Geom Dedicata (2008) 133:169–179
DOI 10.1007/s10711-008-9241-5

ORIGINAL PAPER

Curvature integrals under the Ricci flow on surfaces

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Received: 4 December 2007 / Accepted: 13 February 2008 / Published online: 5 March 2008

Abstract In this paper, we consider the behavior of the total absolute and the total curvature under the Ricci flow on complete surfaces with bounded curvature. It is shown that they are monotone non-increasing and constant in time, respectively, if they exist and are finite at the initial time. As a related result, we prove that the asymptotic volume ratio is constant under the Ricci flow with non-negative Ricci curvature, at the end of the paper.

Keywords Ricci flow · Total absolute curvature · Total curvature · Asymptotic volume ratio

Mathematics Subject Classification (2000) 53C44

1 Introduction

The Ricci flow equation $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$, where R_{ij} denotes the Ricci tensor of $g(t)$, which was introduced by Hamilton, has been and will be a powerful tool in Riemannian geometry. In the present paper, we are concerned with the behavior of the Ricci flow mainly on non-compact surfaces.

We define the Ricci flow on surfaces. Let (M, g_0) be a Riemannian surface. A one-parameter family of Riemannian metrics $g(t),\ t \in [0, T)$, on M is called the Ricci flow with the initial metric g_0 if it satisfies the following evolution equation:

$$\frac{\partial}{\partial t} g = -Rg, \quad g(0) = g_0, \tag{1}$$

where $R = R(x, t)$ denotes the scalar curvature at the point $x \in M$ for $g(t)$. The scalar curvature is twice the Gaussian curvature on surfaces.

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Since the Ricci flow on closed manifolds does not preserve its volume in general, we can rescale the flow so that the volume is constant in time. The resulting flow is called the normalized Ricci flow. Hamilton and Chow proved the long time existence and convergence of the normalized Ricci flow on closed surfaces:

Theorem 1 (Hamilton [7], Chow [1]) *For any closed Riemannian surface (M, g_0), the normalized Ricci flow $g(t)$ with the initial metric g_0 exists for $t \in [0, \infty)$ and converges to the constant curvature metric as $t \to \infty$.*

We refer the reader to Hamilton's original paper [7] and [3, Chap. 5] for the basic properties of the Ricci flow on surfaces.

On the other hand, there are few results about the behavior of the Ricci flow on non-compact surfaces ([10]). The main interest of this paper lies in a better understanding of it. We consider the unnormalized Ricci flow $g(t)$ which is complete as a Riemannian metric with sectional curvature bounded in absolute value by $K(t) > 0$ at each time. Here $K(t)$ depends on t continuously. The existence of such a Ricci flow for complete initial metric with bounded curvature was established by Shi [11].

Our main theorems are as follows.

Theorem 2 *Let $g(t)$ be a Ricci flow which is complete with bounded curvature on a possibly compact surface M. Assume that the scalar curvature $R(\cdot, 0)$ of the initial metric $g(0)$ is an L^1-function. Then the total absolute curvature $\int_M |R(\cdot, t)|d\mu$ is monotone non-increasing. In particular, the Ricci flow on surfaces preserves the L^1-property of its scalar curvature.*

Theorem 3 *Let $g(t)$ be a Ricci flow which is complete with bounded curvature on a surface M. Assume that the initial metric $g(0)$ admits finite total scalar curvature $\int_M R(\cdot, 0)d\mu$. Then the total scalar curvature $\int_M R(\cdot, t)d\mu$ exists and remains constant for $t \geq 0$.*

As far as the author knows, the Ricci flow on non-compact surfaces was studied at first in [13], where some claims similar to our main theorems are stated. She argued in the proof of Proposition 7.4. in [13] as follows:

$$\begin{aligned}
\frac{d}{dt}\int_M |R| d\mu &= \frac{d}{dt}\int_{M_+} R d\mu - \frac{d}{dt}\int_{M_-} R d\mu \\
&= \int_{M_+} \Delta R d\mu - \int_{M_-} \Delta R d\mu \\
&= \int_{\partial M_+} \langle \nabla R, \nu_+ \rangle - \int_{\partial M_-} \langle \nabla R, \nu_- \rangle \\
&\leq 0,
\end{aligned}$$

where M_+ and M_- are the regions where R is non-negative or non-positive, respectively, and $\nu_\pm$ denote the outward normal vectors along the boundaries $\partial M_\pm$. It is not clear to the author whether her argument works well because the integrability of ΔR and the smoothness of the boundary $\partial M_\pm$ are not discussed there.

Although Theorems 2 and 3 may be proved by using tools of the PDE theory, we are interested in these invariants from the geometrical view point. The proof given below consists of geometrical techniques. The author expects that our proof will contribute to further study of the behavior of the Ricci flow on surfaces.

2 Preliminaries

In this section, we collect some formulae and properties of the Ricci flow. These are also found in the book [3].

The evolution equation of the scalar curvature under the Ricci flow on surfaces is given by

$$\frac{\partial}{\partial t} R = \Delta R + R^2. \tag{2}$$

The time-derivative of the volume form $d\mu = d\mu_t$ induced by the Ricci flow $g(t)$ is

$$\frac{\partial}{\partial t} d\mu = -R d\mu. \tag{3}$$

Next, we present some properties of the Ricci flow which are useful for studying the behavior of it. One can see that many quantities increase at most or decrease at worst exponentially, and hence change continuously with respect to t under the Ricci flow with bounded Ricci curvature.

Proposition 1 *Let $g(t),\ t \in [0, T]$, be a Ricci flow with bounded Ricci curvature $|\mathrm{Ric}| \le K$ on an n-manifold M. Then the following hold: for all $p \in M,\ r > 0$ and $t_1, t_2 \in [0, T]$ with $t_2 - t_1 = \Delta t$,*

(1)

$$B_{t_2}(p, re^{-K|\Delta t|}) \subset B_{t_1}(p, r) \subset B_{t_2}(p, re^{K|\Delta t|}), \tag{4}$$

where $B_t(p, r)$ denotes the open metric ball of center $p \in M$ and radius r with respect to $g(t)$.

(2) *If $g(t_1)$ is complete, then so is $g(t_2)$.*

(3) *For any time-independent measurable set $S \subset M$,*

$$\mathrm{Vol}_{t_2}(S)e^{-nK|\Delta t|} \le \mathrm{Vol}_{t_1}(S) \le \mathrm{Vol}_{t_2}(S)e^{nK|\Delta t|}, \tag{5}$$

where we used $\mathrm{Vol}_t(S)$ to denote the volume of $S \subset M$ with respect to $g(t)$.

(4) *Put $\overline{\nu}(t) := \limsup_{r\to\infty} \mathrm{Vol}_t(B_t(p, r))/r^n$. Then,*

$$\overline{\nu}(t_2)e^{-2nK|\Delta t|} \le \overline{\nu}(t_1) \le \overline{\nu}(t_2)e^{2nK|\Delta t|}. \tag{6}$$

The same is true for $\underline{\nu}(t) := \liminf_{r\to\infty} \mathrm{Vol}_t(B_t(p, r))/r^n$.

Proof These properties follow form the fact that the metrics $g(t_1)$ and $g(t_2)$ are equivalent to each other:

$$e^{-2K|\Delta t|} g(t_2) \le g(t_1) \le e^{2K|\Delta t|} g(t_2), \tag{7}$$

which simply follows from the Ricci flow equation and the bound on the Ricci curvature $|\mathrm{Ric}| \le K$. □

A slightly weaker result than the following lemma is stated in [8, Theorem 18.3]. It is assumed there that the curvature operator is non-negative.

Lemma 1 *Let $g(t),\ t \in [0, T)$, be a Ricci flow which is complete with bounded curvature on a non-compact n-manifold M. Suppose that $|\mathrm{Ric}|(x, t) \to 0$ as $d_t(x, p) \to \infty$ for all $t \in [0, T)$, where $p \in M$ is a fixed point. Then $\overline{\nu}(t)$ and $\underline{\nu}(t)$ are constant on $[0, T)$. In particular, if the asymptotic volume ratio $\nu(t) := \lim_{r\to\infty} \mathrm{Vol}_t(B_t(p, r))/r^n$ exists at the initial time, then $\nu(t)$ exists and is constant for all $t \in [0, T)$.*

Proof We only prove that $\overline{\nu}(t)$ is constant. All we have to do is to modify the proof of Proposition 1.(1) by utilizing that the Ricci curvature is arbitrarily small outside a compact set.

For any $t_1, t_2 \in [0, T)$ and $\varepsilon > 0$, take $\sigma, K > 0$ and T_1, T_2 satisfying that $t_1, t_2 \in [T_1, T_2] \subset [0, T)$ and that for all $t \in [T_1, T_2]$,

$$|\mathrm{Ric}| \leq \varepsilon \text{ on } M \setminus B_t(p,\sigma), \quad \text{and } |\mathrm{Ric}| \leq (K+1)\varepsilon \text{ on } B_t(p,\sigma). \tag{8}$$

Then, for any $q \in M$,

$$\begin{aligned}
\frac{d^+}{dt} d_t(p,q) &:= \limsup_{t' \searrow t} \frac{d_{t'}(p,q) - d_t(p,q)}{t'-t} \\
&\leq \lim_{t' \searrow t} \frac{\mathrm{Leng}_{t'}(\gamma_t) - \mathrm{Leng}_t(\gamma_t)}{t'-t} \\
&= \int_\gamma -\mathrm{Ric}(\gamma_t', \gamma_t')ds \\
&\leq (K+1)\varepsilon\sigma + \varepsilon(d_t(p,q) - \sigma) \\
&= \varepsilon(K\sigma + d_t(p,q)),
\end{aligned}$$

where $d_t(p,q)$ and γ_t denote the distance and a minimal geodesic for $g(t)$ between p and q, respectively.

Similarly,

$$\frac{d^-}{dt} d_t(p,q) := \liminf_{t' \nearrow t} \frac{d_t(p,q) - d_{t'}(p,q)}{t-t'} \geq -\varepsilon(K\sigma + d_t(p,q)). \tag{9}$$

Then, putting $\Delta t := t_2 - t_1$,

$$d_{t_2}(p,q) \leq d_{t_2}(p,q) + K\sigma \leq (d_{t_1}(p,q) + K\sigma)e^{\varepsilon|\Delta t|}, \tag{10}$$

or equivalently,

$$B_{t_2}(p,r) \subset B_{t_1}\left(p, (r+K\sigma)e^{\varepsilon|\Delta t|}\right), \tag{11}$$

for all $r > 0$.

Using this and Proposition 1.(3),

$$\begin{aligned}
\overline{\nu}(t_2) &= \limsup_{r\to\infty} \frac{\mathrm{Vol}_{t_2}\left(B_{t_2}(p,r) \setminus \cup_{s\in[T_1,T_2]} B_s(p,\sigma)\right)}{r^n} \\
&\leq \limsup_{r\to\infty} \frac{\mathrm{Vol}_{t_1}\left(B_{t_2}(p,r) \setminus \cup_{s\in[T_1,T_2]} B_s(p,\sigma)\right) e^{n\varepsilon|\Delta t|}}{r^n} \\
&\leq \limsup_{r\to\infty} \frac{\mathrm{Vol}_{t_1}\left(B_{t_1}\left(p, (r+K\sigma)e^{\varepsilon|\Delta t|}\right)\right) e^{n\varepsilon|\Delta t|}}{r^n} \\
&= \overline{\nu}(t_1) e^{2n\varepsilon|\Delta t|}.
\end{aligned}$$

Since $\varepsilon > 0$ and $t_1, t_2 \in [0, T)$ are arbitrary, we have that $\overline{\nu}(t)$ is constant. This proves Lemma 1. □

We need Shi's derivative estimate below. For the convenience of the reader, we state it here in the form we will apply later.

Theorem 4 (Shi's gradient estimate [8, Theorem 13.1]) *Let $g(t)$, $t \in [0, T]$, be a Ricci flow which is complete with bounded curvature $|R| \le K$ on a surface M. Then,*

$$|\nabla R(p,t)| \le \mathrm{Const.} K/\sqrt{t}, \tag{12}$$

for any $p \in M$, and $t \in (0, \min\{1/K, T\}]$.

3 Proof of Theorem 2

Let us put $R_\pm := \max\{\pm R, 0\}$. Then

$$\int_M R d\mu = \int_M R_+ d\mu - \int_M R_- d\mu, \text{ and } \int_M |R| d\mu = \int_M R_+ d\mu + \int_M R_- d\mu. \tag{13}$$

The total curvature $\int_M R d\mu$ exists if and only if at least one of the integrals of R_+ and R_- is finite. $\int_M |R| d\mu < \infty$ is equivalent to that $\int_M R d\mu$ exists and is finite.

The geometry of complete open surfaces and its total curvature have been studied actively by many people. We refer the reader to [12] for these topics. The following theorem tells a lot about complete non-compact surfaces which admit total curvature.

Theorem 5 (Shiohama [12, Theorem 5.2.1]) *Let (M, g) be a complete non-compact surface which is finitely connected, i.e., M is homeomorphic to a closed surface with finitely many points deleted. Suppose that the total curvature of (M, g) exists. Then the following holds*:

$$4\pi\chi(M) - \int_M R d\mu = 2 \lim_{r\to\infty} \frac{\mathrm{Leng}(\partial B(p,r))}{r} = 4 \lim_{r\to\infty} \frac{\mathrm{Area}(B(p,r))}{r^2}, \tag{14}$$

where $\chi(M)$ denotes the Euler number of M.

It follows from Cohn-Vossen's inequality that $4\pi\chi(M) - \int_M R d\mu$ is non-negative. In the sequel, we shall call $\lim_{r\to\infty} \mathrm{Area}(B(p,r))/r^2$ the asymptotic area ratio, or shortly AAR of (M, g).

For complete surfaces which are not finitely connected, Huber proved

Theorem 6 ([9]) *If a complete infinitely connected surface (M, g) admit total curvature, then its value is $-\infty$.*

It is a corollary of Theorem 5 and 6 that the total curvature exists if and only if $\int_M R_+ d\mu < \infty$.

X. Dai and Li Ma stated in [5, Theorem 11] that the Ricci flow preserves the L^1-property of the scalar curvature under the condition that $\int_0^T \int_M |\mathrm{Ric}|^2 d\mu dt < \infty$ and $R = O(r^{-\sigma})$ for $r = d_t(\cdot, p)$ and some $\sigma \ge n-2$. We use their technique but do not need a curvature decay condition. We invoke Theorem 5 instead.

Proof (of Theorem 2) Take any $p > 1$, $\varepsilon > 0$, and $T_0 < \infty$ such that $g(t)$ exists and $|R| \le K$ on $t \in [0, T_0]$. Put $u := \sqrt{R^2 + \varepsilon}$. Then $u \ge \sqrt{\varepsilon} > 0$ and u satisfies

$$\frac{\partial}{\partial t} u \le \Delta u + Ru + \varepsilon. \tag{15}$$

For $r > 0$, take a time-independent cut-off function $\eta = \eta_r : M \to [0, 1]$ satisfying the following properties:

$$\eta(x) = \begin{cases} 1 & \text{if } d_0(x, p) \le r/2 \\ 0 & \text{if } d_0(x, p) \ge r, \end{cases} \tag{16}$$

$$|\nabla \eta|^2_{g(0)} \le \frac{5}{r^2}, \text{ and hence } |\nabla \eta|^2_{g(t)} \le \frac{5}{r^2} e^{Kt} \tag{17}$$

Claim *If $R(\cdot, 0) \in L^1 \cap L^\infty \subset L^p$, then $R(\cdot, t) \in L^p$ for all $p > 1$ and $t \in [0, T_0]$.*

Proof To begin with, note that the bounded function $R(\cdot, 0)$ is L^p-function for all $p > 1$.

$$\begin{aligned} \frac{d}{dt} \int_M \eta^2 u^p d\mu &\le \int_M p\eta^2 u^{p-1} (\Delta u + Ru + \varepsilon) - \eta^2 u^p R d\mu \\ &\le \int_M -2p\eta u^{p-1} \langle \nabla \eta, \nabla u \rangle - p(p-1)\eta^2 u^{p-2} |\nabla u|^2 d\mu \\ &\quad + (p-1) \int_M \eta^2 u^p R d\mu + \varepsilon \int_M p\eta^2 u^{p-1} d\mu \\ &\le \frac{p}{p-1} \int_M u^p |\nabla \eta|^2 d\mu + (p-1) \int_M \eta^2 u^p R d\mu + \varepsilon \int_M p\eta^2 u^{p-1} d\mu \\ &\le \frac{5pe^{Kt}}{(p-1)r^2} \int_{B_0(p,r)} u^p d\mu + \big((p-1)K + p\sqrt{\varepsilon}\big) \int_M \eta^2 u^p d\mu, \end{aligned}$$

where we have used Cauchy's inequality to derive the third inequality:

$$-2\eta u^{p-1} \langle \nabla \eta, \nabla u \rangle \le \frac{1}{p-1} u^p |\nabla \eta|^2 + (p-1)\eta^2 u^{p-2} |\nabla u|^2. \tag{18}$$

Letting $\varepsilon \to 0$ yields

$$\frac{d^+}{dt} \int_M \eta^2 |R|^p d\mu \le \int_{B_0(p,r)} \frac{5pe^{Kt}}{(p-1)r^2} |R|^p d\mu + (p-1)K \int_M \eta^2 |R|^p d\mu. \tag{19}$$

The assumption $R(\cdot, 0) \in L^1$ implies the existences of finite asymptotic area ratio $\nu(0)$ of $g(0)$ by Theorem 5, and hence by Proposition 1.(3), the first term of the right hand side of (19) is bounded above by $10pe^{2KT_0} K^p \nu(0)/p - 1$ on $[0, T_0]$, for sufficiently large r.

Solving ordinary differential inequality (19) gives

$$\int_M \eta^2 |R(\cdot, t)|^p d\mu \le \left[\int_M \eta^2 |R(\cdot, 0)|^p d\mu + \frac{10pe^{2KT_0} K^{p-1} \nu(0)}{(p-1)^2} \right] \exp(p-1)Kt. \tag{20}$$

We let $r \to \infty$ to see that L^p-properties of the scalar curvature are preserved on $[0, T_0]$ for all $p > 1$:

$$\int_M |R(\cdot, t)|^p d\mu \le \left[\int_M |R(\cdot, 0)|^p d\mu + \frac{10pe^{2KT_0} K^{p-1} \nu(0)}{(p-1)^2} \right] \exp(p-1)Kt. \tag{21}$$

□

Next, we see that the L^1-norm of the scalar curvature is non-increasing.

Let us return to the inequality (19) above again. Integrating both sides of (19) on $[0, t] \subset [0, T_0]$ gives

$$\int_M \eta^2 |R(\cdot, t)|^p d\mu - \int_M \eta^2 |R(\cdot, 0)|^p d\mu \\ \leq \int_0^t dt \left[\frac{5pe^{Kt}}{(p-1)r^2} \int_{B_0(p,r)} |R|^p d\mu + (p-1)K \int_M \eta^2 |R|^p d\mu \right].$$

We let $r \to \infty$ and $p \to 1$ keeping in mind that $R(\cdot, t) \in L^p$ for any $p > 1$, and apply the monotone convergence theorem to obtain the desired inequality

$$\int_M |R(\cdot, t)| d\mu \leq \int_M |R(\cdot, 0)| d\mu. \tag{22}$$

This completes the proof of Theorem 2 since $T_0 < \infty$ is arbitrary. □

4 Proof of Theorem 3

Before we get into the proof of Theorem 3, we would like to remark that it is proved in [4] that Theorem 3 holds for ancient solutions to the Ricci flow on surfaces. Ancient solution is the solution $g(t)$ to the Ricci flow equation which exists on $(-\infty, T)$. It is known that non trivial ancient solutions have positive scalar curvature, and hence the curvature is integrable due to Cohn-Vossen's inequality. We extend the argument to general finitely connected surfaces.

Proof (of Theorem 3) First of all, according to Theorem 2, we know that the total curvature exists and remains finite on $[0, T)$. The finiteness of the total curvature implies that M is finitely connected by Huber's theorem.

By Theorem 5, we only have to show that the asymptotic area ratio remains constant under the Ricci flow as in Theorem 3. Since we consider an asymptotic invariant on manifolds with finitely many ends, we may assume that M has only one end without loss of generality.

The proof consists of several steps.

At first, we suppose that $\int_M R(\cdot, 0) d\mu = 4\pi\chi(M)$. In this case, the AAR exists and is equal to 0 on $[0, T)$ by Theorem 5 and Lemma 1.(4). Thus the total curvature remains constant on $[0, T)$ by Theorem 5 again.

In the case where $\int_M R(\cdot, 0) d\mu < 4\pi\chi(M)$, we need the following injectivity radius estimate, which can be immediately generalized to any finitely connected surface. Note that we can apply this lemma to our situation where AAR > 0. If a non-compact manifold M would have positive sectional curvature bounded above by $K > 0$ as in [4], we have $\mathrm{inj}(M) \geq \pi/\sqrt{K}$ [14].

Lemma 2 *Let (M, g) be a complete Riemannian surface with only one end and bounded curvature $|R| \leq K$. Assume that the total curvature of (M, g) exists and that*

$$\mathrm{Leng}(\partial B(p, r)) \geq \alpha \textit{ for all sufficiently large } r > 0 \textit{ and some } \alpha > 0. \tag{23}$$

Then the injectivity radius $\mathrm{inj}(M, g)$ is positive.

Proof We argue by contradiction, so assume that $\mathrm{inj}(M, g) = 0$. Then, by the upper bound of the curvature, we can take a sequence $\{\gamma_i\}$ of geodesic loops at $q_i \in M$ with $\mathrm{Leng}(\gamma_i) \searrow 0$

and $\gamma_i \subset \overline{B(p, r_{i+1})} \setminus B(p, r_i)$, where $r_i \nearrow \infty$. Here, a geodesic loop at $q \in M$ means a geodesic path which starts and ends at $q \in M$. Since we have large $R_0 > 0$ such that $M \setminus \overline{B(p, R_0)}$ is homeomorphic to the complement of some compact subset of $\mathbb{R}^2$, each γ_i encloses the domain D_i with $D_i \subset \overline{B(p, r_{i+1})} \setminus B(p, r_i)$ or $D_i \supset B(p, r_i)$, for large i. We divide the argument into two cases.

At first, we assume that we can take a subsequence of $\{\gamma_i\}$, still denoted by $\{\gamma_i\}$, such that each D_i is homeomorphic to a disk in $\mathbb{R}^2$ and D_i's are pairwise disjoint. In this case, by the Gauss-Bonnet formula,

$$2\pi = \int_{D_i} \frac{R}{2} d\mu + (\text{the outer angle at } q_i) \leq \int_{D_i} \frac{R}{2} d\mu + \pi, \tag{24}$$

so

$$2\pi \leq \int_{D_i} R d\mu \leq \int_{D_i} R_+ d\mu, \quad \text{for all } i = 1, 2, \ldots. \tag{25}$$

This contradicts to the finiteness of $\int_M R_+ d\mu$, which is equivalent to the existence of the total curvature.

If this is not the case, we have $\gamma = \gamma_i$ with $\text{Leng}(\gamma) = 2\varepsilon$ and $p \in D_i$ for small $\varepsilon > 0$.

Put $r_{\min} := \min\{d(x, p) \mid x \in \gamma\}$ and $r_{\max} := \max\{d(x, p) \mid x \in \gamma\}$. Now we observe that

$$\operatorname{diam} \partial B(p, r_{\max}) \leq 3\varepsilon. \tag{26}$$

To see this, take any two points x and y from $\partial B(p, r_{\max})$. The shortest paths conecting x and y with p intersect γ at x' and y', respectively. Since $r_{\max} - r_{\min} \leq \varepsilon$, we have $d(x, x'),\ d(y, y') \leq \varepsilon$, and $d(x', y') \leq \varepsilon$ because they are on γ. Thus, $\operatorname{diam} \partial B(p, r_{\max}) \leq 3\varepsilon$.

We consider the ε-neighborhood W of $\partial B(p, r_{\max})$. Since W contains $B(p, r_{\max} + \varepsilon) \setminus B(p, r_{\max})$, we have

$$\text{Area}(W) \geq \int_{r_{\max}}^{r_{\max}+\varepsilon} \text{Leng}(\partial B(p, r)) dr \geq \int_{r_{\max}}^{r_{\max}+\varepsilon} \alpha dr = \alpha\varepsilon. \tag{27}$$

On the other hand, the diameter of W is not larger than 5ε, and thus, we have

$$\text{Area}(W) \leq \frac{2\pi}{\sqrt{K/2}} \left(\cosh\left(5\sqrt{K/2}\,\varepsilon\right) - 1\right) \tag{28}$$

by the lower bound of the curvature and the Bishop inequality. These lead us to the contradiction, if $\varepsilon > 0$ is sufficiently small. □

We go back to the proof of Theorem 3. The above injectivity radius estimate, the upper curvature bound and Shi's gradient estimate combined with the existence of finite total scalar curvature let us conclude that $|R(x, t)| \to 0$ as $d_t(x, p) \to \infty$, at the time $t > 0$. To see this, we assume that we have a sequence of points $\{x_i\} \subset M$ satisfying $d_{t_0}(x_i, p) \to \infty$ and $|R(x_i, t_0)| \geq C_0 > 0$ at some time $t_0 > 0$. Then, the estimates $\text{inj}(g(t_0)) \geq C_1 > 0$ and $|\nabla R(\cdot, t_0)| \leq C_2$ yield that

$$\int_{B_{t_0}(x_i, C_3)} |R| d\mu \geq \frac{C_0}{2} \text{Area}(B_{t_0}(x_i, C_3)) \geq C_4 > 0, \text{ for all } i = 1, 2, \ldots, \tag{29}$$

which is in conflict with the existence of finite total scalar curvature. Thus, $|R(\cdot, t)|$ is arbitrarily small outside a sufficiently large compact set for any $t > 0$.

Therefore, by Lemma 1, the AAR is constant on $(0, T)$, and hence on $[0, T)$ by the continuity of AAR. In the end, the total curvature is constant for $t \geq 0$ by Theorem 5. This concludes the proof of Theorem 3. □

5 Asymptotic volume ratio under the Ricci flow

In this section, we prove

Theorem 7 *Let $g(t)$ be a Ricci flow which is complete with non-negative bounded curvature operator on an n-manifold M. Then its asymptotic volume ratio $\nu(t) := \lim_{r\to\infty} \mathrm{Vol}_t(B_t(p, r))/r^n$ is independent on t.*

In the above theorem, the non-negativity of the curvature operator is assumed because it is preserved under the Ricci flow with bounded curvature [8]. What we really need is the non-negativity of the Ricci curvature, under which an asymptotic cone as well as the asymptotic volume ratio of $(M, g(t))$ always exist.

Before we begin the proof of Theorem 7, let us recall some definitions.

Definition 1 Let (X, x) and (Y, y) be pointed metric spaces. A map $f : (X, p) \to (Y, q)$ with $f(p) = q$ is called an ε-approximation map if

$$\text{the } \varepsilon\text{-neighborhood of } f(B_X(p, 1/\varepsilon)) \text{ contains } B_Y(q, 1/\varepsilon), \tag{30}$$

and

$$|d(x, y) - d(f(x), f(y))| < \varepsilon, \text{ for any } x, y \in B_X(p, 1/\varepsilon). \tag{31}$$

An ε-approximation map does not need to be continuous.

We say that a sequence $\{(X_i, x_i)\}$ of pointed metric spaces converges to (X, x) in the Gromov-Hausdorff topology if for any $\varepsilon > 0$, there exists an ε-approximation map from (X, x) to (X_i, x_i) for all large i.

When (M, g) has non-negative Ricci curvature, $\{(M, r_i^{-2} g, p)\}$ subconverges to a metric space (X, x) in the pointed Gromov-Hausdorff topology, for any $r_i \nearrow \infty$, by Gromov's precompactness theorem [6]. This (X, x) is called an asymptotic cone of (M, g).

Our proof of Theorem 7 relies on the following volume convergence theorem.

Theorem 8 (Volume convergence theorem [2]) *Let $\{(M_i, g_i, p_i)\}$ be a sequence of pointed n-dimensional complete Riemannian manifolds with uniform lower Ricci curvature bounds $\mathrm{Ric}_{g_i} \geq -(n-1)K$. Suppose that $\mathrm{Vol}(B(p_i, 1)) \geq v > 0$ for all $i = 1, 2, \ldots$, and $\{(M_i, g_i, p_i)\}$ converges to a metric space (X, x) in the pointed Gromov-Hausdorff topology. Then, for any $r > 0$,*

$$\lim_{i\to\infty} \mathrm{Vol}(B(p_i, r)) = \mathcal{H}^n(B_X(x, r)), \tag{32}$$

where $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure which is normalized to agree with the n-dimensional Lebesgue measure on $\mathbb{R}^n$.

Proof (of Theorem 7) We fix $0 \leq t_1 < t_2 < T$ and $K < \infty$ such that $0 \leq \mathrm{Ric} \leq K$ on $M \times [t_1, t_2]$. Since $\nu(t_1) = 0$ if and only if $\nu(t_2) = 0$, due to Lemma 1.(4), we may assume that $\nu(t) > 0$.

We recall the following lemma which can be deduced from the second variational formula.

Lemma 3 ([8]) *Let* $g(t),\ t \in [0, T]$*, be a Ricci flow which is complete with Ricci curvature bounded above* $\mathrm{Ric} \leq K$ *by* $K \geq 0$*. Then, for any two points* $x, y \in M$*,*

$$\frac{d^-}{dt} d_t(x, y) \geq -\mathrm{Const.}\sqrt{K}. \tag{33}$$

This lemma tells us that when the Ricci flow as in the lemma shrinks the metric, the distance does not shrink so much. From this lemma and the non-negativity of the Ricci curvature, we have

$$d_{t_1}(x, y) \geq d_{t_2}(x, y) \geq d_{t_1}(x, y) - \mathrm{Const.}\sqrt{K}(t_2 - t_1), \tag{34}$$

for any $x, y \in M$.

By assumption, $(M, g(t_1))$ has an asymptotic cone (X, x), and according to (34), (X, x) is also an asymptotic cone of $(M, g(t_2))$, because the composition of an ε-approximation map of (X, x) into $(M, r_i^{-2} g(t_1), p)$ with the identity map on M is a 2ε-approximation map of (X, x) into $(M, r_i^{-2} g(t_2), p)$ for $r_i \nearrow \infty$ and sufficiently large i.

Under this setting, Theorem 7 follows from the volume convergence theorem quoted above. In fact,

$$\begin{aligned} \nu(t_1) = \lim_{i\to\infty} \mathrm{Vol}\Big(B(p, 1; r_i^{-2} g(t_1))\Big) &= \mathcal{H}^n(B_X(x, 1)) \\ &= \lim_{i\to\infty} \mathrm{Vol}\Big(B(p, 1; r_i^{-2} g(t_2))\Big) = \nu(t_2). \end{aligned}$$

This proves Theorem 7. □

Acknowledgments The author would like to express his gratitude to Professor Takao Yamaguchi for his constant advice and suggestions. The author is also grateful to Professor Satoshi Ishiwata for numerous stimulating discussions. This work was partially supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists.

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