}L)(X,Y)-(\nabla_{Y}L)(X,Z)\right]\\ &\quad+(n-1)\left[(\nabla_{Z}L)(IX,IY)-(\nabla_{Y}L)(IX,IZ)\right]=0. \end{aligned}\tag{6.17}$$

Substitute $X$ by $Z$, and $Z$ by $X$ in (6.17). The sum of the obtained equalities and (6.16) yield

$$\begin{aligned}&\left[(\nabla_X L)(Y,Z)-(\nabla_Y L)(X,Z)\right]+\left[(\nabla_{IX}L)(IY,Z)-(\nabla_{IY}L)(IX,Z)\right]-g(X,Z)\mathbb{B}(IY,\xi)\\ &\quad+g(Y,Z)\mathbb{B}(IX,\xi)-\omega(X,Z)\mathbb{B}(Y,\xi)+\omega(Y,Z)\mathbb{B}(X,\xi)=0. \end{aligned}\tag{6.18}$$

The cyclic sum in (6.18) gives

$$\begin{aligned}\left[(\nabla_{IZ}L)(X,IY)-(\nabla_{IZ}L)(IX,Y)\right]&=\left[(\nabla_{IX}L)(IY,Z)-(\nabla_{IY}L)(IX,Z)\right]\\ &\quad-\left[(\nabla_{IX}L)(Y,IZ)-(\nabla_{IY}L)(X,IZ)\right]+2\omega(Z,X)\mathbb{B}(Y,\xi)\\ &\quad+2\omega(Y,Z)\mathbb{B}(X,\xi)+2\omega(X,Y)\mathbb{B}(Z,\xi). \end{aligned}\tag{6.19}$$

Now, identity (6.6) follows from (6.14), (6.18) and (6.19). □

**Case 2** $[A, B \in \mathbb{H}, \quad C = \xi]$. In this case, with the help of (2.5), (6.4) turns into the equation

$$\begin{aligned}\nabla du(Z,X,\xi)-\nabla du(X,Z,\xi)&=-R(Z,X,\xi,\nabla u)-\nabla du(T(Z,X),\xi)\\ &=2\omega(Z,X)\nabla du(\xi,\xi). \end{aligned}\tag{6.20}$$

Take a covariant derivative of (6.2) along $Z \in \mathbb{H}$, substitute (6.1) and (6.2) in the obtained equality, then anticommute the covariant derivatives and substitute the result in (6.20) together with the already established (6.6), (6.3) and (5.10) to get after some standard calculations that the integrability condition in this case is

$$(\nabla_Z\mathbb{B})(X,\xi)-(\nabla_X\mathbb{B})(Z,\xi)+L(Z,IL(X))-L(X,IL(Z))=2\mathbb{B}(\xi,\xi)\omega(Z,X).\tag{6.21}$$

Here, the function $\mathbb{B}(\xi,\xi)$ is independent of $u$ and is uniquely determined by

$$\mathbb{B}(\xi,\xi)=-\frac{1}{2n}\left[(\nabla_{e_a}\mathbb{B})(Ie_a,\xi)+L(e_a,IL(Ie_a))\right].\tag{6.22}$$

**Lemma 6.2** *If $W^{pc}=0$ and the dimension is bigger than three, then* (6.21) *holds.*

*Proof* Differentiate the already proved (6.6), take the corresponding traces and use the symmetry of $L$ to see

$$\begin{aligned}(\nabla^2_{e_a,Ie_a}L)(Y,Z)-(\nabla^2_{e_a,Y}L)(Ie_a,Z)&=(\nabla_Z\mathbb{B})(Y,\xi)-\omega(Y,Z)(\nabla_{e_a}\mathbb{B})(Ie_a,\xi)\\ &\quad+2(\nabla_Y\mathbb{B})(Z,\xi) \end{aligned}\tag{6.23}$$

$$\begin{aligned}-(\nabla^2_{e_a,Y}L)(Ie_a,Z)+(\nabla^2_{e_a,Z}L)(Ie_a,Y)&=-(\nabla_Z\mathbb{B})(Y,\xi)-2\omega(Y,Z)(\nabla_{e_a}\mathbb{B})(Ie_a,\xi)\\ &\quad+(\nabla_Y\mathbb{B})(Z,\xi) \end{aligned}\tag{6.24}$$

$$(\nabla^2_{Y,e_a}L)(Ie_a,Z)=-(2n+1)(\nabla_Y\mathbb{B})(Z,\xi).\tag{6.25}$$

A combination of (6.23), (6.25) and (6.24) yields

$$\begin{aligned}&\left[(\nabla^2_{Y,e_a}L)-(\nabla^2_{e_a,Y}L)\right](Ie_a,Z)-\left[(\nabla^2_{Z,e_a}L)-(\nabla^2_{e_a,Z}L)\right](Ie_a,Y)\\ &\quad=2n(\nabla_Z\mathbb{B})(Y,\xi)-\omega(Y,Z)(\nabla_{e_a}\mathbb{B})(Ie_a,\xi)-2n(\nabla_Y\mathbb{B})(Z,\xi). \end{aligned}\tag{6.26}$$

The Ricci identities, (2.5), (3.5), (6.11), (6.12) and (6.13) give

$$\begin{aligned}\left[(\nabla^2_{Y,e_a}L)-(\nabla^2_{e_a,Y}L)\right](Ie_a,Z) &= 2(\nabla_\xi L)(Y,Z)-(trL)[L(Y,IZ)+L(IY,Z)]\\ &+(2n+1)L(Y,IL(Z))-3L(IY,L(Z))-3L(IZ,L(Y))\\ &+L(Z,IL(Y))+\omega(Y,Z)L(e_a,IL(e_a)).\end{aligned}\tag{6.27}$$

$$\begin{aligned}(\nabla^2_{e_a,Ie_a}L)(Y,Z) &= (n+2)[L(IY,L(Z))-L(Y,IL(Z))]-(n+2)L(Z,IL(Y))\\ &+(n+2)L(IZ,L(Y))-2n(\nabla_\xi L)(Y,Z)+(trL)(L(IY,Z)+L(Y,IZ)).\end{aligned}\tag{6.28}$$

The identity (6.21) follows from (6.26) and (6.27). □

**Case 3** $[A=\xi,\quad B,C\in\mathbb{H}]$. In this case (6.4) becomes

$$\nabla du(\xi,X,Y)-\nabla du(X,\xi,Y)+R(\xi,X,Y,\nabla u)+\nabla du(T(\xi,X),Y)=0.\tag{6.29}$$

Take the covariant derivative of (6.1) along $\xi$ and a covariant derivative of (6.2) along a horizontal direction, apply (6.2), (6.1), (6.3), use (3.3) and a suitable traces of (6.1) and (6.13) to get from (6.29) with the help of (6.13), (5.10) and the already proved (6.6) that the integrability condition (6.29) becomes

$$\begin{aligned}(\nabla_X\mathbb{B})(Y,\xi)-(\nabla_\xi L)(X,Y) &= L(Y,IL(X))+\tau(X,L(Y))+\tau(Y,L(X))\\ &+\mathbb{B}(\xi,\xi)\omega(X,Y).\end{aligned}\tag{6.30}$$

Notice that Case 3 implies Case 2 since (6.21) is the skew-symmetric part of (6.30).

**Lemma 6.3** *Suppose $W^{pc}=0$ and dimension is bigger than 3. Then* (6.30) *holds.*

*Proof* Combine (6.23), (6.25), (6.24) and the already proved (6.21) to obtain

$$\begin{aligned}(\nabla^2_{e_a,Ie_a}L)(Y,Z)+\left[(\nabla^2_{Y,e_a}L)-(\nabla^2_{e_a,Y}L)\right](Ie_a,Z) &= -2(n-1)(\nabla_Y\mathbb{B})(Z,\xi)\\ -2\omega(Y,Z)\mathbb{B}(\xi,\xi)+L(Y,IL(Z))-L(Z,IL(Y))&-\omega(Y,Z)(\nabla_{e_a}\mathbb{B})(Ie_a,\xi)\end{aligned}\tag{6.31}$$

Now, (6.27), (6.28) and (6.31) imply (6.30). □

**Case 4** $[A\in\mathbb{H},\quad B=C=\xi]$. In this case (6.4) has the form

$$\begin{aligned}\nabla du(X,\xi,\xi)-\nabla du(\xi,X,\xi) &= -R(X,\xi,\xi,\nabla u)+\nabla du(T(\xi,X),\xi)\\ &= \tau(X,e_a)\nabla du(e_a,\xi).\end{aligned}\tag{6.32}$$

Take the covariant derivative of (6.2) along $\xi$ and a covariant derivative of (6.3) along a horizontal direction, then use (6.1), the already proved (6.30), apply (6.2) to see that (6.32) is equivalent to

$$(\nabla_\xi\mathbb{B})(X,\xi)-(\nabla_X\mathbb{B})(\xi,\xi)-2\mathbb{B}(e_a,\xi)L(X,Ie_a)+\tau(X,e_a)\mathbb{B}(e_a,\xi)=0.\tag{6.33}$$

**Lemma 6.4** *Suppose $W^{pc}=0$ and dimension is bigger than 3. Then* (6.33) *holds.*

*Proof* Differentiate the already proven (6.21), (6.30), the first equality in (6.7), take the corresponding traces, use the symmetry of $L$, $\tau$ and (6.7) to get

$$\begin{aligned}(\nabla^2_{e_a,Ie_a}\mathbb{B})(Y,\xi) &= (n+2)[L(IY,e_b)-L(Y,Ie_b)]\mathbb{B}(e_b,\xi)\\ &-(trL)\mathbb{B}(IY,\xi)-2n(\nabla_\xi\mathbb{B})(Y,\xi)\end{aligned}\tag{6.34}$$

$$\begin{aligned}(\nabla^2_{e_b,Ie_b}\mathbb{B})(Y,\xi)-(\nabla^2_{e_b,\xi}L)(Ie_b,Y) &= -(2n+1)\mathbb{B}(e_a,\xi)[L(Y,Ie_a)+\tau(Y,e_a)]\\ &+(\nabla_Y\mathbb{B})(\xi,\xi)+[(\nabla_{e_b}L)(Y,Ie_a)+(\nabla_{e_b}\tau)(Y,e_a)]L(Ie_b,e_a)\\ &+(\nabla_{e_b}\tau)(Ie_b,e_a)L(Y,e_a)+\tau(Ie_b,e_a)(\nabla_{e_b}L)(Y,e_a)\end{aligned}\tag{6.35}$$

$$(\nabla^2_{\xi,e_b}L)(Ie_b,Y)=-(2n+1)(\nabla_\xi\mathbb{B})(Y,\xi).\tag{6.36}$$

The Ricci identities, Eq. (3.3) in Proposition 3.1, (2.6), (2.7) and the symmetry of $L$ imply

$$(\nabla^2_{\xi,e_b}L)(Ie_b,Y)-(\nabla^2_{e_b,\xi}L)(Ie_b,Y)= -(\nabla_{Ie_b}\tau)(e_b,e_a)L(Y,e_a) +\tau(e_b,Ie_a)(\nabla_{e_a}L)(e_b,Y)+[(\nabla_{e_a}\tau)(e_b,Y) -(\nabla_Y)\tau)(e_b,e_a)]L(e_a,Ie_b). \tag{6.37}$$

A small calculation taking into account (6.34), (6.35), (6.36), (6.37) and using (6.13) yields

$$(\nabla_\xi\mathbb{B})(Y,\xi)-(\nabla_Y\mathbb{B})(\xi,\xi)-3[L(Y,Ie_a)+\tau(Y,e_a)]\mathbb{B}(e_a,\xi)=-(trL)\mathbb{B}(IY,\xi) +[(\nabla_{e_b}L(Y,Ie_a)+(\nabla_{e_b}\tau(Y,e_a)-(\nabla_{e_a}\tau(Y,e_b)+(\nabla_Y)\tau(e_b,e_a)]L(Ie_b,e_a). \tag{6.38}$$

Now, apply the already proven (6.6) together with (6.13) to (6.38) to get the proof of (6.33). □

Thus, the proof of Theorem 1.2 i) is complete.

### 6.1 The three dimensional case

If the dimension is equal to 3 then it is easy to check that $W^{pc}=0$ and the integrability conditions (6.6) and (6.21) are trivially satisfied. Thus, the existence of a smooth solution depends only on the validity of (6.30) since the proof of Lemma 6.4 shows that (6.33) follows from (6.30) also in dimension three. The next Lemma 6.5 implies Theorem 1.2 ii).

**Lemma 6.5** *If $n=1$ and $F=0$ then* (6.30) *holds.*

*Proof* Suppose $n=1$. Then $r(X,Y)=\rho(X,IY)=\frac{Scal}{2}g(X,Y)$ and (5.10) yields

$$L(X,Y)=\frac{Scal}{16}g(X,Y)-\tau(X,IY). \tag{6.39}$$

Apply (6.39) to (6.28) to get

$$2(\nabla_\xi L)(X,Y)=-(\nabla^2_{e_aIe_a}L)(X,Y)-Scal.\tau(X,Y). \tag{6.40}$$

The skew symmetric part of (6.30) is satisfied because $n=1$. Now, (6.25), (6.40) and (6.39) give that the symmetric part of (6.30) is equivalent to $F(X,Y)=0$. □

The proof of Theorem 1.2 is completed.

## 7 A remark on the Cartan–Chern–Moser theorem in the CR case

A CR manifold is a smooth manifold $M$ of real dimension 2n+1, with a fixed n-dimensional complex subbundle $H$ of the complexified tangent bundle $\mathbb{C}TM$ satisfying $H\cap\overline{H}=0$ and $[H,H]\subset H$. If we let $\mathbb{H}=Re\,H\oplus\overline{H}$, the real subbundle $\mathbb{H}$ is equipped with a formally integrable almost complex structure $J$. We assume that $M$ is oriented and there exists a globally defined contact form $\theta$ such that $\mathbb{H}=Ker\,\theta$. Recall that a 1-form $\theta$ is a contact form if the hermitian bilinear form $2g(X,Y)=-d\theta(JX,Y)$ is non-degenerate. The vector field $\zeta$ dual to $\theta$ with respect to $g$ and satisfying $d\theta(\zeta,.)=0$ is called the Reeb vector field. A CR manifold $(M,\theta,g)$ with fixed contact form $\theta$ is called *a pseudohermitian manifold*. In this case the 2-form $d\theta_{|\mathbb{H}}:=2\Omega$ is called the fundamental form. Note that the contact form is determined up to a conformal factor, i.e. $\bar\theta=\nu\theta$ for a positive smooth function $\nu$, defines another pseudohermitian structure called pseudo-conformal to the original one. A basic geometric tool in investigating pseudohermitian structures is the Webster connection $\nabla^{cr}$ [17,18] (see also Tanaka [14]).

The Cartan–Chern–Moser results [2,3] are that the vanishing of the pseudoconformal invariant Chern-Moser tensor $S$ (resp. Cartan invariant for $n = 1$) is a necessary and sufficient condition a non-degenerate CR-hypersurface in $\mathbb{C}^{n+1}$, to be locally CR equivalent to a hyperquadric in $\mathbb{C}^{n+1}$. A proof of these results can be achieved working similarly to our proof of Theorem 1.2. We outline below the crucial steps.

It is well known that a pseudohermitian manifold with flat Webster connection (and zero torsion if $n = 1$) is locally isomorphic to a Heisenberg group. On the other hand, the Cayley transform is a pseudo-conformal equivalence between the Heisenberg group with its flat pseudo-hermitian structure and a hypersphere $g_{\alpha\bar{\beta}}Z^{\alpha}Z^{\bar{\beta}} + W\bar{W} = 1$ in $\mathbb{C}^{n+1}$ [3, p. 223]. It remains to show that the vanishing of the Chern-Moser tensor, $S = 0$, is a sufficient condition a given pseudohermitian manifold to be locally pseudoconformally flat provided the dimension is bigger than three. In dimension three $S$ vanishes identically and the sufficient condition remains only (7.9) below. The scheme is formally very similar to that used in the proof of Theorem 1.2. Namely in all formulas in the proof of Theorem 1.2 one formally replaces $I$ with $\sqrt{-1}J$ and $\xi$ by $\sqrt{-1}\zeta$. We indicate below the most important steps.

The superscript $cr$ means that the objects are taken with respect to the Webster connection. In particular, the pseudohermitian Ricci 2-form $\rho^{cr}$ and the pseudohermitian scalar curvature $Scal^{cr}$ are defined by $2\rho^{cr}(A,B) = g(R^{cr}(A,B)\epsilon_a, J\epsilon_a)$, $Scal^{cr} = r(\epsilon_a,\epsilon_a)$, $A, B \in \Gamma(TM)$. The Chern-Moser tensor $S$ can be obtained from (5.12) formally replacing $I$, $\rho$, $Scal$ and $\omega$ with $\sqrt{-1}J$, $\rho^{cr}$, $Scal^{cr}$ and $\Omega$, respectively. The system of PDE which guaranties the flatness of the pseudoconformal Webster connection and has to be solved is:

$$\nabla^{cr}dv(X,Y) = -C(X,Y) - dv(X)dv(Y) + dv(JX)dv(JY) + \frac{1}{2}g(X,Y)|\nabla^{cr}v|^2 - dv(\zeta)\Omega(X,Y), \tag{7.1}$$

$$\nabla^{cr}du(X,\zeta) = -\mathbb{D}(X,\zeta) - C(X,J\nabla v) + \frac{1}{2}dv(JX)|\nabla^{cr}v|^2 - dv(X)du(\zeta), \tag{7.2}$$

$$\nabla^{cr}dv(\zeta,\zeta) = -\mathbb{D}(\zeta,\zeta) - \mathbb{D}(J\nabla v,\zeta) + \frac{1}{4}|\nabla^{cr}v|^4 - (dv(\zeta))^2, \tag{7.3}$$

where the symmetric tensor $C(X,Y)$, $\mathbb{D}(X,\zeta)$ and $\mathbb{D}(\zeta,\zeta)$ do not depend on the function $u$ and are determined by

$$C(X,Y) = -\frac{1}{2(n+2)}\rho^{cr}(X,JY) - \frac{Scal^{cr}}{8(n+1)(n+2)}g(X,Y) + A(JX,Y) \tag{7.4}$$

$$(\nabla^{cr}_{\epsilon_a}C)(J\epsilon_a, JX) = -(2n+1)\mathbb{D}(JX,\zeta), \tag{7.5}$$

$$\mathbb{D}(\zeta,\zeta) = -\frac{1}{2n}\left[(\nabla^{cr}_{\epsilon_a}\mathbb{D})(J\epsilon_a,\zeta) + C(\epsilon_b, J\epsilon_a)C(J\epsilon_b,\epsilon_a)\right], \tag{7.6}$$

and the symmetric tensor $A(X,Y)$ is the pseudohermitian torsion [11,17,18].

The integrability conditions for the overdetermined system (7.1)–(7.3) are:

$$(\nabla^{cr}_Z C)(X,Y) - (\nabla^{cr}_X C)(Z,Y) = -\Omega(Z,Y)\mathbb{D}(X,\zeta) + \Omega(X,Y)\mathbb{D}(Z,\zeta) - 2\Omega(Z,X)\mathbb{D}(Y,\zeta); \tag{7.7}$$

$$(\nabla^{cr}_Z\mathbb{D})(X,\zeta) - (\nabla^{cr}_X\mathbb{D})(Z,\zeta) + C(Z,JC(X)) - C(X,JL(Z)) = -2\mathbb{D}(\zeta,\zeta)\Omega(Z,X); \tag{7.8}$$

$$(\nabla^{cr}_X \mathbb{D})(Y,\zeta) - (\nabla^{cr}_\zeta C)(X,Y) = C(Y, JC(X)) + A(X, C(Y))$$
$$+A(Y, C(X)) - \mathbb{D}(\zeta,\zeta)\Omega(X,Y); \tag{7.9}$$
$$(\nabla^{cr}_\zeta \mathbb{D})(X,\zeta) - (\nabla^{cr}_X \mathbb{D})(\zeta,\zeta) - 2\mathbb{D}(\epsilon_a,\zeta)C(X, J\epsilon_a) + A(X,\epsilon_a)\mathbb{D}(\epsilon_a,\zeta) = 0. \tag{7.10}$$

As in the proof of Theorem 1.2 i) we can see that the vanishing of the Chern-Moser tensor, $S = 0$, implies the validity of the integrability conditions (7.7)–(7.10) provided $n > 1$.

For $n = 1$ the Chern-Moser tensor is always zero and, following the proof of Theorem 1.2 ii), one checks that the integrability conditions (7.7) and (7.8) are trivially satisfied and (7.10) is a consequence of (7.9). To have a smooth solution to the system (7.1)–(7.3) one has to have (7.9) which is equivalent to the vanishing of the symmetric (0,2) tensor $F^{car}$ defined on $\mathbb{H}$ by

$$F^{car}(X,Y) = (\nabla^{cr} d(Scal^{cr}))(X, JY) + (\nabla^{cr} d(Scal^{cr}))(Y, JX) + 16((\nabla^{cr})^2_{X\epsilon_a} A)(Y,\epsilon_a)$$
$$+ 16((\nabla^{cr})^2_{Y\epsilon_a} A)(X,\epsilon_a) + 48((\nabla^{cr})^2_{\epsilon_a J\epsilon_a} A)(X, JY)$$
$$+ 36 Scal^{cr} A(X,Y) + 3g(X,Y)(\nabla^{cr} d(Scal^{cr}))(\epsilon_a, J\epsilon_a). \tag{7.11}$$

Let us remark that the vanishing of $F^{car}$ is equivalent to the vanishing of the Cartan curvature, cf. [15, Theorem 12.3].

**Corollary 7.1** *A 3-dimensional Sasakian manifold $(M,\theta,g,\zeta)$ is locally pseudoconformally equivalent to the three dimensional Heisenberg group if and only if its Riemannian scalar curvature $Scal^g$ satisfies*

$$(\nabla^g d(Scal^g))(X, JY) + (\nabla^g d(Scal^g))(Y, JX) = 0, \tag{7.12}$$

*where $\nabla^g$ is the Levi-Civita connection of g.*

*Proof* It is well known that a pseudohermitian structure is Sasakian, i.e. its Riemannian cone is Kähler, exactly when the Webster torsion vanishes, $A = 0$. In particular, the Bianchi identities imply $\zeta(Scal^{cr}) = 0$. Then the second and the third lines in (7.11) disappeared in view of (5.9). On the other hand, for a Sasaki manifold, we have $\nabla^{cr}_X Y = \nabla^g_X Y + g(JX,Y)\zeta$ and the Riemannian scalar curvature and the scalar curvature of the Webster connection differ by an additive constant depending on the dimension, $2Scal^{cr} = Scal^g + 2n$ (see e.g. [5]). Now, (7.12) becomes equivalent to (7.11). Hence, $(M,\theta)$ is locally pseudoconformally flat. □

## 8 The ultrahyperbolic Yamabe equation

Recall that the CR Yamabe problem is to determine if there exists a pseudohermitian structure compatible with a given CR structure such that the pseudohermitian scalar, i.e. the scalar curvature of the Webster connection is constant. If the CR structure is strongly pseudo-convex, i.e. the Levi form is negative definite then the CR Yamabe problem reduces to a subelliptic PDE which can be solved on a compact manifold [9].

Similarly to the CR case one can pose a Yamabe type problem for a para CR manifold. Namely, given a para CR structure is there a compatible para hermitian structure such that the scalar curvature of the canonical connection is a constant.

In the case when the Levi form of a given CR structure has neutral signature of type (n,n) then the CR Yamabe equation is of the same type as the para CR Yamabe equation [11], i.e. one has to consider the sub ultrahyperbolic equation (5.8) with respect to the Webster connection where $\bar{s} = const.$.

Here we show an explicit formula for the regular part of a solution to the ultrahyperbolic Yamabe equation on $G(\mathbb{P})$

$$\mathcal{L}\varphi \equiv \sum_{k=1}^{n} \left(U_k^2 - V_k^2\right)\varphi = -\varphi^{2^*-1}, \tag{8.1}$$

where $2^*-1 = (Q+2)/(Q-2) = (n+2)/n$ with $Q = 2n+2$ the homogenous dimension of the group. When $n = 2m$ (8.1) coincides with the Yamabe equation on the Heisenberg group of (real) signature $(2m, 2m)$ defined by the quadric

$$Q = \left\{(z, w) \in \mathbb{C}^n \times \mathbb{C} : \ \operatorname{Im} w = H(z, z)\right\},$$

where $H(z,z) = \sum_{j=1}^{m}\left(z_j\overline{z}'_j - z_{j+m}\overline{z}'_{j+m}\right)$, with the natural group structure

$$(z'', w'') = (z', w') \circ (z, w) = \left(z' + z, w' + w + 2\operatorname{Im} H(z', z)\right).$$

The left-invariant horizontal vector fields are given by

$$X_j = \frac{\partial}{\partial x_j} - 2y_j\frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} + 2x_j\frac{\partial}{\partial t}$$
$$X_{m+j} = \frac{\partial}{\partial x_{m+j}} + 2y_{j+m}\frac{\partial}{\partial t}, \quad Y_{m+j} = \frac{\partial}{\partial x_{m+j}} - 2x_{j+m}\frac{\partial}{\partial t}, \quad j = 1, \ldots, m,$$

while the left invariant contact form with corresponding metric, for which the above vector fields are an orthonormal frame, is given by

$$\theta = \frac{1}{2}\,dt + \Sigma_{j=1}^{m}(y_j\,dx_j - x_j\,dy_j) - \Sigma_{j=1}^{m}(y_{j+m}\,dx_{j+m} - x_{j+m}\,dy_{j+m})$$

so that $g(X_j, X_j) = -g(Y_j, Y_j) = -g(X_{j+m}, X_{j+m}) = g(Y_{j+m}, Y_{j+m}) = 1$, $j = 1, \ldots, m$.

By the Cartan–Chern–Moser result, the above quadric is the flat CR structure of (hermitian) signature $(m, m)$. Henceforth, for $\mathbf{u} \in \mathbb{R}^n$, $\mathbf{u} = (u_1, \ldots, u_n)$ we set $|\mathbf{u}| = \left(u_1^2 + \ldots u_n^2\right)^{1/2}$. We observe

**Proposition 8.1** *Let $G(\mathbb{P})$ be the Heisenberg group of topological dimension $2n+1$. For every $\epsilon > 0$ the function*

$$\varphi_\epsilon(\mathbf{u}, \mathbf{v}, t) = \left(\frac{4n^2\epsilon^2}{(\epsilon^2 + |\mathbf{u}|^2 - |\mathbf{v}|^2)^2 - t^2}\right)^{\frac{n}{2}}, \qquad g \in \boldsymbol{G}, \tag{8.2}$$

*is a solution of the ultrahyperbolic Yamabe equation* (8.1) *on the set where* $|\epsilon^2 + |\mathbf{u}|^2 - |\mathbf{v}|^2| \neq |t|$.

*Proof* Let $f = \left((1 + |\mathbf{u}|^2 - |\mathbf{v}|^2)^2 - t^2\right)^{-\frac{n}{2}}$. After a straightforward calculation we find $\mathcal{L}f = -4n^2 f^{2^*-1}$, which implies easily the equation for $\varphi_1$. Furthermore, using the dilations on the group $\delta_\lambda(\mathbf{u}, \mathbf{v}, t) = (\lambda u, \lambda v, \lambda^2 t)$ we have that the function $f_\lambda(\mathbf{u}, \mathbf{v}, t) = \lambda^{n/2} f(\lambda u, \lambda v, \lambda^2 t)$ satisfies the same equation as $f$, which implies the equation for $\varphi_\epsilon$ by taking $\epsilon = 1/\lambda$. □

Since the ultra-hyperbolic Yamabe equation is invariant under translations it follows that we can construct other solutions, each being a regular function on a corresponding set. The question whether there is a global solution, in the sense of distributions, will not be considered here. In this respect we note that [13,16] found the fundamental solution of the ultra-hyperbolic operator in the left-hand side of (8.1).

It should be pointed out that there is a correspondence between the regular part of solutions to partial differential equations on the hyperbolic Heisenberg group and solutions of partial differential equations on the Heisenberg group. Let $X_k = \frac{\partial}{\partial x_k} - 2y_k\frac{\partial}{\partial s}$, $Y_k = \frac{\partial}{\partial y_k} + 2x_k\frac{\partial}{\partial s}$ be the horizontal left invariant vector fields on the standard Heisenberg group. Note that the difference between this group and the hyperbolic Heisenberg group is in the metric, while the groups are identical. Given a function $f(x, y, t), t \in \mathbb{R},\ x, y \in \mathbb{R}^n$, let $g(u, v, t) = f(it, u, iv)$, which could be a complex valued function even when $f$ is real-valued. Since

$$(X_k f)(\mathbf{x}, \mathbf{y}, s) = (U_k g)(\mathbf{u}, \mathbf{v}, t), \qquad (Y_k f)(\mathbf{x}, \mathbf{y}, s) = -i(V_k g)(\mathbf{u}, \mathbf{v}, t) \tag{8.3}$$

we have $\sum_{k=1}^n \left(X_k^2 + Y_k^2\right) f = \sum_{k=1}^n \left(U_k^2 - V_k^2\right) f$. In particular, solutions of the Yamabe equation on the Heisenberg group turn into solutions of the ultra-hyperbolic Yamabe equation on the hyperbolic Heisenberg group outside a corresponding singular set.

Consider the (standard) Heisenberg group of dimension $2n+1$ with typical point $(z, s)$, $z \in \mathbb{C}^n$, $s \in \mathbb{R}$, and let $A = |z|^2 + it$. The inversion of the point $(z, s)$ is given by

$$(z', s') \stackrel{def}{=} \left(-\frac{z}{\bar{A}}, -\frac{s}{A\bar{A}}\right), \tag{8.4}$$

which can also be written in real coordinates as

$$\mathbf{x}' = \mathbf{x}'(\mathbf{x}, \mathbf{y}, s), \qquad \mathbf{y}' = \mathbf{y}'(\mathbf{x}, \mathbf{y}, s), \qquad s' = s'(\mathbf{x}, \mathbf{y}, s).$$

If $B \stackrel{def}{=} |z'|^2 + is'$, then $AB = 1$ as $B = \frac{|z|^2}{A\bar{A}} - \frac{is}{A\bar{A}} = \frac{\bar{A}}{A\bar{A}} = \frac{1}{A}$.

Based on the above mentioned formal substitution and the preceding paragraph we define an inversion on the hyperbolic Heisenberg group as follows. Let $\Xi = \{p = (\mathbf{u}, \mathbf{v}, t) \in G(\mathbb{P}) : \left||\mathbf{u}|^2 - |\mathbf{v}|^2\right| = |t|\}$ and $p = (t, u, v) \in G(\mathbb{P}) \setminus \Xi$. We define the inversion on the hyperbolic Heisenberg group letting

$$\mathbf{u}' = \mathbf{x}'(\mathbf{u}, i\mathbf{v}, it), \qquad \mathbf{v}' = -i\mathbf{y}'(\mathbf{u}, i\mathbf{v}, it), \qquad t' = -is'(\mathbf{u}, i\mathbf{v}, it) \tag{8.5}$$

using the real form of the "standard" inversion on the Heisenberg group. In other words, for $k = 1, \ldots, n$ we have

$$\begin{aligned} u_k' &= -\frac{(|u|^2 - |v|^2)u_k + tv_k}{(|u|^2 - |v|^2)^2 - t^2}, \quad v_k' = -\frac{(|u|^2 - |v|^2)v_k + tu_k}{(|u|^2 - |v|^2)^2 - t^2}, \\ t' &= -\frac{t}{(|u|^2 - |v|^2)^2 - t^2}, \end{aligned} \tag{8.6}$$

which defines a point $p' = (\mathbf{u}', \mathbf{v}', t') \in G(\mathbb{P}) \setminus \Xi$, or, using $\mathbf{w} = \mathbf{u} + e\mathbf{v}$ with $e^2 = 1$, $|\mathbf{w}|^2 = |\mathbf{u}|^2 - |\mathbf{v}|^2$,

$$\mathbf{w}' \equiv \mathbf{u}' + e\mathbf{v}' = -\frac{\mathbf{w}}{|\mathbf{w}|^2 - et} \qquad t' = -\frac{t}{|\mathbf{w}|^4 - t^2}.$$

This map will be called the inversion of $G(\mathbb{P})$ centered at $\Xi$. The inverse transformation is found by taking into account that the inversion is an involution.

Recall, see [10], that for a function $f(z, t)$ defined on a domain $\Omega$ in the Heisenberg group we define the Kelvin transform $f^*$ on the image $\Omega^*$ of $\Omega$ under the inversion by the following formula

$$f^* \stackrel{def}{=} A^{\frac{n}{2}} \bar{A}^{\frac{n}{2}} f \text{ i.e. } f^*|B|^n = f.$$

Thus, using the preceding considerations we can define a Kelvin transform on the hyperbolic Heisenberg group as follows

$$(\mathcal{K}\varphi)(\mathbf{u},\mathbf{v},t) = \left((|\mathbf{u}|^2 - |\mathbf{v}|^2)^2 - t^2\right)^{-n/2} \varphi(\mathbf{u}',\mathbf{v}',t'),$$

where $(\mathbf{u}'\mathbf{v}', t')$ are given by (8.6). Given a function $\varphi(u, v, t)$ we consider $\psi(x, y, s) = \varphi(x, -iy, -is)$. Thus, using $s = it$, $\mathbf{x} = \mathbf{u}$ and $\mathbf{y} = i\mathbf{v}$, we have from (8.5)

$$\begin{aligned}\varphi(\mathbf{u}',\mathbf{v}',t') = \psi(\mathbf{u}', i\mathbf{v}', it') &= \psi\left(\mathbf{x}'(\mathbf{u}, i\mathbf{v}, it), \mathbf{y}'(\mathbf{u}, i\mathbf{v}, it), s'(\mathbf{u}, i\mathbf{v}, it)\right)\\ &= \psi\left(\mathbf{x}'(\mathbf{x},\mathbf{y},s), \mathbf{y}'(\mathbf{x},\mathbf{y},s), s'(\mathbf{x},\mathbf{y},s)\right),\end{aligned}$$

which shows that the (hyperbolic) Kelvin transform of $\varphi$ corresponds to the ("standard" Heisenberg ) Kelvin transform of $\psi$. Due to (8.3) and the properties of the Kelvin transform on the Heisenberg group (in fact any group of Iwasawa type), cf. [4] and [7], the hyperbolic Kelvin transform preserves the ultra-hyperbolic functions, i.e., solutions of

$$\mathcal{L}\varphi \equiv \sum_{k=1}^{n} \left(U_k^2 - V_k^2\right)\varphi = 0$$

and the solutions of the ultra-hyperbolic Yamabe equation. Note that the Kelvin transform of the functions in (8.2) is given by the same formula.

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