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Gromov hyperbolicity of Denjoy Domains

Venancio Alvarez · Ana Portilla ·
Jose M. Rodriguez · Eva Touris

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Abstract In this paper we characterize the Gromov hyperbolicity of the double of a metric space. This result allows to give a characterization of the hyperbolic Denjoy domains, in terms of the distance to $\mathbb{R}$ of the points in some geodesics. In the particular case of *trains* (a kind of Riemann surfaces which includes the flute surfaces), we obtain more explicit criteria which depend just on the lengths of what we have called *fundamental geodesics*.

Keywords Denjoy domain · Flute surface · Gromov hyperbolicity · Riemann surface · Schottky double · Train

Mathematical Subject Classifications (2000) 41A10 · 46E35 · 46G10

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A. Portilla · J. M. Rodríguez (✉) · E. Tourís
Departamento de Matemáticas, Escuela Politécnica Superior,
Universidad Carlos III de Madrid,
Avenida de la Universidad, 30, 28911 Leganés, Madrid, Spain
e-mail: jomaro@math.uc3m.es

A. Portilla
e-mail: apferrei@math.uc3m.es

E. Tourís
e-mail: etouris@math.uc3m.es

V. Alvarez
Departamento de Análisis Matemático, Facultad de Ciencias,
Campus de Teatinos, 29071 Málaga, Spain
e-mail: nancho@anamat.cie.uma.es

1 Introduction

To understand the connections between graphs and Potential Theory on Riemannian manifolds (see e.g. [4,13,18,23–25,29]), Gromov hyperbolic spaces are a useful tool. Besides, the concept of Gromov hyperbolicity grasps the essence of negatively curved spaces, and has been successfully used in the theory of groups (see e.g. [20] and the references therein).

A geodesic metric space is called hyperbolic (in the Gromov sense) if there is an upper bound of the distance of every point in a side of any geodesic triangle to the union of the two other sides (see Definition 2.3). The condition above is due to Rips.

But, it is not easy to determine if a given space is Gromov hyperbolic or not. One interesting instance is that of a Riemann surface endowed with the Poincaré metric. With that metric structure a Riemann surface is negatively curved, but not all Riemann surfaces are Gromov hyperbolic, since topological obstacles may impede it: for instance, the two-dimensional jungle-gym (a $\mathbb{Z}^2$-covering of a torus with genus two) is not hyperbolic.

We are interested in studying when Riemann surfaces equipped with their Poincaré metric are Gromov hyperbolic [26–28]. To be more precise, in this paper our main aim is to study the hyperbolicity of Denjoy domains, that is to say, plane domains Ω with $\partial\Omega \subset \mathbb{R}$. This kind of surfaces are becoming more and more important in Geometric Theory of Functions, since, on the one hand, they are a very general type of Riemann surfaces, and, on the other hand, their symmetry simplifies their study. However, our techniques let us get as well several characterizations for a more general kind of space: the Schottky double of a Riemann surface, and even the double of a metric space (see Theorem 3.2). This result gives several characterizations of hyperbolic Denjoy domains (see Theorem 5.1), since every Denjoy domain is also a Schottky double.

One of these characterizations is particularly surprising: it is sufficient to check the Rips condition only on geodesic "bigons" (triangles with two vertices); this is clearly false in the general case: every geodesic bigon in $\mathbb{R}^n$ (with the euclidean distance) is 0-thin, but $\mathbb{R}^n$ is not hyperbolic if $n > 1$. So, in general, it is necessary to check the Rips condition for all triangles.

Our main characterization gives that a Denjoy domain is hyperbolic if and only if the distance to $\mathbb{R}$ of any point in any simple closed geodesic is uniformly bounded.

Nevertheless, Denjoy domains are such a wide class of Riemann surfaces that characterization criteria are not straightforward to apply. That is the main reason because of we decided to focus on two particular types of Denjoy domains, which we have called *trains* (see Definition 5.2) and *generalized trains* (see Definition 5.29). About them, we have been able to obtain both characterizations and sufficient conditions that either guarantee or discard hyperbolicity.

We study the hyperbolicity of trains in terms of the lengths of two types of their simple closed geodesics, which we have named as *fundamental* (see Definition 5.2), and whose lengths are denoted by l_n and r_n. So, for instance, Theorem 5.3 provides a characterization of the hyperbolicity of trains which does not require any other condition.

One of the major novelties of this paper is that most of the hyperbolicity criteria depend on the fundamental geodesics just through their lengths l_n and r_n.

The approximation to the problem of the hyperbolicity of trains requires different strategies according to the behavior of the sequences $\{l_n\}_n$ and $\{r_n\}_n$. So:

1. If $\{l_n\}_n$ is bounded, the train is always hyperbolic, regardless of what happens with $\{r_n\}_n$ (see Theorem 5.25).
2. If $\{l_n\}_n$ is not bounded, in general we are going to require that $\{r_n\}_n$ is bounded in order to guarantee hyperbolicity. In fact, Theorem 5.17 and Corollaries 5.18 and 5.19 discard hyperbolicity in most cases when $\{r_n\}_n$ is not bounded.
   2.1. If $\lim_{n\to\infty} l_n = \infty$, Theorem 5.14 is a characterization of hyperbolicity and Theorems 5.12, 5.21 and 5.24 provide sufficient conditions.
   2.2. Otherwise, we have both a characterization of hyperbolicity (see Theorem 5.26) and a sufficient condition (see Theorem 5.27).

Theorems 5.30 and 5.31 are characterizations for generalized trains. And finally, Theorem 5.33 is a result about stability of hyperbolicity under bounded perturbations of the lengths of the fundamental geodesics, even though the original surface and the modified one are not quasi-isometric.

These results let us get interesting examples of hyperbolic and non-hyperbolic Riemann surfaces.

Notations We denote by X a geodesic metric space. By d_X and L_X we shall denote, respectively, the distance and the length in the metric of X. From now on, when there is no possible confusion, we will not write the subindex X.

We denote by Ω a Denjoy domain with its Poincaré metric.

We denote by $\Re z$ and $\Im z$ the real and imaginary part of z, respectively.

Finally, we denote by c and c_i, positive constants which can assume different values in different theorems.

2 Background in Gromov spaces

In our study of hyperbolic Gromov spaces we use the notations of [20]. We give now the basic facts about these spaces. We refer to [20] for more background and further results.

Definition 2.1 Let us fix a point w in a metric space (X, d). We define the *Gromov product* of $x, y \in X$ with respect to the point w as

$$(x|y)_w := \frac{1}{2}\big(d(x,w) + d(y,w) - d(x,y)\big) \geq 0.$$

We say that the metric space (X, d) is δ-*hyperbolic* ($\delta \geq 0$) if

$$(x|z)_w \geq \min\big\{(x|y)_w, (y|z)_w\big\} - \delta$$

for every $x, y, z, w \in X$. We say that X is *hyperbolic* (in the Gromov sense) if the value of δ is not important.

It is convenient to remark that this definition of hyperbolicity is not universally accepted, since sometimes the word hyperbolic refers to negative curvature or to the existence of Green's function. However, in this paper we only use the word *hyperbolic* in the sense of Definition 2.1.

Examples

(1) Every bounded metric space X is $(diam X)$-hyperbolic (see e.g. [20], p. 29).

(2) Every complete simply connected Riemannian manifold with sectional curvature which is bounded from above by $-k$, with $k > 0$, is hyperbolic (see e.g. [20], p. 52).
(3) Every tree with edges of arbitrary length is 0-hyperbolic (see e.g. [20], p. 29).

Definition 2.2 If $\gamma : [a,b] \longrightarrow X$ is a continuous curve in a metric space (X,d), we can define the length of γ as

$$L(\gamma) := \sup\left\{ \sum_{i=1}^{n} d(\gamma(t_{i-1}),\gamma(t_i)) : a = t_0 < t_1 < \cdots < t_n = b \right\}.$$

We say that γ is a *geodesic* if it is an isometry, i.e. $L(\gamma|_{[t,s]}) = d(\gamma(t),\gamma(s)) = |t-s|$ for every $s,t \in [a,b]$. We say that X is a *geodesic metric space* if for every $x,y \in X$ there exists a geodesic joining x and y; we denote by $[x,y]$ any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient). It is clear that every geodesic metric space is path-connected.

Definition 2.3 If X is a geodesic metric space and J is a polygon whose sides are $J_1, J_2, \ldots, J_n$, we say that J is *δ-thin* if for every $x \in J_i$ we have that $d(x, \cup_{j\neq i} J_j) \leq \delta$. If $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of three geodesics $[x_1, x_2]$, $[x_2, x_3]$ and $[x_3, x_1]$. The space X is *δ-thin* (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ-thin.

If we have a triangle with two identical vertices, we call it a "bigon". Obviously, every bigon in a δ-thin space is δ-thin. It is also clear that every geodesic polygon with n sides in a δ-thin space is $(n-2)\delta$-thin.

Definition 2.4 Given a geodesic triangle $T = \{x,y,z\}$ in a geodesic metric space X, let T_E be a Euclidean triangle with sides of the same length than T. Since there is no possible confusion, we will use the same notation for the corresponding points in T and T_E. The maximum inscribed circle in T_E meets the side $[x,y]$ (respectively $[y,z]$, $[z,x]$) in a point z' (respectively x', y') such that $d(x,z') = d(x,y')$, $d(y,x') = d(y,z')$ and $d(z,x') = d(z,y')$. We call the points x', y', z', the *internal points* of $\{x,y,z\}$. There is a unique isometry f of the triangle $\{x,y,z\}$ onto a tripod (a tree with one vertex w of degree 3, and three vertices x'', y'', z'' of degree one, such that $d(x'',w) = d(x,z') = d(x,y')$, $d(y'',w) = d(y,x') = d(y,z')$ and $d(z'',w) = d(z,x') = d(z,y')$). The triangle $\{x,y,z\}$ is *δ-fine* if $f(p) = f(q)$ implies that $d(p,q) \leq \delta$. The space X is *δ-fine* if every geodesic triangle in X is δ-fine.

A basic result is that hyperbolicity is equivalent to Rips condition and to be fine:

Theorem 2.5 ([20], p. 41) *Let us consider a geodesic metric space X.*

(1) *If X is δ-hyperbolic, then it is 4δ-thin and 4δ-fine.*
(2) *If X is δ-thin, then it is 4δ-hyperbolic and 4δ-fine.*
(3) *If X is δ-fine, then it is 2δ-hyperbolic and δ-thin.*

We present now the class of maps which play the main role in the theory.

Definition 2.6 A function between two metric spaces $f : X \longrightarrow Y$ is a *quasi-isometry* if there are constants $a \geq 1$, $b \geq 0$ with

$$\frac{1}{a}\, d_X(x_1,x_2) - b \leq d_Y(f(x_1),f(x_2)) \leq a d_X(x_1,x_2) + b, \qquad \text{for every } x_1,x_2 \in X.$$

A such function is called an (a,b)-*quasi-isometry*. We say that the image of f is ε-*full* (for some $\varepsilon \geq 0$) if for every $y \in Y$ there exists $x \in X$ with $d_Y(y,f(x)) \leq \varepsilon$. We say that X and Y are *quasi-isometrically equivalent* if there exists a quasi-isometry between X and Y, with image ε-full, for some $\varepsilon \geq 0$. An (a,b)-*quasigeodesic* in X is an (a,b)-quasi-isometry between an interval of $\mathbb{R}$ and X. An (a,b)-*quasigeodesic segment* in X is an (a,b)-quasi-isometry between a compact interval of $\mathbb{R}$ and X.

Let us observe that a quasi-isometry can be discontinuous.

Remark It is well known (see e.g. [24,25]) that quasi-isometrical equivalence is an equivalence relation. In fact, if $f: X \longrightarrow Y$ is an (a,b)-quasi-isometry with image ε-full, then there exists a function $g: Y \longrightarrow X$ which is an $(a, 2a\varepsilon + ab)$-quasi-isometry. In particular, if f is a surjective (a,b)-quasi-isometry, then g is an (a,ab)-quasi-isometry (in this case we can choose as $g(y)$ any point in $f^{-1}(y)$).

Quasi-isometries are important since they are the maps which preserve hyperbolicity:

Theorem 2.7 ([20], p. 88) *Let us consider an (a,b)-quasi-isometry between two geodesic metric spaces $f: X \longrightarrow Y$. If Y is δ-hyperbolic, then X is δ'-hyperbolic, where δ' is a constant which only depends on δ, a and b. Besides, if the image of f is ε-full for some $\varepsilon \geq 0$, then X is hyperbolic if and only if Y is hyperbolic.*

It is well-known that if f is not ε-full, the hyperbolicity of X does not imply the hyperbolicity of Y: it is enough to consider the inclusion of $\mathbb{R}$ in $\mathbb{R}^2$ (which is indeed an isometry).

Definition 2.8 Let us consider $H > 0$, a metric space X, and subsets $Y, Z \subseteq X$. The set $V_H(Y) := \{x \in X : d(x,Y) \leq H\}$ is called the *H-neighborhood* of Y in X. The *Hausdorff distance* of Y to Z is defined by $\mathcal{H}(Y,Z) := \inf\{H > 0 : Y \subseteq V_H(Z),\ Z \subseteq V_H(Y)\}$.

The following is a beautiful and useful result:

Theorem 2.9 ([20], p. 87) *For each $\delta \geq 0$, $a \geq 1$ and $b \geq 0$, there exists a constant $H = H(\delta, a, b)$ with the following property:*

Let us consider a δ-hyperbolic geodesic metric space X and an (a,b)-quasigeodesic g starting in x and finishing in y. If γ is a geodesic joining x and y, then $\mathcal{H}(g,\gamma) \leq H$.

This property is known as geodesic stability. Mario Bonk has proved that, in fact, geodesic stability is equivalent to hyperbolicity [11].

Along this paper we will work with topological subspaces of a geodesic metric space X. There is a natural way to define a distance in these spaces:

Definition 2.10 If X is a path-connected space in which we have defined the length L of any curve, we can consider the *intrinsic distance* with respect to L

$$d_X(x,y) := \inf\big\{L(\gamma) : \gamma \subset X \text{ is a continuous curve joining } x \text{ and } y\big\}.$$

In order to prove Theorem 3.2 below, we need the following elementary results (see e.g. [26], Lemma 2.16 and [26], Lemma 2.24 for some proofs):

Lemma 2.11 *For each $\delta, b \geq 0$ and $a \geq 1$, there exists a constant $K = K(\delta, a, b)$ with the following property:*

If X is a δ-hyperbolic geodesic metric space and $T \subseteq X$ is an (a, b)-quasigeodesic triangle, then T is K-thin. Furthermore, $K = 4\delta + 2H(\delta, a, b)$, where H is the constant in Theorem 2.9.

Lemma 2.12 *Let us consider a metric space X, an interval I, an (a, b)-quasigeodesic $g: I \longrightarrow X$ and a curve $g_1 : I \longrightarrow X$ such that $d(g(t), g_1(t)) \leq c$ for every $t \in I$. Then g_1 is an $(a, b + 2c)$-quasigeodesic.*

3 Results in metric spaces

Let us introduce now the kind of spaces which will be the main topic of the current paper.

Definition 3.1 Given a geodesic metric space X and closed connected pairwise disjoint subsets $\{\eta_j\}_{j \in J}$ of X, we consider another copy X' of X. The *double DX* of X is the union of X and X' obtained by identifying the corresponding points in each η_j and η_j'.

Since X and X' are metric spaces, we have defined the length L of any curve. We always consider DX with its intrinsic distance with respect to this L. If $X = S$ is a bordered surface and $\partial S = \cup_{j \in J} \eta_j$, DS is known as the Schottky double of S (see e.g. [1], p. 119).

The following result gives several characterizations of the hyperbolicity of the double DX. These characterizations mean a new approach to the study of the hyperbolicity: now it is sufficient to bound the distance between some geodesics and $\cup_{j \in J} \eta_j$, and then the amount of geodesics to check is drastically reduced with respect to Rips condition.

Theorem 3.2 *Let us consider a geodesic metric space X and closed connected pairwise disjoint subsets $\{\eta_j\}_{j \in J}$ of X, such that the double DX is a geodesic metric space. Then the following conditions are equivalent:*

(1) *DX is δ-hyperbolic.*
(2) *X is δ_0-hyperbolic and there exists a constant c_1 such that for every $k, l \in J$ and $a \in \eta_k, b \in \eta_l$ we have $d_X(x, \cup_{j \in J} \eta_j) \leq c_1$ for every $x \in [a, b] \subset X$.*
(3) *X is δ_0-hyperbolic and there exists a constant c_2 such that for every $k, l \in J$ and $a \in \eta_k, b \in \eta_l$ there exist $a_0 \in \eta_k, b_0 \in \eta_l$ with $d_X(x, \cup_{j \in J} \eta_j) \leq c_2$ for every $x \in [a, a_0] \cup [a_0, b_0] \cup [b, b_0] \subset X$.*
(4) *X is δ_0-hyperbolic and there exists a constant c_3 such that every geodesic bigon in DX with vertices in $\cup_{j \in J} \eta_j$ is c_3-thin.*
(5) *X is δ_0-hyperbolic and there exist constants c_4, α, β such that for every $k, l \in J$ and $a \in \eta_k, b \in \eta_l$ we have $d_X(x, \cup_{j \in J} \eta_j) \leq c_4$ for every x in some (α, β)-quasigeodesic joining a with b in X.*
(6) *X is δ_0-hyperbolic and there exist constants c_5, α, β such that for every $k, l \in J$ and $a \in \eta_k, b \in \eta_l$ there exist $a_0 \in \eta_k, b_0 \in \eta_l$ with $d_X(x, \cup_{j \in J} \eta_j) \leq c_5$ for every x in some (α, β)-quasigeodesic joining a with a_0 in X and some (α, β)-quasigeodesic joining*

b with b_0 in X, and $d_X(x, \cup_{j\in J}\eta_j) \le c_5$ for every x in some (α,β)-quasigeodesic joining a with b in X.

Furthermore, the constants in each condition only depend on the constants appearing in any other of the conditions.

Remark By Theorem 2.9, by $[a,b]$, $[a,a_0]$, $[b,b_0]$ and $[a_0,b_0]$ in (2) and (3) we can mean some particular choice of these geodesics.

Proof We prove first that (1) implies (4). If DX is δ-hyperbolic, then X is δ-hyperbolic, since X is geodesically convex in DX (recall that X' is isometric to X and that the intrinsic distance in X given by d_{DX} is equal to d_X). It is direct that every bigon is 4δ-thin, by Theorem 2.5.

Let us see that (4) implies (2). Consider $k,l \in J$, $a \in \eta_k, b \in \eta_l$ and $x \in [a,b] \subset X$. Let us denote by $[a,b]'$ the symmetric geodesic in DX of $[a,b]$. Since $[a,b] \cup [a,b]'$ is a geodesic bigon in DX, we have $d_{DX}(x,[a,b]') \le c_3$. Consequently, $d_{DX}(x, \cup_{j\in J}\eta_j) \le c_3$, since $X \cap X' = \cup_{j\in J}\eta_j$.

We prove now that (2) implies (1). Denote by g the isometry of DX which maps the points of X in their symmetric points in X' (and viceversa). Let us consider a geodesic triangle $T = \{a,b,c\}$ in DX and the triangle T_0 in X obtained by changing in T the set $T \cap X'$ by $g(T \cap X')$.

If the three vertices are in X, let us observe that $d_{DX}(x,g(x)) \le 2c_1$ for every $x \in T \cap X'$; it is clear that T is a geodesic triangle $(4\delta_0 + 4c_1)$-thin, since T_0 is $4\delta_0$-thin. In other case, we can assume by symmetry, that $a,b \in X$ and $c \in X'$. The side in T_0 joining a and b is geodesic in DX. Let us denote by a_0 (respectively, b_0) the last point in $[a,c]$ (respectively, $[b,c]$) which belongs to X; it is clear that $a_0,b_0 \in \cup_{j\in J}\eta_j$. The subsets $[a,a_0]$ and $[b,b_0]$ of T_0 are geodesics in DX. It is clear that $[a,a_0] \cup [a_0,c]$ and $[b,b_0] \cup [b_0,c]$ are also geodesics in DX.

We consider a geodesic triangle $T_c = \{a_0,b_0,c\}$ in DX (contained in X') with $[a_0,c],[b_0,c] \subset T$. Let us denote by c^1,b^1,a^1 the internal points (see Definition 2.4) of T_c in the geodesics $[a_0,b_0]$, $[a_0,c]$, $[c,b_0]$, respectively. We define T_1 as the (not necessarily geodesic) triangle with vertices a,b,c^1, obtained from T_0 by replacing $g([a_0,c]) \cup g([c,b_0]) \subset X$ by $[a_0,b_0] = [a_0,c^1] \cup [c^1,b_0] \subset X'$. We have that $d_{DX}(x,g(x)) \le 2c_1$ for every $x \in [a_0,b_0]$. Let us observe that $L([a_0,b^1]) = L([a_0,c^1])$ and $L([b_0,a^1]) = L([b_0,c^1])$. Then, Lemma 2.12 gives that T_1 is $(1, 8\delta_0)$-quasigeodesic in DX, since X' is $4\delta_0$-fine. If $T_2 := g(T_1)$, hypothesis (2) and Lemma 2.12 imply that $T_2 \subset X$ is $(1, 8\delta_0 + 4c_1)$-quasigeodesic in DX. Consequently, Lemma 2.11 implies that T_2 is $(4\delta_0 + 2H(\delta_0, 1, 8\delta_0 + 4c_1))$-thin.

Let us prove now that T is also thin. Observe that any point of $T \backslash ([b^1,c] \cup [c,a^1])$ has a point in T_2 at distance least or equal than $2c_1 + 4\delta_0$, since X' is $4\delta_0$-fine. We also have that the points in $[b^1,c]$ and in $[c,a^1]$ are at distance least or equal than $4\delta_0$. Hence, T is $(4c_1 + 12\delta_0 + 2H(\delta_0, 1, 8\delta_0 + 4c_1))$-thin.

Theorem 2.9 implies that (2) is equivalent to (5) and that (3) is equivalent to (6).

It is clear that (2) implies (3), with $a_0 = a$ and $b_0 = b$. We finish the proof by showing that (3) implies (2). Let us fix $k,l \in J$ and $a \in \eta_k, b \in \eta_l$. Consider $[a,b],[b,b_0],[b_0,a_0],[a_0,a] \subset X$ and $x \in [a,b] \subset X$. Since X is $4\delta_0$-thin and $Q := \{a,b,b_0,a_0\}$ is a geodesic quadrilateral in X, given $x \in [a,b]$, we have $d_X(x,[b,b_0] \cup [b_0,a_0] \cup [a_0,a]) \le 8\delta_0$. Consequently, $d_X(x, \cup_{j\in J}\eta_j) \le 8\delta_0 + c_2$ for every $x \in [a,b]$. □

4 Background in Riemann surfaces

We denote by $\overline{z}$, $\Re z$ and $\Im z$, respectively, the conjugate, the real part and the imaginary part of z.

Both in this section and in the next one we always work with the Poincaré metric; consequently, curvature is always -1. In fact, many concepts appearing here (as punctures) only make sense with the Poincaré metric.

Below we collect some definitions concerning Riemann surfaces which will be referred to afterwards.

A *non-exceptional* Riemann surface S is a Riemann surface whose universal covering space is the unit disk $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$, endowed with its Poincaré metric, i.e. the metric obtained by projecting the Poincaré metric of the unit disk $ds = 2|dz|/(1 - |z|^2)$ or, equivalently, the upper half plane $\mathbb{U} = \{z \in \mathbb{C} : \ \Im z > 0\}$, with the metric $ds = |dz|/\Im z$. With this metric, S is a geodesically complete Riemannian manifold with constant curvature -1, and therefore S is a geodesic metric space. The only Riemann surfaces which are left out are the sphere, the plane, the punctured plane and the tori. It is easy to study the hyperbolicity of these particular cases.

We have used the word *geodesic* in the sense of Definition 2.2, that is to say, as a global geodesic or a minimizing geodesic; however, we need now to deal with a special type of local geodesics: simple closed geodesics, which obviously can not be minimizing geodesics. We will continue using the word geodesic with the meaning of Definition 2.2, unless we are dealing with closed geodesics.

A *Y-piece* is a bordered non-exceptional Riemann surface which is conformally equivalent to a sphere without three open disks and whose boundary curves are simple closed geodesics. Given three positive numbers a, b, c, there is a unique (up to conformal mapping) Y-piece such that their boundary curves have lengths a, b, c (see e.g. [12], p. 109). They are a standard tool for constructing Riemann surfaces. A clear description of these Y-pieces and their use is given in [14], Chapter X.3 and [12], Chapter 3.

A *generalized Y-piece* is a non-exceptional Riemann surface (with or without boundary) which is conformally equivalent to a sphere without n open disks and m points, with integers $n, m \geq 0$ such that $n + m = 3$, the n boundary curves are simple closed geodesics and the m deleted points are punctures. Notice that a generalized Y-piece is topologically the union of a Y-piece and m cylinders, with $0 \leq m \leq 3$.

The following spaces are a specially interesting example of Schottky double.

Definition 4.1 A *Denjoy domain* is a domain Ω in the Riemann sphere with $\partial\Omega \subset \mathbb{R} \cup \{\infty\}$.

Denjoy domains have a growing interest in Geometric Function Theory (see e.g. [2,3,19,21]).

We only consider Denjoy domains Ω with at least three boundary points; this fact guarantees that Ω is a non-exceptional Riemann surface.

If we consider the bordered Riemann surface $X := \Omega \cap \{z \in \mathbb{C}\colon \ \Im z \geq 0\}$ and $\{\eta_j\}_{j \in J}$ the connected components of $X \cap \mathbb{R}$, the Denjoy domain Ω is the double of X. Given a subset A of Ω, we denote by A^+ the set $A^+ := A \cap \{z \in \mathbb{C}\colon \ \Im z \geq 0\}$; then, the Denjoy domain Ω is the double of Ω^+, i.e., $\Omega = D\Omega^+$.

5 Results in Riemann surfaces

The following result gives several characterizations of the hyperbolicity of the Denjoy domains. It is an improvement of Theorem 3.2 in the context of this kind of spaces.

In particular, characterization (5) gives that it is sufficient to check the Rips condition just for bigons.

Characterization (3) is also a remarkable improvement of Rips condition in the context of Riemann surfaces, since the amount of geodesics to check is drastically reduced with respect to Rips condition. For example, let us consider an annulus $A_t := \mathbb{C} \setminus ([-1,0] \cup [t,\infty))$; it is well known that every annulus is conformally equivalent to A_t for some $t > 0$. Fix some geodesic γ_0 joining $(-\infty,-1)$ with $(0,t)$. In order to deal with the Rips condition, we need to consider a generic triangle T in A_t, which is determined by the coordinates of three points, i.e., by six real coordinates; however, (3) allows to deal only with γ_0, which is parameterized by one real coordinate.

Theorem 5.1 *Let us consider a Denjoy domain Ω. Then the following conditions are equivalent:*

(1) *Ω is δ-hyperbolic.*
(2) *There exists a constant c_1 such that for every $k,l \in J$ and $a \in \eta_k, b \in \eta_l$ we have $d_\Omega(z,\mathbb{R}) \le c_1$ for every $z \in [a,b]$.*
(3) *There exists a constant c_2 such that for every $k,l \in J$ there exist $a_0 \in \eta_k, b_0 \in \eta_l$ with $d_\Omega(z,\mathbb{R}) \le c_2$ for every $z \in [a_0,b_0]$.*
(4) *There exist constants c_3,α,β such that for every $k,l \in J$ there exist $a_0 \in \eta_k, b_0 \in \eta_l$ with $d_\Omega(z,\mathbb{R}) \le c_3$ for every z in some (α,β)-quasigeodesic joining a_0 with b_0.*
(5) *There exists a constant c_4 such that every geodesic bigon in Ω with vertices in $\mathbb{R}$ is c_4-thin.*

Furthermore, the constants in each condition only depend on the constants appearing in any other of the conditions.

Proof Theorem 5.1 is a consequence of Theorem 3.2 if we consider the bordered Riemann surface $X := \Omega^+ = \Omega \cap \{z \in \mathbb{C} : \Im z \ge 0\}$ and $\{\eta_j\}_{j\in J}$ the connected components of $X \cap \mathbb{R}$. We only need to remark two facts:

(a) X is hyperbolic since it is isometric to a geodesically convex subset of the unit disk (in fact, there is just one geodesic in X joining two points in X). Therefore, X is $\log(1+\sqrt{2})$-thin, as the unit disk (see, e.g. [5], p. 130).

(b) If $a,a_0 \in \eta_k, b,b_0 \in \eta_l$, then $[a,a_0]$ and $[b,b_0]$ are subsets of $\mathbb{R}$. □

It is obvious that as we focus on more particular kind of surfaces, we can obtain more powerful results. That is the reason because we introduce now a new type of space. However, the following theorems will be extended to a more general context later.

Definition 5.2 A *train* is a Denjoy domain $\Omega \subset \mathbb{C}$ with $\Omega \cap \mathbb{R} = \cup_{n=0}^{\infty}(a_n,b_n)$, such that $-\infty \le a_0$ and $b_n \le a_{n+1}$ for every n. A *flute surface* is a train with $b_n = a_{n+1}$ for every n.

We say that a curve in a train Ω is a *fundamental geodesic* if it is a simple closed geodesic which just intersects $\mathbb{R}$ in (a_0,b_0) and (a_n,b_n) for some $n > 0$; we denote by γ_n the fundamental geodesic corresponding to n and $2l_n := L_\Omega(\gamma_n)$. A curve in a train Ω is a *second fundamental geodesic* if it is a simple closed geodesic which just intersects $\mathbb{R}$ in (a_n, b_n) and (a_{n+1}, b_{n+1}) for some $n \geq 0$; we denote by σ_n the second fundamental geodesic corresponding to n and $2r_n := L_\Omega(\sigma_n)$. If $b_n = a_{n+1}$, we define σ_n as the puncture at this point and $r_n = 0$.

A *fundamental Y-piece* in a train Ω is the generalized Y-piece in Ω bounded by $\gamma_n, \gamma_{n+1}, \sigma_n$ for some $n > 0$; we denote by Y_n the fundamental Y-piece corresponding to n. A *fundamental hexagon* in a train Ω is the intersection $H_n := Y_n^+ = Y_n \cap \{z \in \mathbb{C} : \Im z \geq 0\}$ for some $n > 0$. We denote by α_n the length of the opposite side to σ_n^+ in H_n.

Remarks

1. Observe that $\eta_n = (a_n, b_n)$ is a closed set in Ω with $L_\Omega((a_n, b_n)) = \infty$, since $a_n, b_n \notin \Omega$.
2. A train is a flute surface if and only if every second fundamental geodesic is a puncture. Flute surfaces are a special case of trains, but they are important by themselves (see, e.g. [7,8]), since they are the simplest examples of infinite ends; in a flute surface it is possible to give a fairly precise description of the ending geometry (see, e.g. [22]).

Theorem 5.3 *Let us consider a train Ω. Then the following conditions are equivalent*:

(1) *Ω is δ-hyperbolic.*
(2) *There exists a constant c_1 such that $d_\Omega(z, \mathbb{R}) \leq c_1$ for every $z \in \cup_n \gamma_n$.*
(3) *There exist constants c_2, α, β such that $d_\Omega(z, \mathbb{R}) \leq c_2$ for every $z \in \cup_n g_n$, where g_n is freely homotopic to γ_n, $\overline{g_n} = g_n$ and g_n^+ is an (α, β)-quasigeodesic.*

Furthermore, the constants in each condition only depend on the constants appearing in any other of the conditions.

Remark Recall that $\overline{g_n}$ denotes the conjugate of g_n.

Proof The equivalence between (2) and (3) is a direct consequence of Theorem 2.9. Theorem 5.1 gives that (1) implies (2).

We prove now that (2) implies (1). By Theorem 5.1, it is enough to prove that there exists a constant c_1^* such that $d_\Omega(z, \mathbb{R}) \leq c_1^*$ for every $z \in \cup_n \gamma_{mn}$, where γ_{mn} is the simple closed geodesic which just intersects $\mathbb{R}$ in (a_m, b_m) and (a_n, b_n) for $0 < m < n$. Recall that for each set A in Ω, we denote by A^+ the subset $A^+ := A \cap \{z \in \mathbb{C} : \Im z \geq 0\}$. Consider the geodesic hexagon H_{mn} in $X := \Omega^+$, with sides γ_{mn}^+, γ_m^+, γ_n^+, and the three geodesics joining their endpoints which are contained in (a_0, b_0), (a_m, b_m) and (a_n, b_n). Since X is isometric to a geodesically convex subset of the unit disk, it is $\log(1 + \sqrt{2})$-thin, as the unit disk. Hence, H_{mn} is $4\log(1 + \sqrt{2})$-thin, and given any $z \in \gamma_{mn}^+$, there exists $z_0 \in \gamma_m^+ \cup \gamma_n^+ \cup \mathbb{R}$ with $d_\Omega(z, z_0) \leq 4\log(1 + \sqrt{2})$. Therefore, $d_\Omega(z, \mathbb{R}) \leq c_1 + 4\log(1 + \sqrt{2})$. By symmetry, $d_\Omega(z, \mathbb{R}) \leq c_1 + 4\log(1 + \sqrt{2})$ holds for every $z \in \gamma_{mn}$. □

This proof gives directly the following.

Corollary 5.4 *Let us consider a Denjoy domain Ω such that $\cup_{n=0}^\infty (a_n, b_n) \subseteq \Omega$, with $-\infty \leq a_0$, $b_n \leq a_{n+1}$ and $a_n, b_n \in \partial\Omega$ for every n. We denote by γ_{mn} the simple closed geodesic joining (a_m, b_m) and (a_n, b_n), and $\gamma_n := \gamma_{0n}$. If $d_\Omega(z, \mathbb{R}) \leq c_1$ for every $z \in \cup_n \gamma_n$, then $d_\Omega(z, \mathbb{R}) \leq c_1 + 4\log(1 + \sqrt{2})$ holds for every $z \in \cup_{m \neq n} \gamma_{mn}$.*

Next, some lemmas which will allow us to study the hyperbolicity of trains in terms of the lengths of their fundamental geodesics.

Lemma 5.5 *Let us consider a train* Ω.

(1) *We have for every* n,

$$(\operatorname{Arcsinh} 2)\,(e^{-l_n} + e^{-l_{n+1}}) \le \alpha_n.$$

(2) *If* $r_n \le c_1 + |l_n - l_{n+1}|$ *and* $l_n, l_{n+1} \ge l_0$ *for some fixed* n, *then there exists a constant* c_2, *which only depends on* c_1 *and* l_0, *such that*

$$\alpha_n \le c_2\,(e^{-l_n} + e^{-l_{n+1}}).$$

Proof Standard hyperbolic trigonometry (see e.g. [9], p. 161) gives

$$\cosh \alpha_n = \frac{\cosh r_n + \cosh l_n \cosh l_{n+1}}{\sinh l_n \sinh l_{n+1}}.$$

Since $\coth t \ge 1 + 2e^{-2t}$ if $t \ge 0$, we obtain

$$\begin{aligned}
\cosh \alpha_n &\ge 4e^{-l_n - l_{n+1}} + (1 + 2e^{-2l_n})(1 + 2e^{-2l_{n+1}}),\\
\cosh \alpha_n &\ge 4e^{-l_n - l_{n+1}} + 1 + 2e^{-2l_n} + 2e^{-2l_{n+1}},\\
\sinh^2 \frac{\alpha_n}{2} &= \frac{\cosh \alpha_n - 1}{2} \ge e^{-2l_n} + e^{-2l_{n+1}} + 2e^{-l_n - l_{n+1}},\\
\sinh \frac{\alpha_n}{2} &\ge e^{-l_n} + e^{-l_{n+1}},\\
\alpha_n &\ge 2\operatorname{Arcsinh}(e^{-l_n} + e^{-l_{n+1}}).
\end{aligned}$$

Since the function $t^{-1}\operatorname{Arcsinh} t$ is decreasing in $[0, 2]$, we have

$$2\operatorname{Arcsinh} t \ge t \operatorname{Arcsinh} 2 \quad \forall t \in [0, 2] \quad \text{and} \quad \alpha_n \ge (\operatorname{Arcsinh} 2)(e^{-l_n} + e^{-l_{n+1}}).$$

This finishes the proof of (1).

In order to prove (2), we remark that if $x \ge l_0$, then $e^{-2l_0}e^{2x} \ge 1$ and $e^{2x} - 1 \ge (1 - e^{-2l_0})e^{2x}$. Therefore, if we define $c_3^{-1} := (1 - e^{-2l_0})/2$, we have

$$e^{2x} - 1 \ge 2\,c_3^{-1}e^{2x}, \quad \sinh x \ge c_3^{-1}e^{x}, \quad \coth x = 1 + \frac{2}{e^{2x} - 1} \le 1 + c_3\,e^{-2x},$$

for every $x \ge l_0$.

Hence, we obtain

$$\cosh \alpha_n = \frac{\cosh r_n + \cosh l_n \cosh l_{n+1}}{\sinh l_n \sinh l_{n+1}} \le c_3^2\,e^{r_n - l_n - l_{n+1}} + (1 + c_3\,e^{-2l_n})(1 + c_3\,e^{-2l_{n+1}}).$$

The inequality $r_n - l_n - l_{n+1} \le -2\min\{l_n, l_{n+1}\} + c_1$ (which is equivalent to $r_n \le c_1 + |l_n - l_{n+1}|$) gives

$$c_3^2\,e^{r_n - l_n - l_{n+1}} \le c_3^2\,e^{c_1 - 2\min\{l_n, l_{n+1}\}} \le c_3^2\,e^{c_1}(e^{-2l_n} + e^{-2l_{n+1}}).$$

Then

$$\begin{aligned}
\cosh \alpha_n &\le c_3^2\,e^{c_1}(e^{-2l_n} + e^{-2l_{n+1}}) + 1 + c_3\,e^{-2l_n} + c_3\,e^{-2l_{n+1}} + c_3^2\,e^{-2l_n - 2l_{n+1}},\\
2\sinh^2 \frac{\alpha_n}{2} &= \cosh \alpha_n - 1 \le (c_3^2\,e^{c_1} + c_3 + c_3^2)(e^{-2l_n} + e^{-2l_{n+1}}),\\
\frac{\alpha_n}{2} &\le \sinh \frac{\alpha_n}{2} \le \frac{c_2}{2}(e^{-l_n} + e^{-l_{n+1}}),
\end{aligned}$$

and we obtain $\alpha_n \le c_2\,(e^{-l_n} + e^{-l_{n+1}})$. □

Definition 5.6 Given a train Ω and a point $z \in \Omega$, we define the *height* of z as $h(z) := d_\Omega(z,(a_0,b_0))$. We define z_0 as the point in (a_0,b_0) with $h(z) = d_\Omega(z,(a_0,b_0)) = d_\Omega(z,z_0)$. We denote by $p(z)$ a real number with $d_\Omega(z,p(z)) = d_\Omega(z,\mathbb{R})$. (It is possible that there exist several real numbers with this property; in this case $p(z)$ denotes any choice.)

Lemma 5.7 *Let us consider a train Ω. We have $d_\Omega(z,w) \geq |h(z) - h(w)|$ for every $z,w \in \Omega$. Furthermore, if Ω is δ-hyperbolic, then there exists a constant c, which only depends on δ, such that $|h(z) - h(p(z))| \leq c$ for every $z \in \cup_n \gamma_n$.*

Proof Fix $z \in \Omega$. It is enough to show that $d_\Omega(z,w) \geq |h(z) - h(w)|$; the second part of the lemma is a consequence of this fact (with $w = p(z)$) and Theorem 5.3. Let us consider the geodesic quadrilateral $\{z,w,w_0,z_0\}$. Standard hyperbolic trigonometry (see e.g. [17], p. 88) gives

$$\cosh d_\Omega(z,w) = \cosh d_\Omega(z_0,w_0) \cosh h(z) \cosh h(w) - \sinh h(z) \sinh h(w)$$
$$\geq \cosh h(z) \cosh h(w) - \sinh h(z) \sinh h(w) = \cosh \big(h(z) - h(w)\big),$$

and consequently, $d_\Omega(z,w) \geq |h(z) - h(w)|$. □

Lemma 5.8 *Let us consider a train Ω. If $l_0 \leq l_n < l_{n+1}$ and $r_n \leq c_1$ for some fixed n, then $d_\Omega(z,\mathbb{R}) \leq d_\Omega(z,(a_n,b_n)) \leq c_2$ for every $z \in \gamma_{n+1}$ with $h(z) \in [l_n,l_{n+1}]$, where c_2 only depends on c_1 and l_0. We also have $d_\Omega(z,\mathbb{R} \cup \gamma_n) \leq c_2$ for every $z \in \gamma_{n+1}$.*

Proof By Lemma 5.5 there exists a constant c_3, which only depends on c_1 and l_0, such that $\alpha_n \leq c_3/2(\mathrm{e}^{-l_n} + \mathrm{e}^{-l_{n+1}})$. We have $\alpha_n \leq c_3\, \mathrm{e}^{-l_n}$ and $\sinh \alpha_n \leq \mathrm{e}^{-l_n} \sinh c_3$, since $\sinh at \leq t \sinh a$ for every $t \in [0,1]$.

Fix $z \in \gamma_{n+1}$ with $h(z) \in [l_n,l_{n+1}]$. By symmetry, without loss of generality we can assume that $z \in \gamma_{n+1}^+$.

Let us define $u^n := (a_n,b_n) \cap \gamma_n$, $u^{n+1} := (a_{n+1},b_{n+1}) \cap \gamma_{n+1}$ and v^n as the point in γ_{n+1}^+ with $d_\Omega(u^n,\gamma_{n+1}) = d_\Omega(u^n,v^n)$. By convexity (see e.g. [10], Section 4, or [16], p. 2), it is clear that $d_\Omega(z,(a_n,b_n)) \leq \max\{d_\Omega(u^n,\gamma_{n+1}), d_\Omega(u^{n+1},(a_n,b_n))\}$. It is also clear that $d_\Omega(z,\mathbb{R} \cup \gamma_n) \leq \max\{d_\Omega(u^n,\gamma_{n+1}), d_\Omega(u^{n+1},(a_n,b_n))\}$, if $h(z) \leq l_n$.

Let us consider the geodesic right-angled quadrilateral $\{u^n,u_0^n,v_0^n,v^n\}$. Standard hyperbolic trigonometry (see e.g. [17], p. 88) gives $\sinh d_\Omega(u^n,\gamma_{n+1}) = \sinh \alpha_n \cosh l_n < \mathrm{e}^{-l_n} \sinh c_3\, \mathrm{e}^{l_n} = \sinh c_3$, and consequently, $d_\Omega(u^n,\gamma_{n+1}) < c_3$.

The shortest geodesic in H_n joining (a_n,b_n) with γ_{n+1}^+ separates H_n into two right-angled pentagons: P_n (which contains γ_n^+) and Q_n (which contains σ_n^+). We denote by w^{n+1} the intersection of this geodesic with γ_{n+1}^+. Considering P_n, standard hyperbolic trigonometry (see e.g. [17], p. 87) gives $\sinh l_n \sinh \alpha_n = \cosh d_\Omega(\gamma_{n+1}^+,(a_n,b_n))$, and then

$$\sinh d_\Omega(\gamma_{n+1}^+,(a_n,b_n)) < \cosh d_\Omega(\gamma_{n+1}^+,(a_n,b_n)) = \sinh l_n \sinh \alpha_n$$
$$\leq \mathrm{e}^{l_n} \mathrm{e}^{-l_n} \sinh c_3 = \sinh c_3.$$

Hence, $d_\Omega(\gamma_{n+1}^+,(a_n,b_n)) < c_3$. Considering Q_n, standard hyperbolic trigonometry gives $\sinh d_\Omega(\gamma_{n+1}^+,(a_n,b_n)) \sinh d_\Omega(u^{n+1},w^{n+1}) = \cosh r_n$, and then

$$\sinh d_\Omega(u^{n+1},w^{n+1}) = \frac{\cosh r_n}{\sinh d_\Omega(\gamma_{n+1}^+,(a_n,b_n))} \leq \frac{\cosh c_1}{\sinh d_\Omega(\gamma_{n+1}^+,(a_n,b_n))}.$$

The shortest geodesic in Q_n joining u^{n+1} with (a_n, b_n) separates Q_n into two right-angled quadrilaterals. Considering the right-angled quadrilateral which contains $\gamma_{n+1} \cap Q_n$, standard hyperbolic trigonometry gives

$$\begin{aligned}\sinh d_\Omega(u^{n+1}, (a_n, b_n)) &= \sinh d_\Omega(\gamma^+_{n+1}, (a_n, b_n)) \cosh d_\Omega(u^{n+1}, w^{n+1}) \\ &\le \sinh d_\Omega(\gamma^+_{n+1}, (a_n, b_n)) \sqrt{\frac{\cosh^2 c_1}{\sinh^2 d_\Omega(\gamma^+_{n+1}, (a_n, b_n))} + 1} \\ &\le \sqrt{\cosh^2 c_1 + \sinh^2 d_\Omega(\gamma^+_{n+1}, (a_n, b_n))} \le \sqrt{\cosh^2 c_1 + \sinh^2 c_3},\end{aligned}$$

and consequently, $d_\Omega(u^{n+1}, (a_n, b_n)) \le c_4$. If $c_2 := \max\{c_3, c_4\}$, then $d_\Omega(z, \mathbb{R}) \le d_\Omega(z, (a_n, b_n)) \le c_2$, if $h(z) \in [l_n, l_{n+1}]$, and $d_\Omega(z, \mathbb{R} \cup \gamma_n) \le c_2$, if $h(z) \le l_n$. □

Lemma 5.9 *Let us consider a train* Ω. *If* $r_{n_0} \le c_1$, *then* $d_\Omega(z, \mathbb{R} \cup \gamma_{n_0}) \le 4 \log(1 + \sqrt{2}) + c_1/2$ *for every* $z \in \gamma_{n_0+1}$.

Proof Let us consider $z \in \gamma_{n_0+1}$. Without loss of generality we can assume that $z \in \gamma^+_{n_0+1} \subset H_{n_0} = Y^+_{n_0}$. Since H_{n_0} is a (simply connected) right-angled hexagon, it is isometric to a hexagon in the unit disk. Every hexagon in the unit disk is $4 \log(1 + \sqrt{2})$-thin, since the unit disk is $\log(1 + \sqrt{2})$-thin (see, e.g. [5], p. 130). Let us denote by w a point in $\partial H_{n_0} \setminus \gamma^+_{n_0+1}$ with $d_\Omega(z, w) \le 4 \log(1 + \sqrt{2})$. If $w \in \mathbb{R} \cup \gamma_{n_0}$, the conclusion of the lemma holds. If $w \in \sigma^+_{n_0}$, then $d_\Omega(w, \mathbb{R}) \le c_1/2$ and $d_\Omega(z, \mathbb{R} \cup \gamma_{n_0}) \le 4 \log(1 + \sqrt{2}) + c_1/2$. □

The following lemma gathers the main ideas and computations which will be applied in the theorems below.

Lemma 5.10 *Let us consider a train* Ω.

(a) *If there exists a (finite or infinite) subset* $\{n_k\}_k \subset \mathbb{N}$ *with* $r_n \le c_1 + |l_n - l_{n+1}|$ *for every* $n \in [n_1, \sup_k n_k)$, *and* $l_n \ge l_0 > 0$ *for every* $n \in [n_1, \sup_k n_k]$, $l_{n_1} \le l^0$, $r_{n_k} \le c_1$, $l_{n_k+1} + c_1 \ge l_{n_{k+1}}$ *for every* k, *and*

$$\sum_{n=n_m+1}^{n_k} e^{-l_n} \le c_2\, e^{-l_{n_m+1}}, \qquad \textit{for every } m < k, \tag{5.1}$$

then $d_\Omega(z, \mathbb{R}) \le c$ *for every* $z \in \cup_k \gamma_{n_k}$ *with* $h(z) > l_{n_1}$, *where* c *is a constant which only depends on* c_1, c_2 *and* l_0. *Consequently,* $d_\Omega(z, \mathbb{R}) \le \max\{c, l^0\}$ *for every* $z \in \cup_k \gamma_{n_k}$. *We also have* $d_\Omega(z, \mathbb{R} \cup \gamma_{n_m}) \le c$ *for every* m *and every* $z \in \cup_{k \ge m} \gamma_{n_k}$.

(b) *If* $\lim_{n\to\infty} l_n = \infty$, $\{n_k\}_k$ *is a subsequence with* $l_n + c_3 \ge l_{n_k}$ *for every* k *and every* $n \ge n_k$, *and such that the condition*

$$\sum_{n=n_k}^{\infty} e^{-l_n} \le c_2\, e^{-l_{n_k}}, \qquad \textit{for every } k, \tag{5.2}$$

does not hold for this $\{n_k\}_k$, *then* Ω *is not hyperbolic.*

Proof We prove first (a). Fix $z \in \gamma_{n_k}$ for some k, with $h(z) > l_{n_1}$. By symmetry, without loss of generality we can assume that $z \in \gamma^+_{n_k}$.

By Lemma 5.5, there exists a constant c_4, which only depends on c_1 and l_0, such that $\alpha_n \leq c_4/2\,(\mathrm{e}^{-l_n} + \mathrm{e}^{-l_{n+1}})$ for any $n \in [n_1, \sup_k n_k)$.

Since $h(z) > l_{n_1}$, we can choose $1 \leq m < k$ verifying both $l_{n_m} < l_{n_{m+1}}$ and $h(z) \in [l_{n_m}, l_{n_{m+1}}]$.

If $h(z) \geq l_{n_m+1}$, consider the point $z^* \in (a_{n_m+1}, b_{n_m+1}) \cap H_{n_m+1}$ with $h(z) = h(z^*)$. If $h(z) < l_{n_m+1}$, we consider the point $z^* \in \gamma^+_{n_m+1}$ with $h(z) = h(z^*)$.

In both cases, we take the geodesic quadrilateral $\{z, z^*, z_0^*, z_0\}$. Standard hyperbolic trigonometry gives $\sinh \frac{1}{2} d_\Omega(z, z^*) = \sinh \frac{1}{2} d_\Omega(z_0, z_0^*) \cosh h(z)$. Observe that

$$d_\Omega(z_0, z_0^*) \leq \sum_{n=n_m+1}^{n_k-1} \alpha_n \leq \sum_{n=n_m+1}^{n_k-1} \frac{c_4}{2}\,(\mathrm{e}^{-l_n} + \mathrm{e}^{-l_{n+1}}) \leq \sum_{n=n_m+1}^{n_k} c_4\,\mathrm{e}^{-l_n} \leq c_2 c_4\,\mathrm{e}^{-l_{n_m+1}},$$

and therefore $d_\Omega(z_0, z_0^*)$ is bounded by c_2c_4. Then, $\sinh \frac{1}{2} d_\Omega(z_0, z_0^*) \leq c_5 \mathrm{e}^{-l_{n_m+1}}$ and $\sinh \frac{1}{2} d_\Omega(z, z^*) \leq c_5 \mathrm{e}^{-l_{n_m+1}} \mathrm{e}^{h(z)} \leq c_5 \mathrm{e}^{c_1 - l_{n_{m+1}}} \mathrm{e}^{h(z)} \leq c_5 \mathrm{e}^{c_1}$. If $h(z) \geq l_{n_m+1}$, then $d_\Omega(z, \mathbb{R}) \leq 2\,\mathrm{Arcsinh}(c_5 \mathrm{e}^{c_1})$.

If $h(z) < l_{n_m+1}$, then $z^* \notin \mathbb{R}$, but we have $h(z) \in [l_{n_m}, l_{n_m+1}]$. Hence, Lemma 5.8 gives $d_\Omega(z^*, \mathbb{R}) \leq d_\Omega(z^*, (a_{n_m}, b_{n_m})) \leq c_6$, where c_6 only depends on c_1 and l_0.

Consequently, $d_\Omega(z, \mathbb{R}) \leq d_\Omega(z, z^*) + d_\Omega(z^*, \mathbb{R}) \leq 2\,\mathrm{Arcsinh}(c_5 e^{c_1}) + c_6 =: c$, for every $z \in \cup_k \gamma_{n_k}$ with $h(z) > l_{n_1}$.

The same computations finish the proof of part (*a*) (recall that Lemma 5.8 also covers the case $h(z) \leq l_{n_1}$).

We prove now (*b*). By Lemma 5.7, without loss of generality we can assume that there exists a constant c_7 such that $|h(z) - h(p(z))| \leq c_7$ for every $z \in \cup_n \gamma_n$. Since (5.2) does not hold, given any $M > \mathrm{e}^{2(c_3+c_7)}$, there exist $m < k$ such that

$$\sum_{n=n_m}^{n_k} \mathrm{e}^{-l_n} \geq M\,\mathrm{e}^{-l_{n_m}}.$$

Since $\lim_{n\to\infty} l_n = \infty$, without loss of generality we can take m large enough so that $l_{n_m} \geq \log M$. Consider $z \in \gamma^+_{n_k}$ with $h(z) = l_{n_m} - \frac{1}{2}\log M < l_{n_m} - c_3 - c_7$; hence, $h(p(z)) < l_{n_m} - c_3 \leq l_n$ for every $n \geq n_m$, and $p(z) \in \cup_{n=0}^{n_m-1}(a_n, b_n)$. We also have $h(z) = d_\Omega(z, (a_0, b_0)) = l_{n_m} - \frac{1}{2}\log M \geq \frac{1}{2}\log M > c_7$, and then $p(z) \notin (a_0, b_0)$.

Since $p(z) \in \cup_{n=1}^{n_m-1}(a_n, b_n)$, let us consider the geodesic quadrilateral $\{z, p(z), p(z)_0, z_0\}$. Standard hyperbolic trigonometry gives

$$\begin{aligned}\cosh d_\Omega(z, p(z)) &= \cosh d_\Omega(z_0, p(z)_0) \cosh h(z) \cosh h(p(z)) - \sinh h(z) \sinh h(p(z))\\ &\geq (\cosh d_\Omega(z_0, p(z)_0) - 1) \cosh h(z) \cosh h(p(z))\\ &\geq \frac{1}{8}\, d_\Omega(z_0, p(z)_0)^2 \mathrm{e}^{h(z)} \mathrm{e}^{h(p(z))}.\end{aligned}$$

Observe that, by Lemma 5.5,

$$d_\Omega(z_0, p(z)_0) \geq d_\Omega(z_0, \gamma_{n_m}) \geq \sum_{n=n_m}^{n_k-1} \alpha_n \geq \sum_{n=n_m}^{n_k-1} (\mathrm{e}^{-l_n} + \mathrm{e}^{-l_{n+1}}) > \sum_{n=n_m}^{n_k} \mathrm{e}^{-l_n} \geq M\,\mathrm{e}^{-l_{n_m}}.$$

Consequently,

$$\cosh d_\Omega(z, \mathbb{R}) = \cosh d_\Omega(z, p(z)) \geq \frac{1}{8}\, M^2\,\mathrm{e}^{-2l_{n_m}}\,\mathrm{e}^{l_{n_m} - \frac{1}{2}\log M}\,\mathrm{e}^{l_{n_m} - \frac{1}{2}\log M - c_7} = \frac{1}{8}\, M\,\mathrm{e}^{-c_7}.$$

Since M can be arbitrarily large, Theorem 5.3 gives that Ω is not hyperbolic. □

Corollary 5.11 *Let us consider a train Ω with $r_n \le c_1$ for every $M \le n < N$, $l_M \le l^0$ and*

$$\sum_{k=n}^{N} e^{-l_k} \le c_2\, e^{-l_n}, \qquad \text{for every } M < n \le N. \tag{5.3}$$

Then $d_\Omega(z, \mathbb{R} \cup \gamma_M) \le c$ for every $z \in \cup_{n=M}^{N} \gamma_n$, where c is a constant which only depends on c_1 and c_2. Consequently, $d_\Omega(z, \mathbb{R}) \le \max\{c, l^0\}$ for every $z \in \cup_{n=M}^{N} \gamma_n$.

Proof If $l_n < 1$ for some $M \le n \le N$, then $d_\Omega(z, \mathbb{R} \cup \gamma_M) \le d_\Omega(z, \mathbb{R}) \le 1/2$ for every $z \in \gamma_n$.

Let us consider $M_0 \le N_0$ with the following properties:

(i) $l_n \ge 1$ for every $M_0 \le n \le N_0$,
(ii) $l_{M_0-1} < 1$ or $M_0 = M$,
(iii) $l_{N_0+1} < 1$ or $N_0 = N$.

Part (*a*) of Lemma 5.10, with $\{n_k\}_k = \{M_0, M_0 + 1, \ldots, N_0\}$ (observe that in this case $n_{k+1} = n_k + 1$), gives that $d_\Omega(z, \mathbb{R} \cup \gamma_{M_0}) \le c_3$ for every $z \in \cup_{n=M_0}^{N_0} \gamma_n$, where c_3 is a constant which only depends on c_1 and c_2.

If $M_0 = M$, then $d_\Omega(z, \mathbb{R} \cup \gamma_M) \le c_3$ for every $z \in \cup_{n=M_0}^{N_0} \gamma_n$.

If $M_0 > M$, then $l_{M_0-1} < 1$. Lemma 5.9 gives that $d_\Omega(z, \mathbb{R} \cup \gamma_{M_0-1}) \le 4\log(1 + \sqrt{2}) + c_1/2$ for every $z \in \gamma_{M_0}$. Hence, $d_\Omega(z, \mathbb{R} \cup \gamma_M) \le d_\Omega(z, \mathbb{R}) \le c := c_3 + 4\log(1 + \sqrt{2}) + c_1/2 + 1/2$ for every $z \in \cup_{n=M_0}^{N_0} \gamma_n$.

Since every $M \le n \le N$ holds either $l_n < 1$ or $M_0 \le n \le N_0$, for some $M_0 \le N_0$ verifying (i), (ii) and (iii), then $d_\Omega(z, \mathbb{R} \cup \gamma_M) \le c$ for every $z \in \cup_{n=M}^{N} \gamma_n$. □

Now, we provide the results that study hyperbolicity in terms of $\{l_n\}_n$ and $\{r_n\}_n$. We deal separately the cases when $\lim_{n\to\infty} l_n = \infty$, $\{l_n\}_n$ is bounded, or none of these. Firs of all, we consider when $\lim_{n\to\infty} l_n = \infty$.

Theorem 5.12 *Let us consider a train Ω with $\lim_{n\to\infty} l_n = \infty$.*

(a) *If $l_1 \le l^0$, $r_n \le c_1$ for every n and*

$$\sum_{k=n}^{\infty} e^{-l_k} \le c_2\, e^{-l_n}, \qquad \text{for every } n > 1, \tag{5.4}$$

then Ω is δ-hyperbolic, where δ is a constant which only depends on c_1, c_2 and l^0.

(b) *If $l_n + c_3 \ge l_m$ for every $n \ge m$ and Ω is hyperbolic, then* (5.4) *holds.*

Remarks

1. Condition (5.4) is equivalent to

$$\limsup_{n\to\infty}\ e^{l_n} \sum_{k=n}^{\infty} e^{-l_k} < \infty.$$

2. Examples of sequences verifying this property are $l_n = a^{n^b}$ $(a > 1, b > 0)$, and $l_n = n^a$ $(a \ge 1)$. Examples of sequences that do not verify this property are $l_n = n^a$ $(a < 1)$, and $l_n = a \log n$ $(a > 0)$.

3. Condition $l_n + c_3 \geq l_m$ for every $n \geq m$ holds, for example, if $\{l_n\}$ is a non-decreasing sequence.

Proof By Corollary 5.11, for any fixed N, we have $d_\Omega(z,\mathbb{R}) \leq c_3$ for every $z \in \cup_{n=1}^{N}\gamma_n$, where c_3 is a constant which only depends on c_1, c_2 and l^0. Since c_3 does not depend on N, we obtain $d_\Omega(z,\mathbb{R}) \leq c_3$ for every $z \in \cup_{n=1}^{\infty}\gamma_n$. Then Ω is δ-hyperbolic by Theorem 5.3, where δ is a constant which only depends on c_1, c_2 and l^0.

Assume now that (5.4) does not hold. If we take $\{n_k\}_k = \mathbb{N}$, part (b) of Lemma 5.10 gives that Ω is not hyperbolic. □

Corollary 5.13 *Let us consider a train Ω, with $\lim_{n\to\infty} l_n = \infty$, $l_n + c_1 \geq l_m$ for every $n \geq m$, and $\sum_{n=1}^{\infty} e^{-l_n} = \infty$. Then Ω is not hyperbolic.*

We obtain directly the following characterization.

Theorem 5.14 *Let us consider a train Ω with $\lim_{n\to\infty} l_n = \infty$, $r_n \leq c_1$ for every n, and $l_n + c_1 \geq l_m$ for every $n \geq m$. Then Ω is hyperbolic if and only if* (5.4) *holds.*

The following theorem shows that the hypothesis $r_n \leq c_1$ is not very restrictive if $\lim_{n\to\infty} l_n = \infty$. (This is not the case if we have $l_n \leq c$; see Theorem 5.25.) We need two lemmas.

Lemma 5.15 *Let us consider a train Ω. Assume that $l_{n_0} \leq l_n + c$ for every $n \geq n_0$. Then $d_\Omega(z,\mathbb{R}) \leq d_\Omega(z, \cup_{n=0}^{n_0}(a_n,b_n)) < d_\Omega(z,\mathbb{R}) + c$ for every $z \in \cup_{n=1}^{n_0}\gamma_n$.*

Proof The first inequality is trivial. Let us consider $z \in \gamma_n$, with $1 \leq n \leq n_0$ and such that $d_\Omega(z,\mathbb{R}) < d_\Omega(z,\cup_{n=0}^{n_0}(a_n,b_n))$. Without loss of generality we can assume that $z \in \gamma_n^+$. We have that $p(z) \in (a_{n_1}, b_{n_1})$, with $n_1 > n_0$; therefore $h(p(z)) \geq l_{n_1} \geq l_{n_0} - c$. Let us observe that $h(p(z)) \leq l_{n_0}$, since if $h(p(z)) > l_{n_0}$, then $d_\Omega(z,\mathbb{R}) < d_\Omega(z,(a_{n_0},b_{n_0})) < d_\Omega(z,p(z)) = d_\Omega(z,\mathbb{R})$, which is a contradiction. Consider the point $z' \in \gamma_{n_0}^+$ with $h(z') = h(p(z))$. It is clear that $d_\Omega(z,z') < d_\Omega(z,p(z)) = d_\Omega(z,\mathbb{R})$. We also have $d_\Omega(z',(a_{n_0},b_{n_0})) = l_{n_0} - h(z') = l_{n_0} - h(p(z)) \leq c$. Consequently,

$$d_\Omega(z,\cup_{n=0}^{n_0}(a_n,b_n)) \leq d_\Omega(z,(a_{n_0},b_{n_0})) \leq d_\Omega(z,z') + d_\Omega(z',(a_{n_0},b_{n_0})) < d_\Omega(z,\mathbb{R}) + c.$$

□

Lemma 5.16 *Let us consider a train Ω and some fixed n. We take $z_n \in \gamma_{n+1}^+$ with $h(z_n) = l_{n+1} - s_n$, where $s_n := \log(\min\{l_{n+1}, r_n\})$. Then*

$$d_\Omega(z_n,(a_0,b_0)) \geq l_{n+1} - \log l_{n+1}, \qquad d_\Omega(z_n,(a_n,b_n)) \geq r_n - \log r_n,$$
$$d_\Omega(z_n,\gamma_n) \geq \operatorname{Arcsinh} e^{\frac{1}{2}(r_n + l_{n+1} - l_n - 2\log r_n)}.$$

Proof It is direct that $d_\Omega(z_n,(a_0,b_0)) = h(z_n) = l_{n+1} - s_n \geq l_{n+1} - \log l_{n+1}$. We also have that $r_n = d_\Omega((a_n,b_n),(a_{n+1},b_{n+1})) \leq s_n + d_\Omega(z_n,(a_n,b_n))$, and then

$$d_\Omega(z_n,(a_n,b_n)) \geq r_n - s_n \geq r_n - \log r_n.$$

Standard hyperbolic trigonometry (see e.g. [9], p. 161) in H_n gives

$$\cosh\alpha_n = \frac{\cosh r_n + \cosh l_n \cosh l_{n+1}}{\sinh l_n \sinh l_{n+1}} \geq \frac{\frac{1}{2}\,e^{r_n}}{\frac{1}{2}\,e^{l_n}\,\frac{1}{2}\,e^{l_{n+1}}} + 1 = 1 + 2e^{r_n - l_n - l_{n+1}}.$$

Then, we have

$$\frac{1}{2}\sinh\alpha_n \geq \sinh\frac{\alpha_n}{2} = \sqrt{\frac{\cosh\alpha_n - 1}{2}} \geq e^{\frac{1}{2}(r_n - l_n - l_{n+1})}.$$

Standard hyperbolic trigonometry for right-angled quadrilaterals gives

$$\begin{aligned}\sinh d_\Omega(z_n,\gamma_n) &= \sinh\alpha_n\cosh(l_{n+1}-s_n) \geq 2\, e^{\frac{1}{2}(r_n-l_n-l_{n+1})}\frac{1}{2}\, e^{l_{n+1}-\log r_n}\\ &= e^{\frac{1}{2}(r_n+l_{n+1}-l_n-2\log r_n)}.\end{aligned}$$

□

Theorem 5.17 *Let us consider a train Ω and a subsequence $\{n_k\}_k$ verifying either:*

(a) $\lim_{n\to\infty} l_n = \infty$, $\lim_{k\to\infty} r_{n_k} = \infty$, $l_{n_k} \leq l_{n_k+1} + c$ *for every* k *and* $l_{n_k+1} \leq l_n + c$ *for every* k *and every* $n \geq n_k + 1$,

(b) $\lim_{k\to\infty} l_{n_k+1} = \lim_{k\to\infty} r_{n_k} = \lim_{k\to\infty} r_{n_k+1} = \infty$ *and* $l_{n_k}, l_{n_k+2} \leq l_{n_k+1} + c$ *for every* k.

Then Ω is not hyperbolic.

Remark The conclusion of Theorem 5.17 (with hypothesis (a)) also holds if we change condition "$l_{n_k} \leq l_{n_k+1} + c$ for every k" by "there exists an increasing function F with $\lim_{t\to\infty} F(t) = \lim_{t\to\infty}(t - F(t)) = \infty$ and $\lim_{k\to\infty}(r_{n_k} + l_{n_k+1} - l_{n_k} - 2F(\min\{l_{n_k+1}, r_{n_k}\})) = \infty$" (it is enough to change log by F in the definition of s_{n_k} in the proof below).

Proof Let us assume hypothesis (a). Consider $z_{n_k} \in \gamma^+_{n_k+1}$ with $h(z_{n_k}) = l_{n_k+1} - s_{n_k}$, where $s_{n_k} := \log(\min\{l_{n_k+1}, r_{n_k}\})$.

It is direct that $d_\Omega(z_{n_k}, (a_{n_k+1}, b_{n_k+1})) = s_{n_k}$ and $\lim_{k\to\infty} s_{n_k} = \infty$.

Lemma 5.16 implies the following facts:

$$\begin{aligned}d_\Omega(z_{n_k},(a_0,b_0)) &\geq l_{n_k+1} - \log l_{n_k+1} \longrightarrow \infty,\\ d_\Omega(z_{n_k},(a_{n_k},b_{n_k})) &\geq r_{n_k} - \log r_{n_k} \longrightarrow \infty,\\ d_\Omega(z_{n_k},\gamma_{n_k}) &\geq \operatorname{Arcsinh} e^{\frac{1}{2}(r_{n_k}+l_{n_k+1}-l_{n_k}-2\log r_{n_k})}\\ &\geq \operatorname{Arcsinh} e^{\frac{1}{2}(r_{n_k}-c-2\log r_{n_k})} \longrightarrow \infty,\\ d_\Omega(z_{n_k},\cup_{n=1}^{n_k-1}(a_n,b_n)) &\geq d_\Omega(z_{n_k},\gamma_{n_k}) \longrightarrow \infty,\end{aligned}$$

if $k\to\infty$. Then $\lim_{k\to\infty} d_\Omega(z_{n_k}, \cup_{n=0}^{n_k+1}(a_n,b_n)) = \infty$. Since $l_{n_k+1} \leq l_n + c$ for every k and every $n \geq n_k + 1$, Lemma 5.15 gives that $\lim_{k\to\infty} d_\Omega(z_{n_k}, \mathbb{R}) = \infty$. Hence, Ω is not hyperbolic by Theorem 5.3.

Let us assume now hypothesis (b). Consider $z_{n_k} \in \gamma^+_{n_k+1}$ with $h(z_{n_k}) = l_{n_k+1} - s_{n_k}$, where $s_{n_k} := \log(\min\{l_{n_k+1}, r_{n_k}, r_{n_k+1}\})$. The same argument of (a) gives

$$\begin{aligned}d_\Omega(z_{n_k},(a_{n_k+1},b_{n_k+1})) &= s_{n_k} = \log(\min\{l_{n_k+1}, r_{n_k}, r_{n_k+1}\}) \longrightarrow \infty,\\ d_\Omega(z_{n_k},(a_0,b_0)) &\geq l_{n_k+1} - \log l_{n_k+1} \longrightarrow \infty,\\ d_\Omega(z_{n_k},(a_{n_k},b_{n_k})) &\geq r_{n_k} - \log r_{n_k} \longrightarrow \infty,\\ d_\Omega(z_{n_k},\gamma_{n_k}) &\geq \operatorname{Arcsinh} e^{\frac{1}{2}(r_{n_k}+l_{n_k+1}-l_{n_k}-2\log r_{n_k})}\\ &\geq \operatorname{Arcsinh} e^{\frac{1}{2}(r_{n_k}-c-2\log r_{n_k})} \longrightarrow \infty,\\ d_\Omega(z_{n_k},\cup_{n=1}^{n_k-1}(a_n,b_n)) &\geq d_\Omega(z_{n_k},\gamma_{n_k}) \longrightarrow \infty,\end{aligned}$$

if $k \to \infty$. By symmetry (since r_{n_k+1} appears in the definition of s_{n_k}), we also have

$$\begin{aligned} d_\Omega(z_{n_k}, (a_{n_k+2}, b_{n_k+2})) &\geq r_{n_k+1} - \log r_{n_k+1} \longrightarrow \infty, \\ d_\Omega(z_{n_k}, \gamma_{n_k+2}) &\geq \operatorname{Arcsinh} \mathrm{e}^{\frac{1}{2}(r_{n_k+1}+l_{n_k+1}-l_{n_k+2}-2\log r_{n_k+1})} \\ &\geq \operatorname{Arcsinh} \mathrm{e}^{\frac{1}{2}(r_{n_k+1}-c-2\log r_{n_k+1})} \longrightarrow \infty, \\ d_\Omega(z_{n_k}, \cup_{n=n_k+3}^{\infty}(a_n, b_n)) &\geq d_\Omega(z_{n_k}, \gamma_{n_k+2}) \longrightarrow \infty, \end{aligned}$$

if $k \to \infty$. Then $\lim_{k\to\infty} d_\Omega(z_{n_k}, \mathbb{R}) = \infty$ and Ω is not hyperbolic by Theorem 5.3. □

Corollary 5.18 *Let us consider a train Ω, with $\lim_{n\to\infty} l_n = \infty$, $\{l_n\}_n$ a non-decreasing sequence, and $\{r_n\}_n$ a non-bounded sequence. Then Ω is not hyperbolic.*

Corollary 5.19 *Let us consider a train Ω, with $\lim_{n\to\infty} l_n = \lim_{n\to\infty} r_n = \infty$. Then Ω is not hyperbolic.*

Proof Since $\lim_{n\to\infty} l_n = \infty$, we can choose a subsequence $\{n_k\}_k$ with $l_{n_k+1} \leq l_n$ for every k and every $n \geq n_k + 1$.

If $l_{n_k} \leq l_{n_k+1}$ for infinitely many k's, part (a) of Theorem 5.17 gives that Ω is not hyperbolic.

In other case, we have $l_{n_k} > l_{n_k+1}$ for every k large enough. Then, given any k large enough, it is clear that there exists $n_k \leq m_k \leq n_{k+1}$, with $l_{m_k}, l_{m_k+2} \leq l_{m_k+1}$. Consequently, part ($b$) of Theorem 5.17 gives that Ω is not hyperbolic. □

Sometimes it is convenient to split a train into "blocks" and to study locally the hyperbolicity in each of them. As we will see later, a valuable property of a block is that it is somehow "narrow".

Definition 5.20 Given a train Ω and a subsequence $\{n_k\}_k$, we denote by C_{n_k} the set $C_{n_k} := \cup_{m=n_k}^{n_{k+1}-1} Y_m$. We say that C_{n_k} is c_2*-narrow* if $d_\Omega(z, \mathbb{R} \cup \gamma_{n_k} \cup \gamma_{n_{k+1}}) = d_\Omega(z, (\mathbb{R} \cap C_{n_k}) \cup \gamma_{n_k} \cup \gamma_{n_{k+1}}) \leq c_2$ for every $z \in \cup_{m=n_k+1}^{n_{k+1}-1} \gamma_m$.

Next we study the case when (5.4) is only required for a subsequence.

Theorem 5.21 *Let us consider a train Ω with $\lim_{n\to\infty} l_n = \infty$, and a subsequence $\{n_k\}_k$.*

(a) *Let us assume that $r_n \leq c_1 + |l_n - l_{n+1}|$ and $l_n \geq l_0 > 0$ for every n, $l_{n_1} \leq l^0$, and $r_{n_k} \leq c_1$, $l_{n_k+1} + c_1 \geq l_{n_{k+1}}$ for every k. If C_{n_k} is c_2-narrow and*

$$\sum_{n=n_k+1}^{\infty} \mathrm{e}^{-l_n} \leq c_2\, \mathrm{e}^{-l_{n_k+1}}, \qquad \text{for every } k, \tag{5.5}$$

then Ω is δ-hyperbolic, where δ is a constant which only depends on c_1, c_2, l_0 and l^0.

(b) *Let us assume $l_n + c_3 \geq l_{n_k}$ for every k and every $n \geq n_k$. If Ω is hyperbolic, then there exists a constant c_4 such that C_{n_k} is c_4-narrow and*

$$\sum_{n=n_k}^{\infty} \mathrm{e}^{-l_n} \leq c_4\, \mathrm{e}^{-l_{n_k}}, \qquad \text{for every } k. \tag{5.6}$$

Remarks

1. A natural choice for $\{n_k\}_k$ is the set of indices corresponding to the largest non-decreasing subsequence of $\{l_n\}_n$. Observe that condition $l_{n_k+1}+c_1 \geq l_{n_{k+1}}$ is natural in this context: if $l_{n_k+1}+c_1 < l_{n_{k+1}}$ for some k, then n_k+1 must belong to $\{n_k\}_k$.
2. Condition $l_n \geq l_0 > 0$ in (a) is not restrictive at all since we have $\lim_{n\to\infty} l_n = \infty$.

Proof In order to prove (a), let us consider $z \in \cup_n \gamma_n$. If $z \in \gamma_{n_k}$ for some k, then Lemma 5.10 gives that there exists a constant c_5, which only depends on c_1, c_2, l_0 and l^0, such that $d_\Omega(z,\mathbb{R}) \leq c_5$. If $z \in \gamma_n$, with $n \notin \{n_k\}_k$, then $d_\Omega(z,\mathbb{R}) \leq c_2 + c_5$, since C_{n_k} is c_2-narrow for every k. Therefore, Theorem 5.3 gives the result.

If Ω is hyperbolic, then Theorem 5.3 gives that there exists a constant c_6 such that $d_\Omega(z,\mathbb{R}) \leq c_6$ for every $z \in \cup_n \gamma_n$. Hence C_{n_k} is c_6-narrow for every k. Besides, Lemma 5.10 implies (5.6). □

In order to obtain Lemma 5.23, which gives a criteria which assure that C_{n_k} is c_1-narrow for every k, we need the following definition.

Definition 5.22 Given a subsequence $\{n_k\}_k$ in a train Ω, we say that C_{n_k} is *c-admissible* if there exist $n_k \leq n_k^1 \leq n_k^2 \leq n_k^3 \leq n_k^4 \leq n_{k+1}$ verifying $n_k^1 - n_k \leq c$, $n_k^3 - n_k^2 \leq c$, $n_{k+1} - n_k^4 \leq c$,

$$\sum_{k=n}^{n_k^2} e^{-l_k} \leq c\, e^{-l_n}, \qquad \text{for every } n_k^1 < n \leq n_k^2,$$
$$\sum_{k=n_k^3}^{n} e^{-l_k} \leq c\, e^{-l_n}, \qquad \text{for every } n_k^3 \leq n < n_k^4. \tag{5.7}$$

Observe that n_k^j and n_k^{j+1} might coincide for some (or every) j.

Lemma 5.23 *Let us consider a train Ω and a subsequence $\{n_k\}_k$. Let us assume that, for some k, $r_n \leq c_1$ for every $n_k \leq n < n_{k+1}$ and C_{n_k} is c_2-admissible. Then there exists a constant c_3, which only depends on c_1 and c_2, such that C_{n_k} is c_3-narrow.*

Proof Applying Lemma 5.9 at most c_2-times, we obtain that there exists a constant c_4, which only depends on c_1, such that $d_\Omega(z,\mathbb{R}\cup\gamma_{n_k}) \leq c_2c_4$ for every $z \in \cup_{n=n_k}^{n_k^1}\gamma_n$, and (by symmetry) $d_\Omega(z,\mathbb{R}\cup\gamma_{n_{k+1}}) \leq c_2c_4$ for every $z \in \cup_{n=n_k^4}^{n_{k+1}}\gamma_n$. We also have that $d_\Omega(z,\mathbb{R}\cup\gamma_{n_k^2}) \leq c_2c_4$ for every $z \in \cup_{n=n_k^2}^{n_k^3}\gamma_n$.

By Corollary 5.11, there exists a constant c_5, which only depends on c_1 and c_2, such that $d_\Omega(z,\mathbb{R}\cup\gamma_{n_k^1}) \leq c_5$ for every $z \in \cup_{n=n_k^1}^{n_k^2}\gamma_n$, and (by symmetry) $d_\Omega(z,\mathbb{R}\cup\gamma_{n_k^4}) \leq c_5$ for every $z \in \cup_{n=n_k^3}^{n_k^4}\gamma_n$.

Hence, $d_\Omega(z,\mathbb{R}\cup\gamma_{n_k}\cup\gamma_{n_{k+1}}) \leq c_3 := 2c_2c_4 + c_5$ for every $z \in \cup_{n=n_k}^{n_{k+1}}\gamma_n$, and C_{n_k} is c_3-narrow. □

The following result is a direct consequence of Theorem 5.21 and Lemma 5.23.

Theorem 5.24 *Let us consider a train Ω and a subsequence $\{n_k\}_k$. Let us assume that $l_n \geq l_0$ and $r_n \leq c_1$ for every n, $l_{n_1} \leq l^0$, $l_{n_k+1} + c_1 \geq l_{n_{k+1}}$ and C_{n_k} is c_2-admissible for every k, and*

$$\sum_{n=n_k+1}^{\infty} e^{-l_n} \leq c_2\, e^{-l_{n_k+1}}, \qquad \text{for every } k.$$

Then Ω is δ-hyperbolic, where δ is a constant which only depends on c_1, c_2, l_0 and l^0.

The hypotheses in Theorem 5.24 imply $\lim_{n\to\infty} l_n = \infty$. The ideas developed so far do allow us to deal now with results involving trains which do not hold condition $\lim_{n\to\infty} l_n = \infty$.

The first result uses the hypothesis $l_n \leq c$; it is a direct consequence of Theorem 5.3 (let us observe that in this result there is no condition on $\{r_n\}_n$).

Theorem 5.25 *Let us consider a train Ω, with $l_n \leq c$ for every n. Then Ω is δ-hyperbolic, with δ a constant which only depends on c.*

Proof Fix n and $z \in \gamma_n$. We have $d_\Omega(z,\mathbb{R}) \leq d_\Omega(z,(a_0,b_0) \cup (a_n,b_n)) \leq c/2$. Hence, Theorem 5.3 gives the result. □

The same argument proves the following result, in which only a subsequence of $\{l_n\}$ is required to be bounded.

Theorem 5.26 *Let us consider a train Ω and a subsequence $\{n_k\}_k$. Let us assume that $r_n \leq c_1$ for every n and $l_{n_k} \leq c_1$ for every k. Then Ω is δ-hyperbolic if and only if C_{n_k} is c_2-narrow for every k. Furthermore, if C_{n_k} is c_2-narrow, then δ is a constant which only depends on c_1 and c_2.*

Theorem 5.26 and Lemma 5.23 allows to deduce the following.

Theorem 5.27 *Let us consider a train Ω and a subsequence $\{n_k\}_k$. Let us assume that $r_n \leq c$ for every n, and $l_{n_k} \leq c$ and C_{n_k} is c-admissible for every k. Then Ω is δ-hyperbolic, with δ a constant which only depends on c.*

As a particular case, we obtain the next corollary.

Corollary 5.28 *Let us consider a train Ω and a subsequence $\{n_k\}_k$. Let us assume that $r_n \leq c$ for every n, $l_{n_k} \leq c$ and $n_{k+1} - n_k \leq c$ for every k. Then Ω is δ-hyperbolic, with δ a constant which only depends on c.*

As we mentioned before, our results about trains may be somehow extended to a more general kind of spaces: generalized trains. From this point of view, Theorem 5.30 is the version for generalized trains of Theorem 5.3 for trains. This theorem together with Theorem 5.31 (applied to each $\{l_n^k\}_n$) provide criteria in order to decide about the hyperbolicity of generalized trains.

Definition 5.29 A *generalized train* is a Denjoy domain $\Omega \subset \mathbb{C}$ with $\Omega \cap \mathbb{R} = \cup_k \cup_{n=0}^{\infty} (a_n^k, b_n^k)$, such that $\sup_n b_n^k \leq \inf_n a_n^{k+1}$ for every k or $\sup_n b_n^{k+1} \leq \inf_n a_n^k$ for every k, and for each k we have either $b_n^k \leq a_{n+1}^k$ for every n or $b_{n+1}^k \leq a_n^k$ for every n.

We denote by γ_n^k the simple closed geodesic which just intersects $\mathbb{R}$ in (a_0^k, b_0^k) and (a_n^k, b_n^k). A generalized train is called *c-controlled* if $d_\Omega(z,\mathbb{R}) \leq c$, for every $z \in \cup_{n,k}\gamma_n^k$.

Remark The index k belongs either to $\mathbb{N}$ or to a finite set.

Theorem 5.30 *Let Ω be a generalized train. Then Ω is δ-hyperbolic if and only if there exists a constant c such that Ω is c-controlled, and for every $j \neq k$ there exist some n, m and geodesics g_{jk} joining (a_n^j, b_n^j) and (a_m^k, b_m^k) such that $d_\Omega(z, \mathbb{R}) \leq c$, for every $z \in \cup_{j\neq k} g_{jk}$.*

Furthermore, if Ω is c-controlled and if $d_\Omega(z, \mathbb{R}) \leq c$ for every $z \in \cup_{j\neq k} g_{jk}$, then δ is a constant which only depends on c. If Ω is δ-hyperbolic, then c is a constant which only depends on δ.

Proof If Ω is δ-hyperbolic, then Theorem 5.1 gives directly this implication.

In order to see the other implication, let us consider a geodesic α joining whatever two intervals (a_r^j, b_r^j), $(a_s^k, b_s^k) \subset \Omega$. By Theorem 5.1, it is sufficient to prove that there exists a constant c_1, which only depends on c, such that $d_\Omega(z, \mathbb{R}) \leq c_1$ for every $z \in \alpha$. We can assume that $j \neq k$, since the case $j = k$ is easier.

Let us consider $z \in \alpha$. By symmetry, we can assume that $z \in \alpha^+$. By hypothesis, there exist a geodesic g_{jk} joining (a_n^j, b_n^j) and (a_m^k, b_m^k) for some n, m, such that $d_\Omega(w, \mathbb{R}) \leq c$ for every $w \in g_{jk}$. Without loss of generality we can assume also that $g_{jk} \subset \Omega^+$. Now, we are going to consider the geodesics $\alpha_{rn} \subset \Omega^+$ which join (a_r^j, b_r^j) with (a_n^j, b_n^j) and $\alpha_{sm} \subset \Omega^+$ joining (a_s^k, b_s^k) with (a_m^k, b_m^k). Corollary 5.4 gives that $d_\Omega(w, \mathbb{R}) \leq c + 4\log(1 + \sqrt{2})$ for every $w \in \alpha_{rn} \cup \alpha_{sm}$.

Let us define the following geodesics: $\beta_r^j \subset (a_r^j, b_r^j)$ joining the end points of α and α_{rn} which belong to (a_r^j, b_r^j), $\beta_n^j \subset (a_n^j, b_n^j)$ joining the end points of α_{rn} and g_{jk} which belong to (a_n^j, b_n^j), $\beta_s^k \subset (a_s^k, b_s^k)$ joining the end points of α and α_{sm} which belong to (a_s^k, b_s^k) and $\beta_m^k \subset (a_m^k, b_m^k)$ joining the end points of α_{sm} and g_{jk} which belong to (a_m^k, b_m^k).

So, we have obtained a geodesic polygon $Q \subset \Omega^+$ with at most eight sides; Q is $6\log(1 + \sqrt{2})$-thin, since Ω^+ is isometric to a geodesically convex subset of the unit disk. Let us observe that the geodesic α is one of the sides of Q. Let us denote by A the union of the other sides of the polygon. Then, there exists $w \in A$ with $d_\Omega(z, w) \leq 6\log(1 + \sqrt{2})$, and consequently $d_\Omega(z, \mathbb{R}) \leq d_\Omega(z, w) + d_\Omega(w, \mathbb{R}) \leq c_1 := 10\log(1 + \sqrt{2}) + c$. □

We obtain directly the following result.

Theorem 5.31 *Let Ω be a generalized train with k belonging to a finite set. Then Ω is δ-hyperbolic if and only if Ω is c-controlled.*

Proof If Ω is δ-hyperbolic, then Ω is c-controlled, by Theorem 5.30.

Let us assume now that Ω is c-controlled. For each $j \neq k$, choose geodesics g_{jk} joining (a_0^j, b_0^j) and (a_0^k, b_0^k). Then $d_\Omega(z, \mathbb{R}) \leq \max_{j\neq k} L_\Omega(g_{jk})/2$, for every $z \in \cup_{j\neq k} g_{jk}$, and Theorem 5.30 implies the result. □

Finally, a result which shows that hyperbolicity is stable under bounded perturbations of the lengths of the fundamental geodesics. Theorem 5.33 is particularly interesting since there are very few results on hyperbolic stability which do not involve quasi-isometries. We start with a technical lemma.

Lemma 5.32 *Let us consider two trains* Ω *and* Ω' *with* $r_n = r'_n \le c_1$ *for every* n, $l'_n = l_n + l_0$ *if* $l_n < l_0$ *and* $l'_n = l_n$ *if* $l_n \ge l_0$. *Then* Ω *is hyperbolic if and only if* Ω' *is hyperbolic.*

Furthermore, if Ω *is* δ*-hyperbolic, then* Ω' *is* δ'*-hyperbolic, with* δ' *a constant which only depends on* δ, c_1 *and* l_0*; if* Ω' *is* δ'*-hyperbolic, then* Ω *is* δ*-hyperbolic, with* δ *a constant which only depends on* δ', c_1 *and* l_0.

Remark l'_n and r'_n denote the lengths of the fundamental geodesics in Ω'.

Proof To start with, let us suppose that Ω is δ-hyperbolic and let us prove that Ω' is δ'-hyperbolic.

Let us choose $z' \in \gamma'_r \subset \Omega'$, for some r. By symmetry, without loss of generality we can assume that $z' \in (\gamma'_r)^+$. Now, let us take $z \in \gamma_r^+ \subset \Omega$ with $h(z) = h(z')$. (Notice that if there not exists such z, it is because $d_{\Omega'}(z', \mathbb{R}) \le l_0$.)

Since Ω is δ-hyperbolic, by Theorem 5.3, there exists a constant c_2, which only depends on δ, such that $d_\Omega(z,\mathbb{R}) = d_\Omega(z,p(z)) \le c_2$.

There are two possibilities:

(1) If $p(z) \in \gamma_r$, then there exists $z^* \in \gamma'_r \cap \mathbb{R}$ with $d_{\Omega'}(z',\mathbb{R}) \le d_{\Omega'}(z',z^*) \le d_\Omega(z,p(z)) + l_0 \le c_2 + l_0$.

(2) If $p(z) \notin \gamma_r$, we distinguish two cases.
If $l_r < l_0$, then $l'_r = l_r + l_0 < 2l_0$ and $d_{\Omega'}(z',\mathbb{R}) < l_0$.
If $l_r \ge l_0$, let us denote by g the geodesic joining z and $p(z)$ such that $d_\Omega(z,\mathbb{R}) = L_\Omega(g)$. Let us assume that $p(z) \in \cup_{n=1}^{r-1}(a_n,b_n)$ (if $p(z) \in \cup_{n=r+1}^{\infty}(a_n,b_n)$ the argument is symmetric). If $p(z) \in \gamma_s$ and $l_n \ge l_0$ for every $s \le n \le r$, then $l'_n = l_n$ for every $s \le n \le r$ and $\cup_{n=s}^{r-1} Y_n$ is isometric to $\cup_{n=s}^{r-1} Y'_n$, and then $d_{\Omega'}(z',\mathbb{R}) \le d_\Omega(z,p(z)) \le c_2$. If $p(z) \in \gamma_s$ and $l_n < l_0$ for some $s \le n \le r$, let us define $m := \max\{n < r : l_n < l_0\}$, $x := g \cap \gamma_{m+1}$ and $d := d_\Omega(z,x) \le c_2$. Now, let us choose $x' \in \gamma'_{m+1}$ such that $h(x) = h(x')$ and let us call $d' := d_{\Omega'}(z',x')$. Notice that $d = d' \le c_2$, since $\cup_{n=m+1}^{r-1} Y_n$ is isometric to $\cup_{n=m+1}^{r-1} Y'_n$. Observe that the geodesic hexagon H'_m is $4\log(1+\sqrt{2})$-thin, and therefore, $d_{\Omega'}(x',\mathbb{R}) \le 4\log(1+\sqrt{2}) + c_1/2 + l_0$ (recall that $r'_m \le c_1$ and $l'_m < 2l_0$).
It means that

$$d_{\Omega'}(z',\mathbb{R}) \le d_{\Omega'}(z',x') + d_{\Omega'}(x',\mathbb{R}) \le c_2 + 4\log(1+\sqrt{2}) + c_1/2 + l_0.$$

Consequently, Ω' is δ'-hyperbolic, with δ' a constant which only depends on δ, c_1 and l_0, by Theorem 5.3.

In order to prove that Ω' δ'-hyperbolic implies Ω δ-hyperbolic, we can follow a similar argument. □

Theorem 5.33 *Let us consider two trains* Ω, Ω' *and two constants* c_1, c_2 *such that* $r_n, r'_n \le c_1$, *and* $|l'_n - l_n| \le c_2$. *Then* Ω *is hyperbolic if and only if* Ω' *is hyperbolic.*

Furthermore, if Ω *is* δ*-hyperbolic, then* Ω' *is* δ'*-hyperbolic, with* δ' *a constant which only depends on* δ, c_1 *and* c_2.

Remark Observe that in many cases Ω and Ω' are not quasi-isometric (for example, if there exists a subsequence $\{n_k\}_k$ with $\lim_{k\to\infty} l_{n_k} = 0$ and $l'_{n_k} \ge c > 0$).

Proof By symmetry, it is sufficient to prove that if Ω is δ-hyperbolic, then Ω' is δ'-hyperbolic, with δ' a constant which only depends on δ, c_1 and c_2. Therefore, let us assume that Ω is δ-hyperbolic.

By Lemma 5.32 we can assume that $l'_n, l_n \geq 1$ for every n.

Given any point $z' \in \gamma'_k$, by Theorem 5.3 it is sufficient to prove that there exists a constant c_3, which only depends on δ, c_1 and c_2, such that $d_{\Omega'}(z', \mathbb{R}) \leq c_3$.

By symmetry, without loss of generality we can assume that $z' \in (\gamma'_k)^+$. Now, let us take $z \in \gamma_k^+ \subset \Omega$ with $h(z) = h(z')$. (Notice that if there not exists such z, it is because $d_{\Omega'}(z', \mathbb{R}) \leq c_2$.) Since Ω is δ-hyperbolic, then $d_\Omega(z, \mathbb{R}) \leq c_4$, for some c_4, which only depends on δ, by Theorem 5.3.

There are two possibilities:

If $p(z) \in \gamma_k$, then there exists $z^* \in \gamma'_k \cap \mathbb{R}$ with $d_{\Omega'}(z', \mathbb{R}) \leq d_{\Omega'}(z', z^*) \leq d_\Omega(z, p(z)) + c_2 \leq c_2 + c_4$.

If $p(z) \notin \gamma_k$, then $p(z) \in (a_m, b_m)$, with $m \neq 0, k$. By symmetry we can assume that $0 < m < k$. Let us denote by g the geodesic joining z and $p(z)$ such that $d_\Omega(z, \mathbb{R}) = d_\Omega(z, (a_m, b_m)) = L_\Omega(g)$. Let us denote by x the point $x := g \cap \gamma_{m+1}$; we have $d := d_\Omega(z, x) \leq c_4$ since $d_\Omega(z, \mathbb{R}) = d_\Omega(z, x) + d_\Omega(x, \mathbb{R}) \leq c_4$.

We take $x' \in (\gamma'_{m+1})^+$ with $h(x') = \min\{h(x), l'_{m+1}\}$. By the triangle inequality, $d_{\Omega'}(z', \mathbb{R}) \leq d_{\Omega'}(z', x') + d_{\Omega'}(x', \mathbb{R})$. Now, let us try to get an upper bound for $d' := d_{\Omega'}(z', x')$.

Since $l'_n, l_n \geq 1$ for every n, by Lemma 5.5 we know that there exists a constant c_5, which only depends on c_1, such that for any n,

$$\begin{aligned} e^{-l_n} + e^{-l_{n+1}} &\leq \alpha_n \leq c_5\,(e^{-l_n} + e^{-l_{n+1}}), \\ e^{-l'_n} + e^{-l'_{n+1}} &\leq \alpha'_n \leq c_5\,(e^{-l'_n} + e^{-l'_{n+1}}). \end{aligned}$$

In order to simplify the notation we are going to define B and B' as

$$B := \sum_{n=m+1}^{k-1} (e^{-l_n} + e^{-l_{n+1}}), \qquad B' := \sum_{n=m+1}^{k-1} (e^{-l'_n} + e^{-l'_{n+1}}).$$

It is clear that $e^{-c_2} \leq B/B' \leq e^{c_2}$. By hyperbolic trigonometry,

$$\begin{aligned} \cosh d &= \cosh\left(\sum_{n=m+1}^{k-1} \alpha_n\right) \cosh h(z) \cosh h(x) - \sinh h(z) \sinh h(x) \\ &\geq \cosh B \cosh h(z) \cosh h(x) - \sinh h(z) \sinh h(x) \\ &\geq (\cosh B - 1) \cosh h(z) \cosh h(x). \end{aligned}$$

Let us assume that $h(x') = h(x)$; then we obtain

$$\begin{aligned} \cosh d' &= \cosh\left(\sum_{n=m+1}^{k-1} \alpha'_n\right) \cosh h(z) \cosh h(x) - \sinh h(z) \sinh h(x) \\ &\leq \cosh\left(c_5 B'\right) \cosh h(z) \cosh h(x) - \sinh h(z) \sinh h(x) \end{aligned}$$

$$\le \cosh\left(c_5 e^{c_2} B\right) \cosh h(z) \cosh h(x) - \sinh h(z) \sinh h(x)$$
$$= \frac{\cosh\left(c_5 e^{c_2} B\right) - 1}{\cosh B - 1} (\cosh B - 1) \cosh h(z) \cosh h(x) + \cosh h(z) \cosh h(x)$$
$$- \sinh h(z) \sinh h(x)$$
$$\le \frac{\cosh\left(c_5 e^{c_2} B\right) - 1}{\cosh B - 1} \cosh d + \cosh\left(h(z) - h(x)\right).$$

It is clear that $B \le \sum_{n=m+1}^{k-1} \alpha_n = d_\Omega(z_0, x_0) \le d_\Omega(z, x) \le c_4$. Since $c_5 e^{c_2} > 1$, the function $\frac{\cosh(c_5 e^{c_2} B) - 1}{\cosh B - 1}$ is increasing in B; hence,

$$\frac{\cosh\left(c_5 e^{c_2} B\right) - 1}{\cosh B - 1} \le \frac{\cosh\left(c_5 e^{c_2} c_4\right) - 1}{\cosh c_4 - 1}.$$

Besides, $|h(z) - h(x)| \le d_\Omega(z, x) \le c_4$ by Lemma 5.7. Then

$$d' \le \operatorname{Arccosh}\left(\frac{\cosh\left(c_5 e^{c_2} B\right) - 1}{\cosh B - 1} \cosh c_4 + \cosh c_4\right).$$

If $h(x') < h(x)$, then $h(x') = l'_{m+1}$ and $h(x) - h(x') = h(x) - l'_{m+1} \le l_{m+1} - l'_{m+1} \le c_2$. Hence, $|h(z) - h(x')| \le |h(z) - h(x)| + h(x) - h(x') \le c_4 + c_2$. With the same computations we obtain

$$\cosh d' \le \frac{\cosh\left(c_5 e^{c_2} B\right) - 1}{\cosh B - 1} \cosh d + \cosh\left(h(z) - h(x')\right),$$
$$d' \le c_6 := \operatorname{Arccosh}\left(\frac{\cosh\left(c_5 e^{c_2} B\right) - 1}{\cosh B - 1} \cosh c_4 + \cosh(c_2 + c_4)\right).$$

Now we consider $d_{\Omega'}(x', \mathbb{R})$ (recall that $x' \in \gamma'_{m+1}$).

We can assume that $h(x') = h(x)$, since if $h(x') < h(x)$, then $x' \in \mathbb{R}$ and $d_{\Omega'}(z', \mathbb{R}) \le d_{\Omega'}(z', x') \le c_6$.

There are two possibilities.

If $h(x') \ge l'_m$, then by Lemma 5.8 there exists a constant c_7, which only depends on c_1, such that $d_{\Omega'}(x', \mathbb{R}) \le c_7$.

If $h(x') < l'_m$, then $h(x) = h(x') < l_m + c_2$. If $h(x) \ge l_m$, then $h(x') \ge l'_m - c_2$; so $d_{\Omega'}(x', \mathbb{R}) \le c_2 + c_7$. If $h(x) < l_m$, we have that $l_m - h(x) \le h(p(z)) - h(x) \le d_\Omega(x, \mathbb{R}) \le c_4$ and it is easy to check that $l'_m - h(x') \le l_m + c_2 - h(x) \le c_2 + c_4$; so $d_{\Omega'}(x', \mathbb{R}) \le c_2 + c_4 + c_7$.

Therefore $d_{\Omega'}(z', \mathbb{R}) \le d_{\Omega'}(z', x') + d_{\Omega'}(x', \mathbb{R}) \le c_3 := c_6 + c_2 + c_4 + c_7$.

Consequently Ω' is δ'-hyperbolic with δ' a constant which only depends on δ, c_1 and c_2. □

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