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Quantitative property A, Poincaré inequalities, L^p-compression and L^p-distortion for metric measure spaces

Romain Tessera

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Abstract We introduce a quantitative version of Property A in order to estimate the L^p-compressions of a metric measure space X. We obtain various estimates for spaces with sub-exponential volume growth. This quantitative property A also appears to be useful to yield upper bounds on the L^p-distortion of finite metric spaces. Namely, we obtain new optimal results for finite subsets of homogeneous Riemannian manifolds. We also introduce a general form of Poincaré inequalities that provide constraints on compressions, and lower bounds on distortion. These inequalities are used to prove the optimality of some of our results.

Keywords Uniform embeddings of metric spaces into Banach spaces · Property A · Poincare inequalities · Hilbert compression · Hilbert distortion

Mathematics Subject Classification (2000) 51F99 · 43A85

1 Introduction

In [15], Yu introduced a *weak Følner* property for metric spaces that he called Property A. He proved that a metric space satisfying this property uniformly embeds into a Hilbert space. In [14], it is proved that a discrete metric space with subexponential growth has Property A and therefore, uniformly embeds into a Hilbert space (here, we give a very short proof of this fact when X is assumed to be coarsely geodesic, e.g. if X is a graph). In this paper, we define a quantitative L^p-version of Property A and use it to obtain uniform embeddings of metric measure spaces with subexponential growth into L^p with compressions satisfying some lower estimates. For instance we obtain (new) optimal estimates for uniformly doubling graphs (see Theorem 5).

R. Tessera (✉)
Department of Mathematics, Vanderbilt University, Stevenson Center, Nashville, TN 37240, USA
e-mail: tessera@clipper.ens.fr

More precisely, our constructions yield uniform embeddings of a metric measure space (X, d, μ) into $L^p(X, \mu)$, and are equivariant with respect to the actions of the group of measure-preserving isometries of X on these two spaces (see Sect. 4).

We also applying our quantitative property A to estimate the L^p-distortion of finite metric spaces. The L^p-distortion of a metric space X measures how far X is from being isometric to a finite subspace of L^p (see for instance [6] for a survey on these questions). We obtain various estimates, some of them being optimal. For instance we compute the L^p-distortion of finite subsets of connected Lie groups.

To find either upper bounds on the compression of uniform embeddings, or lower bounds on the L^p-distortion, we introduce a general form of Poincaré inequalities. Similar ideas had been discussed in [6]. Our attempt here is to provide the most general obstructions. We project in a future paper to prove that these inequalities are in fact optimal in that sense.

2 Some preliminaries about uniform embeddings

In this short section, we recall the definitions of a uniform embedding between metric spaces and of the compression function associated to a uniform embedding.

Let us start with some notation. The volume of the closed balls $B(x, r)$ is denoted by $V(x, r)$. An L^p-space will mean a Banach space of the form $L^p(\Omega, \mu)$ where (Ω, μ) is some measure space.

Let $f, g : \mathbf{R}_+ \to \mathbf{R}_+$ be non-decreasing functions. We write respectively $f \preceq g$, $f \prec g$ if there exists $C > 0$ such that $f(t) \leq Cg(Ct) + C$, resp. $f(t) = o(g(ct))$ for any $c > 0$. We write $f \approx g$ if both $f \preceq g$ and $g \preceq f$. The asymptotic behavior of f is its class modulo the equivalence relation $\approx$.

Definition 2.1 Let (X, d_X) and (Y, d_Y) be metric spaces. A map $F : X \to Y$ is called a uniform embedding of X into Y if there exists two increasing, non bounded maps ρ_1 and ρ_2 such that

$$\rho_1(d_X(x, y)) \leq d_Y(F(x), F(y)) \leq \rho_2(d_X(x, y)).$$

A map $F : X \to Y$ is called a quasi-isometric embedding if ρ_1 and ρ_2 can be chosen affine (non-constant). The main purpose of this paper is, given a metric space X, to find "good" uniform embeddings of X into some L^p-space. By good, we mean as close as possible to a quasi-isometric embedding. Hence, the quality of a uniform embedding will be measured by the asymptotics of ρ_1 and ρ_2. More precisely, let us define the compression of F to be the supremum ρ_F of all functions ρ_1 satisfying the above inequality and the dilatation to be the infimum δ_F of all functions ρ_2. Hence, F is quasi-isometric if and only if ρ_F and δ_F are both asymptotically equivalent to affine functions. Throughout this paper we will consider quasi-geodesic spaces, for which the dilatation is always less than an affine function.

Proposition 2.2 [5] *Let X and Y be two metric spaces such that X is quasi-geodesic. Then, any uniform embedding F from X to Y is large-scale Lipschitz.*

Proof Recall (see for instance [5] for an equivalent definition) that a metric space (X, d) is called *quasi-geodesic* if there exist $b > 0$ and $\gamma \geq 1$ such that for all $x, y \in X$, there exists a chain $x = x_0, x_1, \ldots, x_n = y$ satisfying

$$n \leq \gamma d(x, y), \text{ and } \forall k = 1, \ldots, n, \quad d(x_{k-1}, x_k) \leq b.$$

Such a chain is called an b-quasi-geodesic chain between x and y. Let x and y be two elements of X, and let $x = x_0, x_1, \ldots, x_n = y$ be an b-quasi-geodesic chain. Then,

$$d_Y(F(x), F(y)) \leq (n+1)\delta(b) \leq \gamma\delta(b)(d(x,y)+1). \qquad \square$$

Definition 2.3 [5] Fix $p \geq 1$. The L^p-compression rate $R_p(X)$ of a metric space X is the supremum of α such that there exits a large-scale Lipschitz uniform embedding from X into a L^p-space with compression $\rho(t) \succeq t^\alpha$.

Remark 2.4 Note that $R_p(X)$ is invariant under quasi-isometry. More generally, let $u : Y \to X$ be a quasi-isometric embedding from X to Y. Assume that X admits a large-scale Lipschitz uniform embedding F into some L^p-space with compression ρ_F, then $F \circ u$ defines a large-scale Lipschitz uniform embedding of Y whose compression satisfies $\rho_{F \circ u} \succeq \rho_F$.

3 Results

3.1 Quantitative property A and construction of uniform embeddings in L^p

Let us give a definition[1] of Yu's Property A for metric measure spaces that coincides with the usual one in the case of discrete metric spaces.

Definition 3.1 We say that a metric measure space X has Property A if there exists a sequence of families of probability densities on X: $((\psi_{n,x})_{x\in X})_{n\in\mathbf{N}}$ such that

(i) for every $n \in \mathbf{N}$, the support of each $\psi_{n,x}$ lies in the (closed) ball $B(x,n)$ and
(ii) $\|\psi_{n,x} - \psi_{n,y}\|_1$ goes to zero when $n \to \infty$ uniformly on controlled sets $\{(x,y) \in X^2,\ d(x,y) \leq r\}$.

The following proposition follows immediately from basic L^p-calculus and its proof is left to the reader.

Proposition 3.2 *Property A is equivalent to the following statement. Let $1 \leq p < \infty$. There exists a sequence of families of unit vectors in $L^p(X)$: $((\psi_{n,x})_{x\in X})_{n\in\mathbf{N}}$ such that*

(i) *for every $n \in \mathbf{N}$, the support of each $\psi_{n,x}$ lies in $B(x,n)$ and*
(ii) *$\|\psi_{n,x} - \psi_{n,y}\|_p$ goes to zero when $n \to \infty$ uniformly on controlled sets $\{(x,y) \in X^2,\ d(x,y) \leq r\}$.*

The main conceptual tool in this paper the following quantitative version in L^p of Property A.

Definition 3.3 Let $X = (X, d, \mu)$ be metric measure space, $J : \mathbf{R}_+ \to \mathbf{R}_+$ be some increasing map and let $1 \leq p < \infty$. We say that X has property $A(J, p)$ if for every $n \in \mathbf{N}$, there exists a map $\psi_n : X \to L^p(X)$ such that

- for every $x \in X$, $\|\psi_{n,x}\|_p \geq J(n)$,
- $\|\psi_{n,x} - \psi_{n,y}\|_p \leq d(x,y)$,
- $\psi_{n,x}$ is supported in $B(x,n)$.

The function J, that we call the A-profile in L^p, is a increasing function dominated by the identity. This definition is motivated by the following central observation.

[1] See also [12, Definition 2.1].

Proposition 1 (see Proposition 5.3) *Let X be a metric measure space satisfying Property $A(J, p)$. Then, for every increasing function f satisfying*

$$\int_1^\infty \left(\frac{f(t)}{J(t)}\right)^p \frac{dt}{t} < \infty, \tag{J, p}$$

there exists a large-scale Lipschitz uniform embedding F of X into $\oplus_\infty^{\ell^p} L^p(X, \mu)$ with compression $\rho \succeq f$. In particular,

$$R_p(X) \geq \liminf_{t\to\infty} \frac{\log J(t)}{\log t}.$$

We give estimates of the A-profile in L^p for spaces with subexponential growth.

Proposition 2 (see Proposition 6.3) *Let (X, d, μ) be a metric measure space. Assume that there exists an increasing function v and constants $C \geq 1$ and $d > 0$ such that*

$$1 \leq v(r) \leq V(x, r) \leq Cv(r), \quad \forall x \in X, \forall r \geq 1;$$

Then for every $p \geq 1$, X satisfies equivariant property $A(J, p)$ with $J(t) \approx t/\log v(t)$.

For instance, we obtain the following corollary.

Corollary 3 (see Proposition 6.4) *Keep the same hypothesis as in Proposition 6.3 and assume that $v(t) \preceq e^{t^\beta}$, for some $\beta < 1$. Then, for every $p \geq 1$,*

$$R_p(X) \geq 1 - \beta.$$

Recall that a metric measure space X is called doubling if there exists a constant C such that $V(x, 2r) \leq CV(x, r)$ for every $r > 0$, $x \in X$. We say that X is uniformly doubling if there exists an increasing function v satisfying $v(r) \leq V(x, r) \leq Cv(r)$ for every $r > 0$, $x \in X$, and a doubling property $v(2r) \leq C'v(r)$.

Proposition 4 (see Proposition 6.5) *Let (X, d, μ) be a uniformly doubling metric measure space. Then for every $p \geq 1$, X satisfies the property $A(J, p)$ with $J(t) \approx t$.*

Corollary 5 (see Corollary 6.6) *Let X be uniformly doubling graph and let $p \geq 2$. Then, for every increasing $f : \mathbf{R}_+ \to \mathbf{R}_+$ satisfying*

$$\int_1^\infty \left(\frac{f(t)}{t}\right)^p \frac{dt}{t} < \infty, \quad (C_p)$$

there exists a uniform embedding F of X into $\oplus_\infty^{\ell^p} L^p(X, \mu)$ with compression $\rho_F \succeq f$. In particular,

$$R_p(X) = 1.$$

About Condition (C_p)

- First, note that if $p \leq q$, then (C_p) implies (C_q): this immediately follows from the fact that a nondecreasing function f satisfying (C_p) also satisfies $f(t)/t = O(1)$.
- If f and h are two increasing functions such that $f \preceq h$ and h satisfies (C_p), then f satisfies (C_p).

- The function $f(t) = t^a$ satisfies (C_p) for every $a < 1$ but not for $a = 1$. More precisely, the function
$$f(t) = \frac{t}{(\log t)^{1/p}}$$
does not satisfy (C_p) but
$$f(t) = \frac{t}{(\log t)^{a/p}}$$
satisfies (C_p) for every $a > 1$.
- Let us call a function f sublinear if $f(t)/t \to 0$ when $t \to \infty$. Surprisingly, one can easily check [13] that there exists no sublinear function that dominate all functions satisfying Property C_p. Hence, by Corollary 5, a function that dominates all the compression functions associated to uniform embeddings of a uniformly doubling space into L^p is at least linear.

In [13], we proved that Corollary 5 is actually true for a large variety of metric spaces, such as homogeneous Riemannian manifolds, 3-regular trees, etc. In the case of a 3-regular tree, the result is tight since in turn, the compression function associated to a uniform embedding in L^p has to satisfy condition (C_p). More surprising is that there exists a doubling metric measure space having such a property.

Proposition 6 (see the remark preceding Proposition 6.7) *There exists an infinite uniformly doubling graph such that for any uniform embedding F of X into an Hilbert space, the compression function of F has to satisfy condition (C_2), i.e.*
$$\int_1^{\infty} \left(\frac{\rho_F(t)}{t}\right)^2 \frac{dt}{t} < \infty.$$

Remark 3.4 Note that Corollary 5 should remain true if we merely assume that X is doubling as suggested by the result of Assouad [2] that $R_p(X) = 1$ for any doubling metric measure space. On the other hand, this lack of generality is partly compensated by the "equivariant" property of our constructions (see Sect. 4).

3.2 L^p-distortion of finite metric spaces

We also relate the quantitative property A to the L^p-distortion of finite metric spaces.

Definition 3.5

- The distortion of an injection between two metric spaces $F : (X, d) \to (Z, d)$ is the number (possibly infinite)
$$dist(F) = \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x, y)} \cdot \sup_{x \neq y} \frac{d(x, y)}{d(f(x), f(y))}.$$
- The ℓ^p-distortion $D_p(X)$ of a finite metric space X is the infimum of all $dist_F$ over all possible injections F from X to L^p.

Our main result concerning the L^p-distortion is that a finite 1-discrete[2] subset of a homogeneous manifold satisfies the following inequality

[2] By 1-discrete, we mean that the distance between two points of X is at least 1.

$$D_p(X) \leq C(\log \mathrm{Diam}(X))^{1/p}, \tag{3.1}$$

where $p \geq 2$, and C is a constant depending on the group. This result is optimal in the sense that for any Lie group (or any homogeneous Riemannian manifold) with exponential volume growth, there exists an increasing sequence of 1-discrete subsets X_n of diameter n satisfying

$$D_p(X_n) \geq c(\log \mathrm{Diam}(X_n))^{1/p} \quad \forall n \in \mathbf{N}. \tag{3.2}$$

Note that Bourgain [3] proved the inequalities (3.1) and (3.2) in case X is a finite binary rooted tree. We deduce (3.2) for Lie groups with exponential growth from Bourgain's theorem and from the fact [4] that any Lie group with exponential growth contains a quasi-isometrically embedded infinite binary rooted tree.

We also reprove [6, Theorem 4.1] the optimal upper bound (3.1) for the L^p-distortion of a uniform[3] doubling metric spaces, the constant C only depending on the doubling constant of the metric space. Again, we loose some generality by assuming uniform doubling property instead of doubling property, but in counterpart, we get very explicit embeddings, defined by simple expressions involving only the metric[4] and the measure (which makes them equivariant).

3.3 Optimality of the constructions and Poincaré inequalities

The graph of Proposition 6 is a planar self-similar graph introduced and studied in [7,8]. In [6], the authors show that this graph satisfies a "Poincaré-style" inequality (for short, let us say Poincaré inequality) and they deduce lower bounds on their L^p-distortions. Here, we use this Poincaré inequality to prove Proposition 6. The crucial role of Poincaré inequalities for obtaining lower bounds on Hilbert distortion has already been noticed in [11]. Here, we try to define the "more general possible" Poincaré inequalities that could be used to obtain, either constraints on the compression of uniform embeddings into L^p-space, or lower bounds on the L^p-distortion, for $1 \leq p < \infty$. We also propose a generalization of these inequalities in order to treat uniform embeddings into more general Banach spaces. We hope that these definitions will be helpful in the future. In particular, proving that Heisenberg satisfies a cumulated Poincaré inequalities[5] CP(J,p) with $J(t) = ct$ would provide optimal constraints on its L^p-compressions, for $p > 1$. Another (weaker) consequence would be that the L^p-distortion of balls of radius r of the standard Cayley graph of the discrete Heisenberg group is larger than $c(\log r)^{\min(1/2,1/p)}$, which is not known, even for $p = 2$, at least to our knowledge.

3.4 Organization of the paper

- In Sect. 4, we introduce the equivariant property A and give its interpretation in terms of the quasi-regular representation of Aut(X) on $L^p(X)$. We also prove that spaces with subexponential volume growth have equivariant property A.
- The central part of the paper is Sect. 5. In Sect. 5.1, we show how the A-profile can be used to construct uniform embeddings with "good" compression. In Sect. 5.2, we introduce general forms of Poincaré inequalities that provide constraints on the compression of uniform embeddings.

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3 This bound actually holds for all doubling metric spaces.

4 In [6], the constructions involve choices, either of nets at various scales, for the Bourgain-style embeddings, or of partitions for the Rao-style embeddings, which, at least at first sight, prevent them from being equivariant.

5 See Sect. 5.2.

- Finally, in Sect. 6, we estimate the A-profile for spaces with subexponential volume growth and Homogeneous Riemannian manifolds. Applying the results of Sect. 5, we obtain explicit constructions of uniform embeddings of these spaces into L^p-spaces.

4 Equivariant property A

4.1 Equivariant property A and quasi-regular representations of Aut(X)

Let us denote by Aut(X) the group of measure-preserving isometries of X. We define a notion of "equivariant" property A, which means that it behaves well under the action of Aut(X). We will see that this property implies that the quasi-regular representation of Aut(X) in $L^p(X)$ has almost invariant vectors for every $1 \leq p < \infty$.

Definition 4.1 (*Equivariant property A*) Let G be a group of isometries of X. We say that a metric measure space X has G-equivariant property A if there exists a sequence of families of unit vectors in $L^p(X)$ for one (equivalently for any) $1 \leq p < \infty$: $((\psi_{n,x})_{x\in X})_{n\in \mathbf{N}}$ satisfying the conditions of Definition 3.1 and the following additional one. For every $n \in \mathbf{N}$, $x, y \in X$ and $g \in G$,

$$\psi_{n,gx}(y) = \psi_{n,x}(g^{-1}y). \tag{4.1}$$

If G is the entire group of isometries of X, then we just say that X has equivariant property A (the same if $X = (X, d, \mu)$ is a metric measure space, and G is the group of measure-preserving isometries of X).

Remark 4.2 Note that if $f_{n,x}$ is defined only in terms of metric measure properties around the point x, such as $V(x,r)$ or $1_{B(x,r)}$ where r is a constant for instance, then it satisfies (4.1). This will be the case of all our constructions.

Proposition 4.3 *Assume that X has G-equivariant property A, then the quasi-regular representation of G on $L^p(X)$ for any $1 \leq p < \infty$ has almost invariant vectors. Moreover, if G acts transitively on X, then the converse is also true.*

Proof Let us prove the first assertion for $p = 1$. Let $((\psi_{n,x})_{x\in X})_{n\in\mathbf{N}}$ satisfy the assumptions of Definition 4.1. Then by (4.1), the sequence $h_n = \psi_{n,x}$ for any fixed x is almost-G-invariant. Conversely, if X is homogeneous and if h_n is an almost-G-invariant sequence in $L^p(X)$, then, given some $x_0 \in X$, we can define a sequence of families of unit vectors in $L^p(X)$: $((\psi_{n,x})_{x\in X})_{n\in\mathbf{N}}$ satisfying the conditions of Definition 4.1, by $\psi_{n,gx_0}(y) = h_n(g^{-1}y)$. □

4.2 Equivariant property A for metric measure spaces with subexponential growth

In this section, we give a short proof of the fact that subexponential growth implies Property A. This is originally due to Tu [14]. Tu's theorem works for any discrete metric space, so a slight adaptation makes it work for any metric measure space. The counterpart of this generality is that the proof is quite complicated and does not yield any equivariance. Here, restricting ourself to a certain class of metric measure spaces that includes all graphs and Riemannian manifolds for instance, we give a short proof that subexponential growth implies equivariant property A.

Recall that a metric measure space X has bounded geometry if for every $r > 0$, there exists $C_r < \infty$ such that

$$C_r^{-1} \leq V(x,r) \leq C_r \quad \forall x \in X.$$

Consider some $b > 0$ and define a b-geodesic distance on X by setting

$$d_b(x, y) = \inf_{\gamma} l(\gamma),$$

where γ runs over all chains $x = x_0, \ldots, x_m = y$ such that $d(x_{i-1}, x_i) \le b$ for all $1 \le i \le m$, and where $l(\gamma) = \sum_{i=1}^{m} d(x_i, x_{i-1})$ denotes the length of γ.

Definition 4.4 We say that a metric space (X, d) is coarsely geodesic if there exists $b > 0$ such that the identity map $(X, d_b) \to (X, d)$ is a uniform embedding.

Let (X, d, μ) be a metric measure space such that (X, d) is coarsely geodesic. As $d \ge d_b$, we have $V_b(x, r) \le V(x, r)$, where V_b denotes the volume of balls in (X, d_b, μ).

Proposition 4.5 *Let (X, d, μ) be a coarsely geodesic metric measure space. Assume that there exists a subexponential function $v : \mathbf{R}_+ \to \mathbf{R}_+$ such that $1 \le V(x, r) \le v(r)$ for every $x \in X$ and $r \ge 0$. Then X has equivariant property A.*

Proof Since X is coarsely geodesic, we can assume without loss of generality that d is a 1-geodesic distance (replacing d with d_1). It is then easy to see by a covering argument that there exists a constant $C < \infty$ such that $V(x, r+1) \le C V(x, r)$ for every $r > 0$. We define a sequence of families of probability densities $(\psi_{n,x})$ by

$$\psi_{n,x} = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{V(x, k)} 1_{B(x,k)} \quad \forall x \in X.$$

Let x and y be such that $d(x, y) \le 1$. For every $h > 0$, denote $S_h(x, r) = V(x, r+h) - V(x, r)$. We have

$$\begin{aligned}
&\left\| \frac{1}{V(x, k+1)} 1_{B(x,k+1)} - \frac{1}{V(y, k+1)} 1_{B(y,k+1)} \right\| \\
&\le \left\| \frac{1}{V(x, k+1)} (1_{B(x,k+1)} - 1_{B(y,k+1)}) \right\| + V(y, k+1) \left| \frac{1}{V(x, k+1)} - \frac{1}{V(y, k+1)} \right| \\
&\le 2 \frac{S_2(x, k)}{V(x, k+1)},
\end{aligned}$$

where the norm considered is the L^1-norm. Thus,

$$\begin{aligned}
\|\psi_{n,x} - \psi_{n,y}\| &\le \frac{1}{n} \sum_{k=1}^{n} \left\| \frac{1}{V(x, k+1)} 1_{B(x,k+1)} - \frac{1}{V(y, k+1)} 1_{B(y,k+1)} \right\| \\
&\le \frac{2}{n} \sum_{k=1}^{n} \frac{S_2(x, k)}{V(x, k+1)} \\
&\le \frac{2C}{n} \sum_{k=1}^{n} \frac{S_2(x, k)}{V(x, k+2)}
\end{aligned}$$

But,

$$\frac{S_2(x, k)}{V(x, k+2)} = \frac{S_1(x, k)}{V(x, k+2)} + \frac{S_1(x, k+1)}{V(x, k+2)} \le \frac{S_1(x, k)}{V(x, k+1)} + \frac{S_1(x, k+1)}{V(x, k+2)}.$$

Hence,

$$\begin{aligned}\|\psi_{n,x}-\psi_{n,y}\| &\leq \frac{4C}{n}\sum_{k=1}^{n+2}\frac{S_1(x,k)}{V(x,k+1)}\\ &= \frac{4C}{n}\sum_{k=1}^{n+2}\frac{V(x,k+1)-V(x,k)}{V(x,k+1)}\\ &\leq \frac{4C}{n}\log\left(\frac{V(x,n+3)}{V(x,1)}\right)\\ &\leq \frac{4C}{n}\log v(n+3)\end{aligned}$$

We conclude since v is subexponential. □

5 Geometric conditions to control compression and distortion

5.1 Quantitative property A, construction of uniform embeddings and upper bounds on distortion

Definition 5.1 Let $X=(X,d,\mu)$ be metric measure space, $J:\mathbf{R}_+\to\mathbf{R}_+$ be some increasing map and let $1\leq p<\infty$. We say that X has property $A(J,p)$ if for every $n\in\mathbf{N}$, there exists a map $\psi_n:X\to L^p(X)$ such that

- for every $x\in X$, $\|\psi_{n,x}\|_p\geq J(n)$,
- $\|\psi_{n,x}-\psi_{n,y}\|_p\leq d(x,y)$,
- $\psi_{n,x}$ is supported in $B(x,n)$.

Remark 5.2 Basic L^p-calculus shows that if $q\geq p\geq 1$, then Property $A(J,q)$ implies Property $A(J,p)$ and Property $A(J,p)$ (only) implies Property $A(J^{p/q},q)$.

This definition is motivated by the following two propositions.

Proposition 5.3 *Let X be a metric measure space satisfying Property $A(J,p)$. Then, for every increasing function f satisfying*

$$\int_1^\infty\left(\frac{f(t)}{J(t)}\right)^p\frac{dt}{t}<\infty, \qquad (J,p)$$

there exists a large-scale Lipschitz uniform embedding F of X into $\oplus_\infty^{\ell^p}L^p(X,\mu)$ with compression $\rho\succeq f$. In particular,

$$R_p(X)\geq\liminf_{t\to\infty}\log\frac{J(t)}{\log t}.$$

Proof Choose a sequence $(\psi_{n,x})$ like in Definition 5.1. Fix an element o in X and define

$$F(x)=\bigoplus_{k\in\mathbf{N}}^{\ell^p}F_k(x)$$

where

$$F_k(x)=\left(\frac{f(2^k)}{J(2^k)}\right)(\psi_{2^k,x}-\psi_{2^k,o}).$$

The fact that F exists and is Lipschitz follows from the fact that Condition (J, p) is equivalent to

$$\sum_k \left(\frac{f(2^k)}{J(2^k)}\right)^p < \infty.$$

Hence, a direct computation yields

$$\begin{aligned}\|F_k(x) - F_k(y)\|_p &\leq \left(\sum_k \left(\frac{f(2^k)}{J(2^k)}\right)^p \|\psi_{2^k,x} - \psi_{2^k,y}\|_p^p\right)^{1/p} \\ &\leq d(x, y)\left(\sum_k \left(\frac{f(2^k)}{J(2^k)}\right)^p\right)^{1/p}.\end{aligned}$$

On the other hand, since $\psi_{2^k,x}$ is supported in $B(x, 2^k)$, if $d(x, y) > 2.2^k$, then the supports of $\psi_{2^k,x}$ and $\psi_{2^k,y}$ are disjoint. Thus

$$\begin{aligned}\|F(x) - F(y)\|_p \geq \|F_k(x) - F_k(y)\|_p &\geq \left(\|\psi_{2^k,x}\|_p^p + \|\psi_{2^k,y}\|_p^p\right)^{1/p} / J(2^k) f(2^k) \\ &\geq 2^{1/p} f(2^k).\end{aligned}$$

whenever $d(x, y) > 2.2^k$. So we are done. □

Remark 5.4 Note that the proposition does not provide a uniform embedding for any unbounded J as it might happen that $\int_1^\infty (\frac{1}{J(t)})^p \frac{dt}{t} = \infty$. Nevertheless, as soon as J is not bounded and $f(t) = o(J(t))$, one can choose an increasing injection $i : \mathbf{N} \to \mathbf{N}$ such that

$$\sum_n \left(\frac{f(2^{i(n)})}{J(2^{i(n)})}\right)^p < \infty,$$

so that

$$F(x) = \bigoplus_n^{\ell^p} \left(\frac{f(2^{i(n)})}{J(2^{i(n)})}\right)(\psi_{2^{i(n)}}(x) - \psi_{2^{i(n)}}(o))$$

defines a uniform embedding of X whose compression satisfies $\rho \succeq f \circ i^{-1}$.

Proposition 5.5 *Let X be a finite metric space satisfying Property $A(J, p)$. Then,*

$$D_p(X) \leq 2\left(\int_1^{\mathrm{Diam}(X)/4} \left(\frac{t}{J(t)}\right)^p \frac{dt}{t}\right)^{1/p}.$$

In particular, if $J(t) \geq ct$, then

$$D_p(X) \leq C\,(\log(\mathrm{Diam}(X)))^{1/p}.$$

Proof Fix an element o in X, set $n = [\log(\mathrm{Diam}(X))/2]$ and define

$$F(x) = \bigoplus_{k\in\mathbf{N}}^{\ell^p} F_k(x)$$

where

$$F_k(x) = \left(\frac{2^k}{J(2^k)}\right)(\psi_{2^k,x} - \psi_{2^k,o}).$$

We have

$$\begin{aligned}\|F(x) - F(y)\|_p &\le d(x,y)\left(\sum_{k=0}^{n}\left(\frac{2^k}{J(2^k)}\right)^p\right)^{1/p}\\ &\le d(x,y)\left(\int_1^{\mathrm{Diam}(X)/2}\left(\frac{t}{J(t/2)}\right)^p \frac{dt}{t}\right)^{1/p}\\ &= 2^{2/p} d(x,y)\left(\int_1^{\mathrm{Diam}(X)/4}\left(\frac{t}{J(t)}\right)^p \frac{dt}{t}\right)^{1/p}.\end{aligned}$$

On the other hand, since $\psi_{2^k,x}$ is supported in $B(x, 2^k)$, if $d(x, y) > 2.2^k$, then the supports of $\psi_{2^k,x}$ and $\psi_{2^k,y}$ are disjoint. Thus

$$\|F(x) - F(y)\|_p \ge \|F_k(x) - F_k(y)\|_p \ge 2^k \left(\|\psi_{2^k,x}\|_p^p + \|\psi_{2^k,y}\|_p^p\right)^{1/p} / J(2^k) \ge 2^{1/p}2^k.$$

whenever $d(x, y) > 2.2^k$. So we are done. □

Remark 5.6 [**Equivariance**] This remark concerns the embeddings F constructed in both Propositions 5.3 and 5.5.

Let $G = \mathrm{Aut}(X)$ be the group of measure preserving isometries of X. Assume that in Propositions 5.3 and 5.5, the metric measure space $X = (X, d, \mu)$ actually satisfies the equivariant property A(J,p), i.e. if

$$\psi_{n,gx}(y) = \psi_{n,x}(g^{-1}y).$$

Then, the maps F constructed in the proofs of those propositions are G-equivariant, according to Definition 4.1. More precisely, there exists an affine isometric action σ_F of G on $\oplus_{\infty}^{\ell^p} L^p(X, \mu))$, whose linear part is the action by composition (which is isometric since the elements of G preserve the measure), such that for every $g \in G$ and every $x \in X$,

$$\sigma_F(g)F(x) = F(gx).$$

In particular, Hence for every $g \in G$, we have

$$\forall x, y \in X, \quad \|F(gx) - F(gy)\|_p = \|F(x) - F(y)\|_p.$$

In particular, if $X = G$ is a compactly generated, locally compact group, then $b(g) = F(g) - F(1)$ defines a 1-cocycle of G on the infinite direct sum of the left regular representation (see [13]).

5.2 Poincaré inequalities, constraints on uniform embeddings and lower bounds on distortion

In this section, we introduce general "Poincaré-like" inequalities in order to provide obstructions to embed a metric space into an L^p-space, for $1 \le p < \infty$. The reader will note that these inequalities are trivially inherited from a subspace (that is a subset equipped with the induced metric).

In the sequel, let (X, d) be a metric space and let $1 \le p < \infty$. For any $r > 0$, we denote

$$E_r = \{(x, y) \in X^2,\ d(x, y) \ge r\}.$$

5.2.1 The Poincaré inequality P(J,p)

Definition 5.7 Let $J : \mathbf{R}_+ \to \mathbf{R}_+$ be a increasing function and let $r > 0$. We say that X satisfies a Poincaré inequalities P(J,p) at scale r if the following holds. There exists a Borel probability P_r on E_r and a Borel probabilily Q_r on X^2 such that for every compactly supported continuous functions $\varphi : X \to \mathbf{R}$,

$$\int_{E_r} \left(\frac{|\varphi(x) - \varphi(y)|}{J(r)}\right)^p dP_r(x, y) \le \int_{E_1} \left(\frac{|\varphi(x) - \varphi(y)|}{d(x, y)}\right)^p dQ_r(x, y).$$

This definition is motivated by the following simple proposition.

Proposition 5.8 *Let $p > 1$. If a Metric space X satisfies a Poincaré inequality P(J,p) at scale r, then for any measurable large-scale Lipschitz map F of X to a L^p-space, the compression ρ_F has to satisfy*

$$\rho_F(t) \le J(t)$$

for $t \le r$. If X is finite, then,

$$D_p(X) \ge \frac{r}{J(r)}.$$

Proof Let $F : X \to L^p([0, 1], \lambda)$ be a measurable large-scale Lipschitz map. For almost every $t \in [0, 1]$, the map $F_t(x) = |F(x)(t)|^p$ defines a measurable map from X to $\mathbf{R}$. Applying the Poincaré inequality to this map and then integrating over t yields, by Fubini Theorem,

$$\int_{E_r} \left(\frac{\|F(x) - F(y)\|_p}{J(r)}\right)^p dP_r(x, y) \le \int_{E_1} \left(\frac{\|F(x) - F(y)\|_p}{d(x, y)}\right)^p dQ_r(x, y).$$

Now, the bounds for ρ and $D_p(X)$ follow easily. □

The skew cube inequality: In [1], upper bounds on the Hilbert compression rate are proved for a wide variety of finitely generated groups including Thompson's group F, $\mathbf{Z} \wr \mathbf{Z}$, etc. To show these bounds, they consider for all $n \in \mathbf{N}$, injective group morphisms $j : \mathbf{Z}^n \to G$. Then, they focus on the image, say C_n, of the n-dimensional cube $\{-1, 1\}^n$. Let F be a uniform embedding from G into a Hilbert space $\mathcal{H}$. They apply the well-known screw-cube inequality in Hilbert spaces to $F(C_n)$. This inequality says that sum of squares of edges of a cube is less or equal than the sum of squares of its diagonals. To conclude something about the compression of F, they need an upper bound (depending on n) on the length of diagonals of C_n and a lower bounds on the length of its edges. It is easy to check that this actually remains to prove a Poincaré inequality P(J,2) for a certain function J (for instance, $J(t) = t^{1/2} \log t$ for Thompson's group; and $J(t) = t^{3/4}$ for $Z \wr \mathbf{Z}$).

Let us briefly explain how one can deduce a Poincaré inequality from the skew cube inequality. Denote by Δ_n the set of edges of C_n (seen as a cube embedded in G), and by D_n the set of diagonals. Assume that for all $(x, y) \in \Delta_n$, $d(x, y) \le l_n$ and for all $(x, y) \in D_n$, $d(x, y) \ge L_n$, which actually means that $D_n \subset E_{L_n}$. We have $|\Delta_n| = n2^{n-1}$ and $|D_n| =$

2^{n-1}. Take a function $\varphi : G \to \mathbf{R}$. The skew cube inequality for the image of C_n under φ yields

$$\sum_{(x,y)\in D_n} |\varphi(x)-\varphi(y)|^2 \le \sum_{(x,y)\in \Delta_n} |\varphi(x)-\varphi(y)|^2.$$

An easy computation shows that this implies the following inequality

$$\frac{1}{|D_n|}\sum_{(x,y)\in D_n}\left(\frac{|D_n|^{1/2}|\varphi(x)-\varphi(y)|}{l_n|\Delta_n|^{1/2}}\right)^2 \le \sum_{(x,y)\in \Delta_n}\left(\frac{|\varphi(x)-\varphi(y)|}{d(x,y)}\right)^2,$$

which is nothing but P(J,2) with $J(L_n) = l_n|\Delta_n|^{1/2}/|D_n|^{1/2} = l_n n^{1/2}$.

Expanders: Note that a metric space satisfying P(J,p) with a constant function J does not admit any uniform embedding in any L^p-space. This is the case of families of expanders when $p = 2$. Recall [10] that a sequence of finite graphs $(X_i)_{i\in\mathbf{N}}$ is called a family of expanders if

- for every $i \in I$, the degree of X_i is bounded by a constant d;
- the cardinal $|X_i|$ of X_i tends to infinity when n goes to infinity;
- there is a constant $C > 0$ such that for all $i \in I$, and every function $f : X \to \mathbf{R}$,

$$\frac{1}{|X_i|^2}\sum_{(x,y)\in X_i^2} |f(x)-f(y)|^2 \le \frac{C}{|X_i|}\sum_{x\sim y}|f(x)-f(y)|^2.$$

As the volume of a ball of radius r in X_i is less than d^r, for i large enough, we have

$$|E_r| \ge |X_i|(|X_i|-d^r) \ge |X_i|^2/2.$$

Hence, the third property of expanders is equivalent to property P(J,2), where $J \approx 1$, P_r is the average over E_r, and Q_r is the average over $\{(x,y)\in X_i^2,\ d(x,y)=1\} \subset E_1$.

5.2.2 The cumulated poincaré inequality CP(J,p)

More subtle, the following definition will provide a finer control on distortions and compressions.

Definition 5.9 Let $K : \mathbf{R}_+ \to \mathbf{R}_+$ be a increasing function and let $r > 0$. We say that X satisfies a cumulated Poincaré inequalities CP(J,p) at scale r if the following holds. There exist Borel probabilities $P_{r,k}$ on E_{2^k} for $k = 1, 2, \ldots [\log_2 r]$ and a Borel probabilily Q_r on E_1 such that for every measurable function,

$$\sum_{k=1}^{[\log_2 r]}\int_{E_{2^k}}\left(\frac{|\varphi(x)-\varphi(y)|}{J(2^k)}\right)^p dP_{r,k}(x,y) \le \int_{E_1}\left(\frac{|\varphi(x)-\varphi(y)|}{d(x,y)}\right)^p dQ_r(x,y).$$

Here is the main application of these inequalities.

Proposition 5.10 *Let $p > 1$. If a Metric space X satisfies a cumulated Poincaré inequality CP(J,p) at scale r, then for any Lipschitz map F of X to a L^p-space, the compression ρ_F has to satisfy*

$$\int_1^r\left(\frac{\rho_F(t)}{J(t)}\right)^q \frac{dt}{t} \le 1,$$

where $q = \max(2, p)$. *If X is finite, then,*

$$D_p(X) \geq \left(\int_1^r \left(\frac{t}{J(t)}\right)^p \frac{dt}{t}\right)^{\min(1/p,1/2)}.$$

Proof By a similar argument as for last proposition, we obtain the following inequality

$$\sum_{k=1}^{[\log_2 r]} \int_{E_{2^k}} \left(\frac{\|F(x)-F(y)\|_p}{J(2^k)}\right)^p dP_{r,k}(x,y) \leq \int_{E_1} \left(\frac{\|F(x)-F(y)\|_p}{d(x,y)}\right)^p dQ_r(x,y).$$

And, again, the proposition follows easily. □

Trees and doubling graphs: For a tree or for the doubling graph of Proposition 6, the fact that any uniform embedding into an L^p-space satisfies Property C_p is a consequence of the inequality[6] CP(J,p) for $J(t) = ct$.

Note that this inequality does not say anything for $p = 1$ since the 3-regular tree T admits a (trivial) bi-Lipschitz embedding into ℓ^1.

5.2.3 Poincaré inequalities with values in a Banach space

To generalize these inequalities in order to treat embeddings into more general Banach spaces, we can define P(J,p) (resp. CP(J,p)) with values in a Banach space E to be the same inequalities applied to elements in the Banach space $L^p(X, E)$ consisting of functions $\varphi : X \to E$ such that $x \to \|\varphi(x)\|$ is in $L^p(X)$. We equip $L^p(X, E)$ with the norm

$$\|\varphi\|_p = \left(\int_X \|\varphi(x)\|^p d\mu(x)\right)^{1/p}.$$

In [3], Bourgain proves that if E is a uniformly p-convex Banach space, then the E-distortion of the binary tree T_n of dept n is more than a constant times $(\log n)^{1/q}$ where $q = \max\{p, 2\}$. To obtain this result, he actually proves that T_n satisfies CP(J,p), with values in E, and with $J(t) = ct$.

In [9], Lafforgue constructs a sequence of expanders satisfying P(J,2) with values in any uniformly convex Banach space E, and with $J = constant$. In particular his expanders do not uniformly embed into E.

6 Application to certain classes of metric spaces

6.1 Spaces with subexponential growth

The proof of Proposition 4.5 yields the following proposition.

Proposition 6.1 *Let* (X, d, μ) *be a quasi-geodesic metric measure space with bounded geometry. Assume that there exists a subexponential function* $v : \mathbf{R}_+ \to \mathbf{R}_+$ *such that* $1 \leq V(x,r) \leq v(r)$ *for every* $x \in X$ *and* $r \geq 1$. *Then X has equivariant property* $A(J_p, p)$ *for every* $1 \leq p < \infty$, *with* $J_p(t) \approx (t/\log v(t))^{1/p}$.

Proof The proof of Proposition 4.5 gives the result for $p = 1$. Then, by Remark 5.2, we deduce it for all $1 \leq p < \infty$. □

[6] Proved in [3] for the trees, and in [6] for the doubling graph.

Corollary 6.2 *Assume that* $v(t) \preceq e^{t^\beta}$*, for some* $\beta < 1$*. Then, for every* $p \geq 1$*,*

$$R_p(X) \geq (1-\beta)/p.$$

We can also improve Proposition 6.1 by assuming some uniformity on the volume of balls.

Proposition 6.3 *Let* (X, d, μ) *be a metric measure space. Assume that there exists an increasing function* v *and* $C \geq 1$ *such that*

$$1 \leq v(r) \leq V(x, r) \leq Cv(r), \quad \forall x \in X, \quad \forall r \geq 1;$$

Then for every $p \geq 1$*,* X *satisfies equivariant property* $A(J, p)$ *with* $J(t) \approx t/\log v(t)$*.*

Corollary 6.4 *Keep the same hypothesis as in Proposition 6.3 and assume that* $v(t) \preceq e^{t^\beta}$*, for some* $\beta < 1$*. Then, for every* $p \geq 1$*,*

$$R_p(X) \geq 1 - \beta.$$

Proof of Proposition 6.3 Define

$$k(n) = \sup\{k;\ v(n-k) \geq v(n)/2\}$$

and

$$j(n) = \sup_{1 \leq j \leq n} k(j).$$

Note that as v is sub-exponential, $j(n) \geq 1$ for n large enough. We have

$$v(n) \geq 2^{n/j(n)} v(1)$$

which implies

$$j(n) \geq \frac{n}{\log v(n)}.$$

Let $q_n \leq n$ be such that $j(n) = k(q_n)$. Now define

$$\psi_{n,x} = \frac{1}{v(q_n)^{1/p}} \sum_{k=1}^{q_n - 1} 1_{B(x,k)}.$$

If $d(x, y) \leq 1$, we have

$$\|\psi_{n,x} - \psi_{n,y}\|_p^p \leq \frac{\mu(B(x, q_n))}{v(q_n)} \leq C.$$

On the other hand

$$\|\psi_{n,x}\|_p \geq j(n) \left(\frac{\mu(B(x, q_n - j(n)))}{v(q_n)} \right)^{1/p} \geq (1/2)^{1/p} \frac{n}{\log v(n)}$$

so we are done. □

Proposition 6.5 *Let* (X, d, μ) *be a uniformly doubling metric measure space. Then for every* $p \geq 1$*,* X *satisfies the property* $A(J, p)$ *with* $J(t) \approx t$*.*

Corollary 6.6 *Let (X, d, μ) be a uniformly doubling metric measure space. Let $p \geq 1$ and $q = \max(p, 2)$. Then, for every increasing $f : \mathbf{R}_+ \to \mathbf{R}_+$ satisfying*

$$\int_1^\infty \left(\frac{f(t)}{t}\right)^q \frac{dt}{t} < \infty,$$

there exists a uniform embedding F of X into $\oplus_\infty^{\ell^p} L^p(X, \mu)$ with compression $\rho \succeq f$. In particular,

$$R_p(X) = 1.$$

Moreover, if X is finite, then

$$D_p(X) \leq C\,(\log \mathrm{Diam}(X))^{1/q}$$

where C only depends on the doubling constant of X.

Proof of Proposition 6.5 Recall that being uniformly doubling implies that there exists a function v and $C, C' > 0$ such that $v(2r) \leq Cv(r)$ and $v(r) \leq V(x, r) \leq C'v(r)$ and for all $x \in X$, and $r > 0$.

Define

$$\psi_{n,x} = \sum_{k=[n/2]}^{n} \frac{1}{v(k)^{1/p}} 1_{B(x,k)} \quad \forall x \in X.$$

We have

$$\|\psi_{n,x}\|_p \geq \frac{n}{2}\left(\frac{v(n)}{C'v([n/2])}\right)^{1/p} \geq \frac{n}{2(CC')^{1/p}}.$$

On the other hand, for x and y in X at distance d, we want to prove that

$$\|\psi_{n,x} - \psi_{y,n}\|_p \leq C''d$$

for some constant C''. For obvious reasons, we can assume that $d \leq n/4$. Hence, we can write this difference as follows

$$\psi_{n,x} - \psi_{y,n} = \sum_{i=1}^{2d}\left(\sum_{j=1}^{[n/4d]} \frac{1}{v(2jd+i)^{1/p}}\left(1_{B(x,2jd+i)} - 1_{B(y,2jd+i)}\right)\right).$$

But note that for any i, j,

$$B(x, 2jd+i) \vartriangle B(y, 2jd+i) \subset B(x, 2(j+1)d+i) \smallsetminus B(x, 2jd+i).$$

Hence, taking the absolute value, we obtain

$$|\psi_{n,x} - \psi_{y,n}| \leq \sum_{i=1}^{2d}\left(\sum_{j=1}^{[n/4d]} \frac{1}{v(2jd+i)^{1/p}} 1_{B(x,2(j+1)d+i) \smallsetminus B(x,2jd+i)}\right).$$

Moreover, for distinct j and j', $B(x, 2(j+1)+i) \setminus B(x, 2jd+i)$ and $B(x, 2(j'+1)d+i) \setminus B(x, 2j'd+i)$ are disjoint. Hence, taking the L^p-norm, we get

$$\begin{aligned}\|\psi_{n,x} - \psi_{y,n}\|_p &\leq \sum_{i=1}^{2d} \left(\sum_{j=1}^{[n/4d]} \frac{1}{v(jd+i)} |B(x, 2(j+1)d+i) \setminus B(x, 2jd+i)| \right)^{1/p} \\ &\leq \sum_{i=1}^{2d} \frac{1}{v([n/4]^{1/p})} \left(\sum_{j=1}^{[n/4d]} |B(x, 2(j+1)d+i) \setminus B(x, 2jd+i)| \right)^{1/p} \\ &= d \frac{V(x, 2n+2d)^{1/p}}{v([n/2]^{1/p})} \\ &= C''d. \end{aligned}$$

□

Now, the optimality of Corollary 6.6 follows from Proposition 5.10 and the following result, essentially remarked in [6, Theorem 4.1].

Proposition 6.7 *There exists an infinite uniformly doubling graph satisfying CP(J,2) for a linear increasing function J.*

6.2 Homogeneous Riemannian manifolds

Theorem 6.8 *Let X be a homogeneous Riemannian manifold. Then, X has Property A(J,p) for all $p \geq 1$ and $J = ct$ where c depends only on X. Moreover, X has Equivariant Property A if and only if Isom(X) is amenable.*

Proof In [13], we prove that for any amenable Lie group G, equipped with a left Haar measure and with a left-invariant Riemannian metric[7] and for every $1 \leq p < \infty$ and every $n \in \mathbf{N}$, there exists a measurable function $h_n : G \to \mathbf{R}$ whose support lies in the ball $B(1, n)$ and such that for every element $g \in G$ of length less than 1,

$$\|h_n(g\cdot) - h_n\|_p \leq 1$$

and

$$\|h_n\|_p \geq cn$$

for a constant c only depending on G. To see that G satisfies equivariant A(J,p) with $J(t) = ct$, we construct as in the proof of Proposition 4.3, a sequence

$$\psi_{n,g}(y) = h_n(g^{-1}y).$$

Now, let X be a homogeneous manifold and let G be its group of isometries. We have $X = G/K$ where K is a compact subgroup of G.

First, assume that G is amenable. Averaging them over K, we can assume that the h_n are K-bi-invariant and then we can push them through the projection $G \to X$. We therefore get equivariant property A(J,p) for X. Conversely, if X has equivariant property A, then by Proposition 4.3, the quasi-regular representation of G on $L^p(X)$ has almost-invariant vectors. Lifting them to G, we obtain almost-invariant vectors on the left regular representation of G on $L^p(G)$, which implies that G is amenable.

[7] Actually, we prove it using a word length metric on G, but such a metric is quasi-isometric to any Riemannian one.

We are left to prove that even when G is not amenable, X satisfies A(J,p). Recall that every connected Lie group has a connected solvable co-compact subgroup. So any homogeneous Riemannian manifold is actually quasi-isometric to some amenable connected Lie group. Hence the first statement of the theorem follows from Lemma 6.9. □

Lemma 6.9 *If $F : X \to Y$ is a quasi-isometry between two metric measure spaces with bounded geometry, then if X satisfies A(J,p), then Y satisfies A(J',p) with $J' \approx J$ for some constant $c > 0$.*

Proof of the Lemma If we assume that X and Y are discrete spaces equipped with the counting measure and that F is a bi-Lipschitz map, then the claim is obvious. Now, to reduce to this case, we just have to prove that we can replace X and Y by any of their nets, which is essentially proved in [12, Lemma 2.2]. □

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