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Higher Dimensional Thompson Groups

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1. Introduction

Three groups $F \subseteq T \subseteq V$, known as Thompson groups, have generated interest since R. J. Thompson introduced them in the late 1960s. Part of their initial interest was the fact that T and V supplied the first known examples of infinite, simple, finitely presented groups. Since then, other properties of the groups have been studied as well as their interaction with other areas of mathematics. The standard reference for the groups $F \subseteq T \subseteq V$ is [1]. It is not necessary to have [1] in hand while reading this paper, but it would not hurt.

The largest group V can be described as a subgroup of the homeomorphism group of the Cantor set C. Intrinsic to this description is the standard 'deleted middle thirds' construction of the Cantor set as a subset of the unit interval. Since the unit interval is a one-dimensional object, we will refer to V as a one-dimensional Thompson group.

In this paper, we describe an intrinsically two-dimensional group $2V$ that is more naturally described as a subgroup of the homeomorphism group of $C \times C$. We will show in this paper that $2V$ is infinite, simple and finitely generated, and in another paper [2], that $2V$ is finitely presented.

In spite of the fact that $C \times C$ is homeomorphic to C, we will also show in this paper that $2V$ is not isomorphic to V. In fact, we will show that $2V$ is not isomorphic to any group in a list of other infinite, simple, finitely presented groups that are closely related to T and V. However, we do not show that $2V$ is not isomorphic to all known infinite, simple, finitely presented groups.

The group $2V$ is a member of a family of groups nV of which $1V = V$. We describe these groups for $n > 2$, but prove little about them. They are all infinite and it is a reasonable guess that they are also simple and finitely presented. Part of the charm of the family of groups nV is the uncanny resemblance they have to the little cube operads of [3, 4]. The importance of this resemblance is speculative.

It would be nice to claim that $2V$ is a *new* infinite, simple, finitely presented group. As the list of such groups grows longer, such claims become less interesting and harder to support. The difficulty of establishing such a claim is increased by the lack of an official list. So we make no such claim and introduce the groups here for their pleasant properties and their potential applicability to other areas of mathematics. See the discussion in Sections 9.2 and 10.1 for references to all infinite, simple, finitely presented groups that the author is aware of.

The proof that $2V$ is not isomorphic to V and other simple groups is not algebraic. We invoke a difficult theorem of M. Rubin that gives conditions under which isomorphism implies topological conjugacy. Then we demonstrate irreconcilable differences between the dynamics of V and $2V$: there is an element of $2V$ exhibiting 'chaotic' behavior and there is no such element of V. Thus part of this paper studies properties of the older group V. Dynamical properties of Thompson groups have been studied before in [5, 6].

The chaotic element in $2V$ is a well-known map often referred to as the baker's map. Some aspects of the baker's map are discussed briefly in the last two sections of the paper. In particular we show that the baker's map is a product of commutators of some rather easily described homeomorphisms of the Cantor set.

The paper [1] (in Sections 1 and 2 and the first few pages of 5 and 6) covers the representation of the groups F, T and V as homeomorphism groups and also the encoding of the elements of the groups by pairs of trees. We will need the pairs of trees point of view in analyzing the dynamics of elements of V.

2. Defining $2V$

We will define the group $2V$ as a subgroup of the group $H(C^2)$ of self-homeomorphisms of $C^2 = C \times C$. We introduce some conventions that we will use.

In this paper, homeomorphisms will act on the left and will be composed from right to left. The *support* of an element h of $2V$, denoted $\mathrm{Supp}(h)$, will be the closure of $\{x \in C \times C \mid x \neq h(x)\}$. We will use h^g to denote ghg^{-1} and from this, we will have $\mathrm{Supp}(h^g) = g(\mathrm{Supp}(h))$. We will use $[h, g]$ to denote the commutator $hgh^{-1}g^{-1} = h(h^{-1})^g$.

We will think of the Cantor set C as the usual deleted middle thirds subset of the unit interval $I = [0, 1]$ and we will identify elements of C with infinite words in the alphabet $\{0, 1\}$. These words are just functions from the natural numbers $\mathbf{N} = \{0, 1, 2, \ldots\}$ to $\{0, 1\}$. The identification of these words with elements of C takes words beginning with 0 to points in $[0, \frac{1}{3}]$, words beginning with 1 to points in $[\frac{2}{3}, 1]$, words beginning with 00 to points in $[0, \frac{1}{9}]$ and so forth. If w is an infinite word in $\{0, 1\}$, then doubling all the 'digits' in w gives the ternary expansion as a real number of the image of w in C.

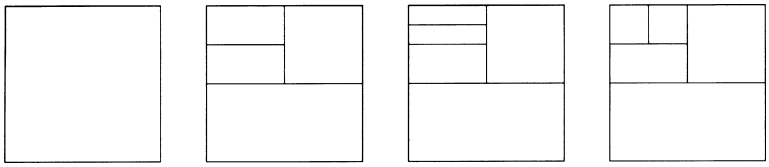
We view $C \times C$ as a subset of the unit square $I^2 = [0, 1] \times [0, 1]$. We next develop methods of describing individual elements of $2V$.

2.1 PATTERNS OF RECTANGLES

We will describe certain partitions of $C \times C$ by patterns of rectangles in the unit square. First we describe the patterns, and then we say what partitions of $C \times C$ the patterns are describing.

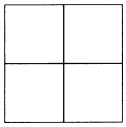
We inductively define what we call *patterns* in I^2. Each pattern is a finite set of rectangles in I^2 with pairwise disjoint, nonempty interiors, with sides parallel to the sides of I^2 and whose union is all of I^2. The first pattern is I^2 itself and will be called the *trivial pattern*. If P is a pattern and R is one of the rectangles in P, then we can create a new pattern by removing R from P and replacing it with two congruent rectangles R' and R'' obtained from R by dividing R exactly in half by either a horizontal line, or a vertical line. A pattern is one that can be obtained from the trivial pattern in this way in a finite number of steps.

Below we show four patterns. Pattern (a) is the trivial pattern and patterns (c) and (d) are obtained from pattern (b) by the two possible ways of dividing the upper left rectangle in (b).

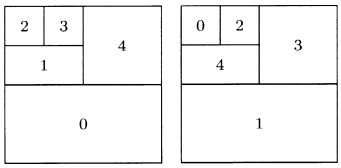


(a) (b) (c) (d)

We note that different sequences of modifications to the trivial pattern can result in the same pattern. Several of the fundamental relations that we will work with come from the several sequences of operations that yield the pattern below when starting from the trivial pattern.



We next number our patterns. A *numbered pattern* is a pattern with a one-to-one correspondence between $\{0, 1, \ldots, n-1\}$ and the rectangles in the pattern where n is the number of rectangles in the pattern. Below we show two different numbered patterns based on the same pattern.



2.2. PARTITIONS FROM PATTERNS

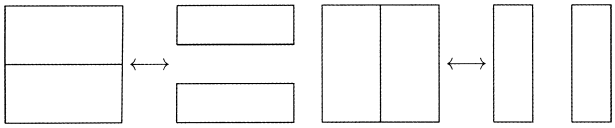
Each rectangle in a pattern corresponds to a closed and open subset of $C \times C$. We describe this correspondence inductively.

The rectangle I^2 corresponds to all of $C \times C$.

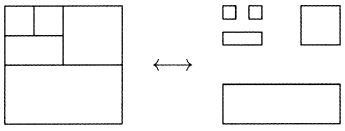
Let a rectangle R in a pattern correspond to the subset $A(R)$ in $C \times C$. Let R' and R'' be obtained from R by dividing R equally by a vertical line, with R' the left rectangle and R'' the right. Then R' corresponds to the left third of $A(R)$ and R'' corresponds to the right third of $A(R)$.

Similarly, if R' and R'' be obtained from R by dividing R equally by a horizontal line, with R' the bottom rectangle and R'' the top. Then R' corresponds to the bottom third of $A(R)$ and R'' corresponds to the top third of $A(R)$.

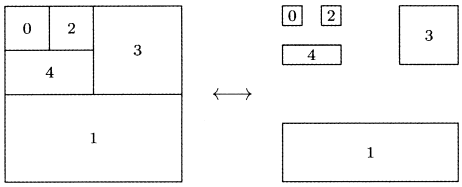
The next figures give the basic correspondences for the patterns obtainable from the trivial pattern I^2 using one division.



Thus it is seen that a pattern is just a lazy way of drawing a particular partition of $C \times C$ into closed and open sets in $C \times C$. Each rectangle in the pattern gives one set in the partition. An example is pictured below.



Each numbered pattern corresponds to a numbered partition of $C \times C$ in the obvious way. To make sure that we agree on this, we give the above example a numbering.



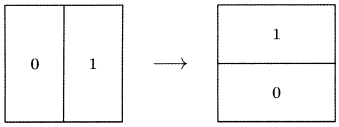
2.3. HOMEOMORPHISMS FROM NUMBERED PATTERN PAIRS

Let P_1 and P_2 be numbered patterns with the same number n of rectangles in each. Let the rectangles in P_1 be $R_0, R_1, \ldots, R_{n-1}$, where the subscripts reflect the numbering, and let the rectangles in P_2 be $T_0, T_1, \ldots, T_{n-1}$ with the same comment. We will get a self-homeomorphism $h = h(P_1, P_2)$ of $C \times C$ from the pair (P_1, P_2).

We let $A(R_i)$ and $A(T_i)$ be the closed and open sets in $C \times C$ associated, respectively, to R_i and T_i, and we let h take $A(R_i)$ onto $A(T_i)$ affinely so as to preserve the

orientation in each coordinate. By this we mean that the restriction of h to $A(R_i)$ is the restriction of the unique self homeomorphism of $\mathbf{R}^2$ of the form $(x, y) \mapsto (a + 3^j x, b + 3^k y)$ that maps $A(R_i)$ onto $A(T_i)$. Thus, the four corners of $A(R_i)$, the lower left, lower right, upper right, and upper left, are each carried to the corner of $A(T_i)$ with the same description. Doing this for each i in $\{0, 1, \ldots, n-1\}$ defines a homeomorphism $h(P_1, P_2)$ from $C \times C$ to itself.

We picture such homeomorphisms by giving a pair of numbered patterns with an arrow from the numbered pattern describing the domain to the numbered pattern describing the range. The pair below represents an element called the 'baker's map'. This particular map will be discussed later in this paper.



2.4. THE GROUP $2V$

The group $2V$ is the set of all self homeomorphisms of $C \times C$ of the form $h(P_1, P_2)$, where P_1 and P_2 are numbered patterns with the same number of rectangles. The group operation is composition. Closure under inversion is immediate and closure under composition is a pleasant exercise. Those that wish to defer the exercise can wait until Section 5.2 where techniques for multiplying elements will be given.

The following is clear.

LEMMA 2.1. *The group $2V$ is countably infinite and contains all finite groups.*

3. Simplicity of the Commutator Subgroup

It is usually the case that a sufficiently transitive permutation group generated by elements 'of small support' has a simple commutator subgroup (see [7] or [8]). This section contains the arguments to show the simplicity of the commutator subgroup of $2V$. This section is self-contained since it is often just as easy to apply the techniques of [7] and [8] as it is to quote them.

Later in the paper, we will know that $2V$ equals its commutator subgroup. Thus, we will have proven that $2V$ is simple.

3.1. TRANSITIVITY

We establish that $2V$ is sufficiently 'transitive'.

LEMMA 3.1. *Let K be a closed, proper subset of $C \times C$ and let U be a non-empty open set in $C \times C$. Then there is an element h of $2V$ so that $h(K)$ is contained in U.*

Proof. Since K is closed and not all of $C \times C$, there is a pattern P_1 where not all the rectangles are needed to get a set S of rectangles whose corresponding closed and open sets in $C \times C$ cover K. There is a pattern P_2 with more than one rectangle and with a

rectangle R so that $A(R)$ is contained in U. If n is the number of rectangles in S, then we can apply $n-1$ subdivision operations at random to the rectangle R to create a new pattern P_2' containing n different rectangles that are contained in R whose corresponding sets in C are thus contained in U. Using further subdivisions if necessary in some rectangle not in S or in R, we can get the number of rectangles in P_1 and P_2' to be the same. Now an element of $2V$ can be built that carries the rectangles in S into R. □

3.2. SMALL GENERATORS

Our next step is to argue that $2V$ is generated by elements 'of small support'.

The word small is misleading. All that we will need from the notion of small is that a composition of a fixed number of elements of sufficiently small support yields an element whose support is a closed, proper subset of $C \times C$. Our notion of small will also cooperate equally well with C^n and $C^{\mathbf{N}}$.

We will measure size in one coordinate only. Since C has measure 0, we will not measure in C itself, but in I. We use the notion of patterns in I in a manner parallel to patterns in I^2. The unit interval $I = [0, 1]$ is the trivial pattern and all other patterns are derived from the trivial pattern by dividing intervals at their midpoints.

Given an $\epsilon > 0$, we say that a set in C is *of size no more than* ϵ if it is in a collection of disjoint closed and open sets in C corresponding to a set of intervals in a pattern on I whose lengths sum to no more than ϵ. Now we say that a set in $C \times C$ is *of size no more than* ϵ if the projection of the set to the first coordinate is of size no more than ϵ. Given $\epsilon > 0$, let B_ϵ be the set of those elements of $2V$ that have support of size no more than ϵ.

PROPOSITION 3.2. *Given* $\epsilon > 0$, *the group* $2V$ *is generated by* B_ϵ.

Proof. Let A_0 be the elements in $2V$ with support in the left half of $C \times C$, and let A_1 be the elements with support in the right half of $C \times C$. The elements of A_0 and A_1 have support of size no more than $\frac{1}{2}$.

Let h be in $2V$. We will show first that h is the composition of elements of $B_\epsilon \cup A_0 \cup A_1$. The claimed result will follow from this because an identical exercise will then show that elements in each A_i will be compositions of elements in $B_\epsilon \cup A_{i0} \cup A_{i1}$ where A_{i0} and A_{i1} are the sets of elements whose supports are contained in, respectively, the left and right halves of the half of $C \times C$ containing the support of the elements of A_i. Thus, elements of the A_{ij} have supports of size no more than $\frac{1}{4}$. Continuing in this way, we get h to be a composition of elements in B_ϵ.

We will alter h and then work with the alterations. To simplify the notation, we will use h to refer to the current alteration. The alterations will be to compose h with elements of A_0 and A_1, and the goal will be to reduce h to an element of B_ϵ.

We may assume that h is not trivial and thus $x \neq h(x)$ for some x. Let us assume for now that x is in the left half of $C \times C$. There are two disjoint closed and open sets E and F corresponding to rectangles in patterns with $x \in E$, $h(x) \in F$ so that h carries

E affinely onto F. We can choose E to be contained in the left half of $C \times C$. We do not care about the location of F.

At the expense of making the sets smaller, we can find a pattern so that both E and F correspond to rectangles in the pattern. This allows us to build an element h_1 that agrees with h on E, and that does nothing but interchange E and F so that $\mathrm{Supp}(h_1)$ is in $E \cup F$. Using the uniform continuity of h, we can choose E small enough so that h_1 is in B_ϵ. Because E and F are disjoint, we note that $h_1^{-1}h$ fixes the nonempty open set E and that $\mathrm{Supp}(h_1^{-1}h)$ is contained in $\mathrm{Supp}(h)$. We replace h by $h_1^{-1}h$.

Using Lemma 3.1, we can build an element g of A_0 that carries the subset E onto the complement in the left half of $C \times C$ of an open set U whose closure has size no more than $\epsilon/2$. Now $\mathrm{Supp}(h^g) = g(\mathrm{Supp}(h))$, so we know that the size of $\mathrm{Supp}(h^g)$ in the left half of $C \times C$ is of size no more than $\epsilon/2$. We replace h by h^g.

We now move our attention to the right half of $C \times C$. If there is some x there with $x \neq h(x)$, then we repeat the above operations except that the conjugating element g is from A_1. The result is an h whose support in the right half of $C \times C$ is of size no more than $\epsilon/2$. Of the two steps in the modification, the first (replacing h by $h_1^{-1}h$) does not increase support, and the second (conjugating by g) only modifies the support in the right half of $C \times C$. Thus, these modifications will not affect the fact that the support of the new h in the left half of $C \times C$ is of size no more than $\epsilon/2$. This reduces the support of h to have size no more than ϵ. □

3.3. SIMPLICITY OF THE COMMUTATOR SUBGROUP

The ideas in the next lemma are to be found in [7] and [8].

PROPOSITION 3.3. *The commutator subgroup of* $2V$ *is simple.*

Proof. Let W be the commutator subgroup of $2V$. Let j be a nontrivial element of W and let N be the normal closure of j in W. We must show that elements of $2V$ commute modulo N. It suffices to show that any two generators of $2V$ commute modulo N. Thus we let h and g be elements of $2V$ with very small supports.

There is some open and closed set E in the support of j so that E and $j(E)$ are disjoint. If some element k of W carries the union of the supports of g and h into E, then

$$\mathrm{Supp}(h^{(j^{k^{-1}})}) = k^{-1}jk(\mathrm{Supp}(h)) \subseteq k^{-1}j(E)$$

is disjoint from $\mathrm{Supp}(g) \subseteq k^{-1}(E)$, and g and h commute modulo N. Thus, we are done if we find such an element k.

Let X be the union of the supports of h and g. By Lemma 3.1, there is an element k' in $2V$ that carries X into E. Since this can be done by interchanging rectangles, we can keep the support of k' contained in a small neighborhood of $X \cup E$. Since this can be kept very small, there is an r in $2V$ that carries $X \cup E$ to a set disjoint from $X \cup E$. Thus, it is seen that the element $k = [k', r] = k'((k')^{-1})^r$ of W agrees with k' on X. This completes the proof. □

4. Groups Related to $2V$

4.1. THE GROUPS nV AND ωV

For each n there is a group nV that acts on the product C^n. There is also a group ωV that acts on the countably infinite product $C^{\mathbf{N}}$. It is the ascending union of the nV. Patterns for nV would consist of rectangular solids in I^n. These are easiest to code by n-tuples of finite (and possibly empty) words in the alphabet $\{0,1\}$. The n-tuple of empty words corresponds to I^n. If a rectangular solid is given by the n-tuple $(w_0, w_1, \ldots, w_{n-1})$, then a division into two solids by a cut perpendicular to coordinate axis i would result in the two solids given by replacing w_i by w_i0 or w_i1.

For ωV, solids in $I^{\mathbf{N}}$ would be given by infinite sequences of finite words in $\{0,1\}$ such that all but finitely many of the words in the sequence are the empty word. A division of a solid would be similar to a division in I^n.

The group $1V$ is the Thompson group V as described in [1]. We show below that V and $2V$ are not isomorphic. The groups $2V$ and ωV are not isomorphic since one is finitely generated and the other is not. It is not known whether any other pairs of the nV with $n > 1$ or ωV are isomorphic to each other, but it would be pleasant if the answer were no.

It is clear that some of the results of this paper and [2] should apply to the nV and ωV, but it is not clear how many. Lemma 2.1 clearly applies. The simplicity results of this paper do not depend on finding a presentation, but they do depend on finding enough relations to prove that the Abelianization is trivial. At this point, one can only say that it is believable that this can be done for the nV and ωV. It can be said that the results of Section 3 apply and that nV and ωV possess simple commutator subgroups. Whether a calculation of a full, finite presentation can be carried out for all these groups comes under other descriptions. For the nV with n finite, it is to be hoped for. The group ωV is another matter since it is not finitely generated. For ωV the hope would be to write down any reasonable presentation.

4.2. OTHER RELATED GROUPS

In [9], groups are defined that are called $G_{n,r}$ there, but which we will relable as $V_{k,r}$ here to conform to the letter V used in [1] and to avoid conflict with our use of n. Groups $nV_{k,r}$ can be defined to parallel the $V_{k,r}$ so that $1V_{k,r} = V_{k,r}$. The group $nV_{k,r}$ would consist of self-homeomorphisms of r copies of C^n. To obtain the defining patterns of these homeomorphisms, the basic inductive step would be to divide a rectangular solid into k congruent pieces by cuts perpendicular to a particular axis. We will not treat the $nV_{k,r}$ here. It should be noted, that not all the $V_{k,r}$ are simple, but in [9] it is shown they have infinte, simple, finitely presented commutator subgroups that are of index 1 when k is even and index 2 when $k > 2$ is odd.

4.3. THE CONTAINING GROUP $\widehat{2V}$ OF $2V$

The group $2V$ is contained in a 'larger' finitely presented group $\widehat{2V}$. The relation between $2V$ and $\widehat{2V}$ is similar to the relation between BV and $\widehat{BV}$ in [10]. In [10], the group $\widehat{BV}$ is analyzed first and this analysis is then used to analyze its subgroup BV. This outline will be partly followed here. We briefly discuss why and why only partly.

Once generators are found for $2V$, the word problem is theoretically solvable. The elements of $2V$ are specific permutations of $C \times C$ and a word in specific elements is trivial if and only if the resulting permutation is trivial. However, there is a large difference between theory and practice and we need a more practical way to identify the trivial element.

The group BV involves braids, and identifying the trivial element in BV is even less trivial. Triviality in BV is detected by its setting as a subgroup of $\widehat{BV}$ which has a known, finite presentation and a strong normal form (Lemma 10.3 of [10]). While $\widehat{2V}$ has a known, finite presentation, it does not have as nice a normal form. This is discussed in [2]. Thus we need another technique for identifying trivial elements. This is solved in [2] by finding a less nice normal form for elements of $\widehat{2V}$ and an algorithm for acheiving the form.

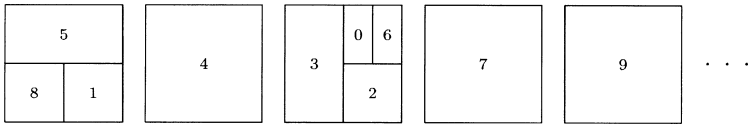
Even before we need to understand trivial elements, we need to be able to multiply easily in $2V$. It turns out that $\widehat{2V}$ has a very easy multiplication since it is a group of fractions of a very well-behaved monoid Π of positive elements with an even easier multiplication. Applying the multiplication in Π to $2V$ will be discussed more fully below (see Section 5.2).

One last reason for discussing $\widehat{2V}$ is that the advertised resemblance to the little cubes operads of [3, 4] is seen most strongly in the monoid Π.

4.4 THE POSITIVE MONOID OF $\widehat{2V}$

We start with the monoid Π. Elements of the monoid will correspond to certain 'numbered sequences of patterns'. We make this specific by starting with an infinite sequence of pairwise disjoint unit squares $(S_0, S_1, \ldots)$ in the plane. To be very specific, we can take them in the upper half plane so that S_i intersects the x-axis in the interval $[2i, 2i+1]$. Now a sequence of patterns (we will number them shortly) is an infinite sequence $(P_0, P_1, \ldots)$ of patterns where P_i is thought of as a pattern in S_i and only finitely many of the P_i are not trivial.

A numbering of such a sequence of patterns is a one-to-one correspondence between $\mathbf{N}$ and the rectangles in the sequence for which there are j and k in $\mathbf{N}$ so that $i > k$ implies that the pattern P_i that is applied to S_i is the trivial pattern and that the number of the rectangle consisting of S_i is $i + j$. In words, the numbering eventually becomes a consecutive numbering, from left to right, of unsubdivided unit squares. An example is given below where it is assumed that all squares not pictured are numbered consecutively from left to right starting with 10. Thus in this example, $k = 3$ and $j = 5$ work for the required restriction.



If we let X be the union of the S_i, then a numbered sequence of patterns $(P_0, P_1, \ldots)$ describes a continuous function from X to itself. The function simply takes the square S_i, affinely and with orientation of each coordinate preserved, to rectangle i in the numbered sequence of patterns. Under composition of functions, this is a submonoid of the monoid $C^0(X)$ of continuous functions from X to itself. The identity element is given by the trivial sequence in which every S_i is a rectangle and the number of S_i is i.

If h is an element of the monoid described by a sequence of patterns $P = (P_0, P_1, \ldots)$, then we can think of h as carrying the trivial sequence to the sequence P. Thus, we can be less efficient and think of two pieces of information as describing h: the sequence P and the trivial sequence. The trivial sequence describes a structure of the domain of h and the sequence P describes the structure of the range of h. This lack of efficiency anticipates what we need to describe elements of the full group $\widehat{2V}$.

It is easy to argue that this monoid is cancellative and has common right multiples (if composition proceeds from right to left) and thus has a group of fractions by Ore's theorem (see Theorem 1.23 of [8]). However, this is not necessary. We can 'invert' the elements by representing them as homeomorphisms rather than continuous functions. This version with invertible elements will be what we call the monoid Π.

4.5. THE GROUP $\widehat{2V}$

Each square S_i contains a copy of $C \times C$ in the obvious way, and we let Y be the union of all these copies of $C \times C$. The numbered sequence of patterns gives a numbering of closed and open sets in Y that correspond to the rectangles in the sequence of patterns. Now a homeomorphism from Y to Y comes out of a numbered sequence by taking the entire copy of $C \times C$ in S_i to the closed and open set numbered i by an affine map that preserves orientation in each coordinate. This is done exactly as described in Section 2.3.

We will use Π to denote this monoid of self-homeomorphisms of Y. The group of fractions that we want is the group generated by the elements of Π and their inverses. That this corresponds to the group of fractions construction follows from Problem 3 on page 37 and Theorem 1.24 of [11].

The group $\widehat{2V}$ is the group just described. Elements can be represented by pairs of numbered sequences of patterns. If P and Q are patterns representing elements g and h, respectively, in Π, then each is a homeomorphism from Y to Y. Each has its domain structure described by the trivial sequence of patterns. The range structure

for g is described by P and the range structure for h is described by Q. If we consider the element gh^{-1} of $\widehat{2V}$, then the resulting homeomorphism from Y to itself is described by using Q to determine the structure of the domain and by using P to determine the structure of the range. For each i, the homeomorphism gh^{-1} takes the set corresponding to rectangle i in Q to the set corresponding to the rectangle i in P.

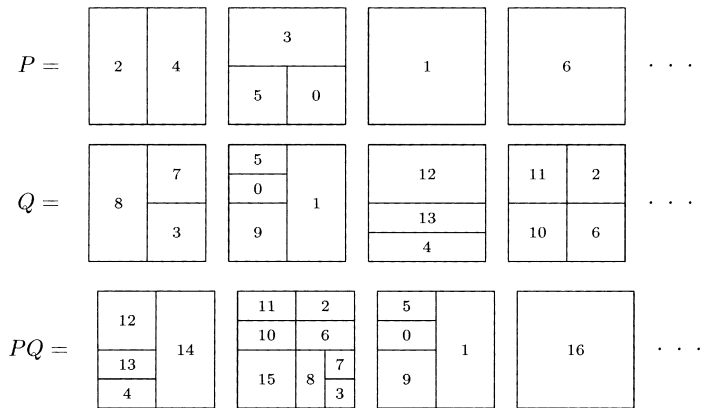
If we use the pair (P, Q) to represent the element gh^{-1} in $\widehat{2V}$, then we can think of this as a map from Q to P. We use the right element of (P, Q) to represent the pattern for the domain so that right to left composition reads nicely as in $(P, Q)(Q, R) = (P, R)$.

One can also define groups $\widehat{nV}$ and $\widehat{\omega V}$ in a similar manner, but we have nothing to say about these groups.

In the next few paragraphs, we will discuss the product in Π, we will describe some elements of Π that clearly generate Π, and we will describe some relations satisfied by those elements. The relations that we give suffice to present Π but this will be less clear and a proof of this fact will be given in [2].

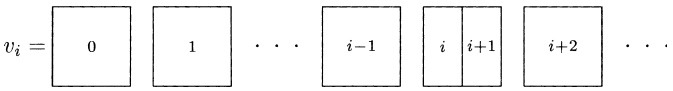
4.6. MULTIPLICATION IN THE POSITIVE MONOID OF $\widehat{2V}$

If a sequence of patterns $P = (P_0, P_1, \ldots)$ represents an element h in the positive monoid Π of $\widehat{2V}$ and $Q = (Q_0, Q_1, \ldots)$ represents $g \in \Pi$, then the sequence of patterns PQ for hg (recall that g is applied first) is gotten by pasting the pattern Q_i affinely into the rectangle i of P for each $i \in \mathbf{N}$. The numbering of the resulting pattern is that of Q. That is, when the pattern Q_i is pasted into rectangle i of P, the rectangle numbers of Q are pasted in along with the pattern. We illustrate this below where the squares not pictured in P are not subdivided and as rectangles they are numbered consecutively, left to right, from 7. Similarly in Q the corresponding number is 14, and in PQ the corresponding number is 17.



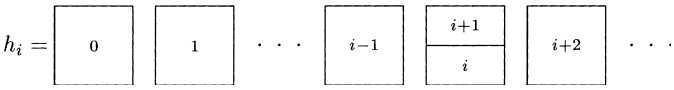
4.7. ELEMENTS OF THE POSITIVE MONOID OF $2\widehat{V}$

For $i \geqslant 0$, let v_i be as pictured below.



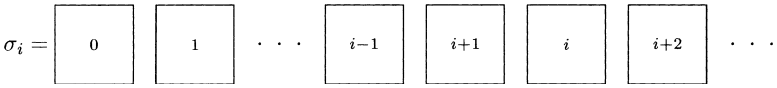
In the above picture, each square S_j with $j \neq i$ has the trivial pattern, each square S_j with $j < i$ is numbered j and each square S_j with $j > i$ is numbered $j+1$.

For $i \geqslant 0$, let h_i be as pictured below.



In the above picture, each square S_j with $j \neq i$ has the trivial pattern, each square S_j with $j < i$ is numbered j and each square S_j with $j > i$ is numbered $j+1$.

For $i \geqslant 0$, let σ_i be as pictured below.



In the above picture, every S_j has the trivial pattern and every S_j with $j \notin \{i, i+1\}$ is numbered j.

We make some remarks about multiplying by these elements on the right. If P is any numbered sequence of patterns, then Pv_i is obtained from P by dividing rectangle i of P vertically, giving the number i to the left half and the number $i+1$ to the right half, preserving the numbers of all rectangles numbered less than i in P, adding 1 to the numbers of all rectangles numbered greater than i in P.

Similar remarks apply to Ph_i except that rectangle i of P is now divided horizontally with the lower half numbered i and the upper half numbered $i+1$. Lastly $P\sigma_i$ is obtained from P by exchanging the numbers of rectangles numbered i and $i+1$ and making no other changes.

It is clear from the description of the multiplication in Π as described in Section 4.6 that any sequence of patterns (with perhaps the wrong numbering) can be obtained as a word in the v_i and h_i. Now the numbering can be fixed up by following this word with a word in the σ_i. Thus Π is generated by $\{v_i, h_i, \sigma_i \mid i \in \mathbf{N}\}$.

The previous paragraph says more. It implies that every element of Π can be written as a word in the v_i and h_i followed by a word in the σ_i. This will be supported by the claims that we make next.

4.8. RELATIONS IN THE POSITIVE MONOID OF $2\widehat{V}$

We give four sets of relations that the elements v_i, h_i and σ_i satisfy. The fact that they hold is important and the reader should verify that they do so by drawing pictures.

The first set is

$$x_j y_i = y_i x_{j+1}, \quad i < j, \tag{1}$$

where the symbols x and y come independently from the set of symbols $\{h, v\}$. We refer to the relations in (1) as the 'Thompson relations' because of their resemblance to the relations in Thompson's group F and because of their power in reducing infinite presentations to finite presentations.

The next set is

$$\sigma_i^2 = 1, \quad i \geqslant 0, \tag{2}$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| \geqslant 2, \tag{3}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad i \geqslant 0, \tag{4}$$

which are simply the relations of the infinite permutation group generated by the transpositions σ_i.

The third set is

$$\sigma_j x_i = \begin{cases} x_i \sigma_{j+1}, & i < j, \\ x_{j+1} \sigma_j \sigma_{j+1}, & i = j, \\ x_j \sigma_{j+1} \sigma_j, & i = j+1, \\ x_i \sigma_j, & i > j+1, \end{cases} \tag{5}$$

where the symbol x comes from $\{v, h\}$. These relations give the interaction between the v_i and h_i on the one hand and the σ_i on the other. The relations (5) tell how to 'switch' an x_i and σ_j that occur in the 'wrong' order. The relations (5) give another argument for the claim that any element of Π can be written as a word in the v_i and h_i followed by a word in the σ_i.

The last set is

$$v_i h_{i+1} h_i = h_i v_{i+1} v_i \sigma_{i+1}, \quad i \geqslant 0, \tag{6}$$

which we refer to as the 'cross relations'. A picture is warranted to explain why. On the left we show square 0 of $v_0 h_1 h_0$ and on the right we show square 0 of $h_0 v_1 v_0$. (Note the omission of σ_1 from the second expression.)

1	3
0	2

2	3
0	1

4.9. REMARK

There seems to be no group $2F$. The construction of the group F uses the natural left–right order in the unit interval to define a subgroup of $V = 1V$ of elements that preserve this order. The author's attempts to select a preferred order for rectangles in the unit square have resulted in failure. The effort was to construct a submonoid of Π that was cancellative, had common right multiples and that respected some selected order of rectangles. The most successful attempt achieved cancellativity and common right multiples, but was not a submonoid of Π since the resulting multiplication was not associative.

5. Generators for $2V$

In this section, we find generators for $2V$ and we will prove that they generate. In the next section, we will find relations for $2V$ but we will not prove that they suffice to give a presentation. However, in order to do what we want in this section and the next, we need to be able to multiply elements. We only have to multiply in practice and not in theory. Thus it will be sufficient to give a method for multiplying elements that works for all the calculations that we need, but we will not have to prove that our method always works. In fact it does, and the curious reader can supply the reasons why.

Before we develop a method for mutliplying elements, we will work out a nice method for representing elements that will make use of our knowledge of the positive monoid Π of $\widehat{2V}$.

5.1. ELEMENTS AS PAIRS OF WORDS

Each element of $2V$ is a homeomorphism from $C \times C$ to itself. Each element of $\widehat{2V}$ is a homeomorphism from Y, a countable union of copies of $C \times C$, to itself. We can think of $2V$ as a subgroup of $\widehat{2V}$ by thinking of $2V$ as those elements of $\widehat{2V}$ that take the copy of $C \times C$ in square 0 onto itself and that is the identity on all other copies of $C \times C$.

If we view elements of $\widehat{2V}$ as pairs of numbered sequences of patterns, then the elements of $2V$ are those pairs in which the patterns are both trivial after square 0, in which the numbering in both patterns of the rectangles is consecutive after square 0, and in which the count of rectangles in square 0 is the same for both patterns. Let Π_0 be the set (it is a submonoid, but in a very uninteresting way) of those elements of Π corresponding to numbered sequences of patterns that are trivial after square 0 and which number the squares consecutively after square 0. Then $2V$ is given by pairs (a, b) with a and b in Π_0 and so that the count of rectangles in square 0 is the same for both a and b.

Let w be a word in the elements of Section 4.7 in the form discussed at the end of Section 4.7. That is, w is a word in $\{v_i, h_i, \sigma_i \mid i \in \mathbf{N}\}$ consisting of a word p in $\{v_i, h_i \mid i \in \mathbf{N}\}$ followed by a word q in the σ_i. From the discussion in Section 4.7, we give sufficient conditions for such a word to give an element of Π_0.

We can build up the word p letter by letter from the trivial word. As we do so, we create approximations to the pattern determined by p. The addition of each letter divides one rectangle in the pattern built to that stage. We assume that each stage is a pattern in Π_0. Because of the nature of multiplying on the right by a letter in $\{v_i, h_i \mid i \in \mathbf{N}\}$, if the next letter subdivides a rectangle in square 0, the result of adding that letter will also give an element of Π_0. The trivial pattern is in Π_0, thus the word p gives an element of Π_0 if adding each letter to the prefix before it represents a division of a rectangle in square 0. It follows from this that if

$$p = x_{i_0} x_{i_1} \ldots x_{i_{k-1}},$$

where each x independently represents either the letter v or the letter h and $i_j \leqslant j$ for $0 \leqslant j < k$, then p represents an element of Π_0.

The letters in q are transpositions. If p is as given above, then the pattern that p represents has $k+1$ rectangles in square 0 numbered from 0 through k. As long as the permutation given by q affects only rectangles 0 through k, the word $w = pq$ will give an element in Π_0. Thus the sufficient condition that we get is as follows.

LEMMA 5.1. *Let $w = pq$ be a word with $p = x_{i_0}x_{i_1}\ldots x_{i_{k-1}}$ a word in $\{v_i, h_i \mid i \in \mathbf{N}\}$ in that each x independently stands for v or h, and $q = \sigma_{m_0}\sigma_{m_1}\ldots\sigma_{m_{n-1}}$ a word in the σ_i. If $i_j \leqslant j$ for $0 \leqslant j < k$ and $m_j < k$ for $0 \leqslant j < n$, then w represents an element of Π_0.*

We need a partial converse to Lemma 5.1.

LEMMA 5.2. *Let a be in Π_0. Then there is a word $w = pq$ satisfying the description in Lemma 5.1 that represents a.*

Proof. We know that there is a word pq' with $p = x_{i_0}x_{i_1}\ldots x_{i_{k-1}}$ a word in $\{v_i, h_i \mid i \in \mathbf{N}\}$ and q' a word in the σ_i representing a. Part of the conclusion is forced. If any i_j is greater than j, then some square other than square 0 is divided into smaller rectangles. This is something that cannot be undone, so we must have $i_j \leqslant j$ for all j. Thus p represents an element of Π_0. This means that the ordering of all the squares other than square 0 is already correct. The only way that this can still be true of pq' is for q' to be the trivial permutation above k. This means that q' is a permutation on $\{0, 1, \ldots, k\}$ and can be rewritten as a word q in $\{\sigma_0, \sigma_1, \ldots, \sigma_{k-1}\}$. The desired word is pq. □

5.2. MULTIPLYING PAIRS OF WORDS

If (a, b) is an element of $\widehat{2V}$ with a and b in Π, then it represents the element ab^{-1} in $\widehat{2V}$. Thus we get $(a, b)^{-1} = (b, a)$ and $(a, b)(b, c) = (a, c)$. This makes it easy to invert pairs (a, b) representing elements of $2V$ with a and b in Π_0 since (b, a) is still a pair of words with both entries in Π_0.

If (a, b) and (c, d) are pairs with all entries in Π_0, then we take advantage of the fact that (ax, bx) represents the same element as (a, b) for any x. We only need to find x and y so that $bx = cy$ and so that all of ax, bx, cy and dy are in Π_0 to write

$$(a, b)(c, d) = (ax, bx)(cy, dy) = (ax, dy).$$

In fact, this can always be done, but we do not have to prove that. As long as it always happens for the calculations that we do, we are in good shape.

5.3. THE GENERATORS

The reader can verify that all of the following represent elements of $2V$:

$$A_i = (v_0^{i+1}v_1,\ v_0^{i+2}), \quad i \geqslant 0,$$
$$B_i = (v_0^{i+1}h_1,\ v_0^{i+2}), \quad i \geqslant 0,$$
$$C_i = (v_0^{i}h_0,\ v_0^{i+1}), \quad i \geqslant 0,$$
$$\pi_i = (v_0^{i+2}\sigma_1,\ v_0^{i+2}), \quad i \geqslant 0,$$
$$\overline{\pi}_i = (v_0^{i+1}\sigma_0,\ v_0^{i+1}), \quad i \geqslant 0.$$

We let

$$\Sigma = \{A_i, B_i, C_i, \pi_i, \overline{\pi}_i \mid i \in \mathbf{N}\}.$$

In the above list, the reader will note that C_0 is the element we called the baker's map in Section 2.3. The reader will also note that all second entries in the pairs are powers of v_0. We let $2V_0$ be the set of elements that can be represented by a pair (a, v_0^k) with a in Π_0. Note that k is determined from a since v_0^k cuts square 0 into $k+1$ rectangles and a will only cut square 0 into $k+1$ rectangles if there are exactly k appearances of the letters h or v in a.

Now if (a, v_0^k) and (b, v_0^k) are elements of $2V_0$ (the fact that they use the same power if v_0 is deliberate), then $(a, v_0^k)(b, v_0^k)^{-1} = (a, v_0^k)(v_0^k, b) = (a, b)$ is an element of $2V$. Conversely, if (a, b) is in $2V$, then a and b divide square 0 into the same number (say $k+1$) of rectangles and (a, v_0^k) and (b, v_0^k) are in $2V$ with $(a, b) = (a, v_0^k)(b, v_0^k)^{-1}$. Thus, if for every a in Π_0 we can show that (a, v_0^k), with $k+1$ the number of rectangles in square 0 of the pattern for a, is a word in the elements of Σ, then we will have shown that Σ is a generating set for $2V$.

In order to do this, we must understand how products of elements in Σ behave.

5.4. BUILDING A PATTERN

Let $w = pq$ be a word as described in Lemma 5.1 representing an element of Π_0. We want to show that (w, v_0^k) can be obtained as a word in Σ.

We will modify some of the descriptions in Section 5.3 to make them more convenient to work with. To do this we take advantage of the fact that for any c, the pair (ac, bc) represents the same element (a, b).

We give two sets of alternate formulations of some of the generators. The second set of formulations is just re-indexing of the first. The first set uses the relations (1) and (5) to replace $v_1 v_0^{j-1}$ by $v_0^{j-1}v_j$ and so forth:

$$A_i = (v_0^{i+j}v_j,\ v_0^{i+j+1}), \quad i \geqslant 0,\ j \geqslant 1,$$
$$B_i = (v_0^{i+j}h_j,\ v_0^{i+j+1}), \quad i \geqslant 0,\ j \geqslant 1,$$
$$\pi_i = (v_0^{i+j+1}\sigma_j,\ v_0^{i+j+1}), \quad i \geqslant 0,\ j \geqslant 1,$$

$$A_i = (v_0^k v_{k-i},\ v_0^{k+1}), \quad i \geqslant 0,\ \ k > i,$$
$$B_i = (v_0^k h_{k-i},\ v_0^{k+1}), \quad i \geqslant 0,\ \ k > i,$$
$$\pi_i = (v_0^{k+1}\sigma_{k-i},\ v_0^{k+1}), \quad i \geqslant 0,\ \ k > i.$$

We are now ready to build (w, v_0^k) with $w = pq$ as in Lemma 5.1. We start with (p, v_0^k). Lemma 5.1 implies that if p is the empty word, then so is q, so we assume that p has at least one letter.

We have $p = x_{i_0}x_{i_1}\ldots x_{i_{k-1}}$ a word in $\{v_i, h_i \,|\, i \in \mathbf{N}\}$ with each $i_j \leqslant j$. This makes x_{i_0} equal to v_0 or h_0. We will build (p, v_0^k) as a word in Σ, and the first letter of this word is the trivial element represented as (v_0, v_0) if $x_{i_0} = v_0$, and the first letter is $C_0 = (h_0, v_0)$ if $x_{i_0} = h_0$.

Now for j with $0 < j < k-1$, we let p_j be the prefix $x_{i_0}x_{i_1}\ldots x_{i_{j-1}}$ of p of length j and assume that (p_j, v_0^j) has been represented as a word in Σ. There are four cases to consider:

$$x_{i_j} = \begin{cases} v_0, & \\ v_m, & 0 < m \leqslant j, \\ h_0, & \\ h_m, & 0 < m \leqslant j. \end{cases}$$

In the first case, we multiply by the identity since (p_j, v_0^j) represents the same element as $(p_j v_0, v_0^{j+1}) = (p_{j+1}, v_0^{j+1})$. In the second case, we write

$$(p_{j+1}, v_0^{j+1}) = (p_j v_m, v_0^{j+1}) = (p_j v_m, v_0^j v_m)(v_0^j v_m, v_0^{j+1}) = (p_j, v_0^j)A_{j-m}.$$

Similarly, the third case leads to $(p_{j+1}, v_0^{j+1}) = (p_j, v_0^j)C_j$ and the last case leads to $(p_{j+1}, v_0^{j+1}) = (p_j, v_0^j)B_{j-m}$. Inductively, we get to (p, v_0^k) as a word in Σ.

Now we know that q is a word in $\{\sigma_0, \sigma_1, \ldots, \sigma_{k-1}\}$. We have available in Σ the elements $\overline{\pi}_{k-1} = (v_0^k\sigma_0, v_0^k)$ and $\pi_i = (v_0^k\sigma_{k-1-i}, v_0^k)$ for $0 \leqslant i < k-1$. Setting $m = k-1-i$, gives $0 < m \leqslant k-1$ and $\pi_{k-1-m} = (v_0^k\sigma_m, v_0^k)$. As three of the four cases just above were handled, it is now elementary that $(w, v_0^k) = (pq, v_0^k)$ can be written as a word in Σ.

From Lemma 5.2, we know that any element of Π_0 is represented by a word as described in Lemma 5.1. Thus, we have shown the following proposition:

PROPOSITION 5.3. *The set Σ is a generating set for the group $2V$.*

6. Relations for $2V$

There is no subtlety in this section. We have the following proposition:

PROPOSITION 6.1. *The following 17 infinite families of relations hold in $2V$. In the following the letters X and Y represent symbols from $\{A, B\}$.*

$$X_q Y_m = Y_m X_{q+1}, \quad m < q,$$
$$\pi_q X_m = X_m \pi_{q+1}, \quad m < q,$$
$$\pi_q X_q = X_{q+1} \pi_q \pi_{q+1}, \quad q \geqslant 0,$$
$$\pi_q X_m = X_m \pi_q, \quad m > q+1,$$
$$\bar{\pi}_q X_m = X_m \bar{\pi}_{q+1}, \quad m < q,$$
$$\bar{\pi}_m A_m = \pi_m \bar{\pi}_{m+1}, \quad m \geqslant 0,$$
$$\bar{\pi}_m B_m = C_{m+1} \pi_m \bar{\pi}_{m+1}, \quad m \geqslant 0,$$
$$C_q X_m = X_m C_{q+1}, \quad m < q,$$
$$C_m A_m = B_m C_{m+2} \pi_{m+1}, \quad m \geqslant 0,$$
$$\pi_q C_m = C_m \pi_q, \quad m > q+1,$$
$$A_m B_{m+1} B_m = B_m A_{m+1} A_m \pi_{m+1}, \quad m \geqslant 0,$$
$$\pi_q \pi_m = \pi_m \pi_q, \quad |m-q| \geqslant 2,$$
$$\pi_m \pi_{m+1} \pi_m = \pi_{m+1} \pi_m \pi_{m+1}, \quad m \geqslant 0,$$
$$\bar{\pi}_q \pi_m = \pi_m \bar{\pi}_q, \quad q \geqslant m+2,$$
$$\pi_m \bar{\pi}_{m+1} \pi_m = \bar{\pi}_{m+1} \pi_m \bar{\pi}_{m+1}, \quad m \geqslant 0,$$
$$\pi_m^2 = 1, \quad m \geqslant 0,$$
$$\bar{\pi}_m^2 = 1, \quad m \geqslant 0.$$

It will be shown in [2] that the above list of relations suffices to present $2V$.

Proof. The above is nothing but calculation. We illustrate some and leave the rest for the reader.

For the first line with $X = Y = A$, we assume $m < q < k$

$$\begin{aligned} A_q A_m &= (v_0^k v_{k-q},\, v_0^{k+1})(v_0^{k+1} v_{k+1-m},\, v_0^{k+2}) \\ &= (v_0^k v_{k-q} v_{k+1-m},\, v_0^{k+2}) \\ &= (v_0^k v_{k-m} v_{k-q},\, v_0^{k+2}) \\ &= (v_0^k v_{k-m},\, v_0^{k+1})(v_0^{k+1} v_{k+1-(q+1)},\, v_0^{k+2}) \\ &= A_m A_{q+1}. \end{aligned}$$

Similarly, we get all of

$$A_q B_m = B_m A_{q+1},$$
$$B_q B_m = B_m B_{q+1},$$
$$B_q A_m = A_m B_{q+1}.$$

We now turn to the π_i. We note that

$$m < q \;\rightarrow\; (k+1-m) > (k-q)+1,$$
$$m = q \;\rightarrow\; (k+1-m) = (k-q)+1,$$
$$m = q+1 \;\rightarrow\; (k+1-m) = (k-q),$$
$$m > q+1 \;\rightarrow\; (k+1-m) < (k-q).$$

Now we get

$$\begin{aligned}
\pi_q A_m &= (v_0^{k+1}\sigma_{k-q},\, v_0^{k+1})(v_0^{k+1}v_{k+1-m},\, v_0^{k+2}) \\
&= (v_0^{k+1}\sigma_{k-q}v_{k+1-m},\, v_0^{k+2}) \\
&= \begin{cases}
(v_0^{k+1}v_{k+1-m}\sigma_{k-q},\, v_0^{k+2}) = A_m\pi_{q+1}, & m < q, \\
(v_0^{k+1}v_{k-m}\sigma_{k+1-m}\sigma_{k-m},\, v_0^{k+2}) = A_{m+1}\pi_m\pi_{m+1} & \\
\qquad\qquad = A_{q+1}\pi_q\pi_{q+1}, & m = q, \\
(v_0 v_{k+2-m}\sigma_{k-q}\sigma_{k+1-q},\, v_0^{k+2}) = A_{m-1}\pi_{q+1}\pi_q & \\
\qquad\qquad = A_q\pi_{q+1}\pi_q, & m = q+1, \\
(v_0^{k+1}v_{k+1-m}\sigma_{k+1-q},\, v_0^{k+2}) = A_m\pi_q, & m > q+1.
\end{cases}
\end{aligned}$$

Note that the case $m = q+1$ (not mentioned in the statement of the proposition) actually follows from the case $m = q$ since each π_i is its own inverse.

We get similar calculations for $\pi_q B_m$.

Now we look at

$$\begin{aligned}
\bar{\pi}_m B_m &= (v_0^{m+1}\sigma_0,\, v_0^{m+1})(v_0^{m+1}h_1,\, v_0^{m+2}) \\
&= (v_0^{m+1}\sigma_0 h_1,\, v_0^{m+2}) \\
&= (v_0^{m+1}h_0\sigma_1\sigma_0,\, v_0^{m+2}) \\
&= (v_0^{m+1}h_0,\, v_0^{m+2})(v_0^{m+2}\sigma_1,\, v_0^{m+2})(v_0^{m+2}\sigma_0,\, v_0^{m+2}) \\
&= C_{m+1}\pi_m\overline{\pi}_{m+1}.
\end{aligned}$$

One last example is

$$\begin{aligned}
C_m A_m &= (v_0^m h_0 v_0,\, v_0^{m+2})(v_0^{m+2}v_2,\, v_0^{m+3}) \\
&= (v_0^m h_0 v_0 v_2,\, v_0^{m+3}) \\
&= (v_0^m h_0 v_1 v_0,\, v_0^{m+3}) \\
&= (v_0^m v_0 h_1 h_0\sigma_1,\, v_0^{m+3}) \\
&= (v_0^{m+1}h_1,\, v_0^{m+2})(v_0^{m+2}h_0,\, v_0^{m+3})(v_0^{m+3}\sigma_1,\, v_0^{m+3}) \\
&= B_m C_{m+2}\pi_{m+1}.
\end{aligned}$$

All of the above are supported by drawing pictures. □

An immediate consequence of Proposition 6.1 is that $2V$ is finitely generated.

PROPOSITION 6.2. *The group $2V$ is generated by $\{A_i, B_i, \pi_i, \bar{\pi}_i \mid i = 0, 1\}$.*

Proof. The relations in Proposition 6.1 include relations of the type $Z_q A_m = A_m Z_{q+1}$ whenever $m < q$ and Z is from $\{A, B, C, \pi, \bar{\pi}\}$. From this $Z_{q+1} = A_0^{-q} Z_1 A_0^q$ follows and we see that $2V$ is generated by $\{A_i, B_i, C_i, \pi_i, \overline{\pi}_i \mid i = 0, 1\}$.

From $C_m A_m = B_m C_{m+2} \pi_{m+1}$ and $C_{m+1} B_m = B_m C_{m+2}$ we get $C_m A_m = C_{m+1} B_m \pi_{m+1}$ which gives $C_m = C_{m+1}(B_m \pi_{m+1} A_m^{-1})$. Now we use $\bar{\pi}_m B_m = C_{m+1} \pi_m \bar{\pi}_{m+1}$ to get $C_{m+1} = \bar{\pi}_m B_m \bar{\pi}_{m+1} \pi_m$ using the fact that the π_i and $\bar{\pi}_i$ are involutions. Combining the two gives $C_m = (\bar{\pi}_m B_m \bar{\pi}_{m+1} \pi_m)(B_m \pi_{m+1} A_m^{-1})$. □

7. Simplicity of $2V$

From Proposition 3.3, we must show the following:

PROPOSITION 7.1. *The group $2V$ equals its commutator subgroup.*

Proof. This is a very straightforward calculation showing that the eight generators of $2V$ are all products of commutators. To eliminate a few words, we use $V \simeq W$ to mean that $V = W$ modulo the commutator subgroup.

From the proof of Proposition 6.2, we already know that $Z_q \simeq Z_1$ for all $q > 1$ and Z in $\{A, B, C, \pi, \bar{\pi}\}$.

From $\pi_0 A_0 = A_1 \pi_1 \pi_0$, we get $A_0 \simeq A_1 \pi_1$. Similarly, $B_0 \simeq B_1 \pi_1$.

Now $\pi_1 A_1 = A_2 \pi_1 \pi_2$ gives $\pi_1 A_1 \simeq A_1 \pi_1 \pi_1 = A_1$ and $\pi_1 \simeq 1$. Thus $A_0 \simeq A_1$ and $B_0 \simeq B_1$. Also $\pi_0 \pi_1 \pi_0 = \pi_1 \pi_0 \pi_1$ gives $\pi_0 \simeq \pi_1$, so $\pi_0 \simeq 1$.

From $\bar{\pi}_1 A_1 = \pi_1 \bar{\pi}_2 \simeq \bar{\pi}_1$ we get $A_0 \simeq A_1 \simeq 1$.

The pair of relations $\bar{\pi}_0 A_0 = \pi_0 \bar{\pi}_1$ and $\pi_0 \bar{\pi}_1 \pi_0 = \bar{\pi}_1 \pi_0 \bar{\pi}_1$ give $\bar{\pi}_0 \simeq \bar{\pi}_1 \simeq 1$.

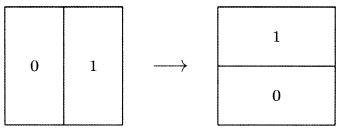
Lastly $A_0 B_1 B_0 = B_0 A_1 A_0 \pi_1$ gives $B_1 \simeq 1$ so $B_0 \simeq B_1 \simeq 1$. This shows that all eight generators from Proposition 6.2 are in the commutator subgroup. □

From Propositions 3.3 and 7.1 we get the following theorem:

THEOREM 1. *The group $2V$ is simple.*

8. On the Baker's Map

The baker's map given by



distinguishes $2V$ sharply from V. The following lemma and corollary are well known. We learned of them from Dennis Pixton.

Let elements of $C \times C$ be represented by based doubly infinite strings in $\{0,1\}$ in the following manner. The first coordinate in $C \times C$ will be represented by an infinite sequence of elements of $\{0,1\}$ written from left to right. The second coordinate in $C \times C$ will be represented by an infinite sequence of elements of $\{0,1\}$ written from right to left. The two sequences are then written on the same line separated by a 'binary point'. Obviously, this can be viewed as a function from the integers $\mathbf{Z}$ to $\{0,1\}$. If this function is written so that x_i is the image of i, then the binary point can be viewed as coming between x_{-1} and x_0 so that the sequence $(x_i)_{i \geqslant 0}$ gives the first coordinate in $C \times C$ and $(x_i)_{i<0}$ gives the second coordinate.

LEMMA 8.1. *The baker's map corresponds to shifting a based doubly infinite sequence from* $\{0,1\}$ *one position. Specifically, if* b *is the baker's map and* $x : \mathbf{Z} \to \{0,1\}$ *is a sequence representing a point in* $C \times C$, *then* $(b(x))_i = x_{i+1}$.

Proof. There are two cases to consider. If the first coordinate starts with 0, then the point is from the left half of the unit square in the figure above, and if the first coordinate starts with 1, then the point is from the right half of the unit square. The remaining details are easy. □

COROLLARY 8.1.1. *There is no bound on the size of the finite orbits of the baker's map*.

Proof. A periodic function $x : \mathbf{Z} \to \{0,1\}$ with period p lies in a finite orbit of the baker's map of size p. □

9. Rubin's Theorem

The previous section shows that the baker's map has complex dynamics. In the next section we will show that no element of V or a related group has such dynamics. We will make use of these observations by applying results of Mati Rubin. Generally, these say that under a set of hypotheses, the groups involved in two group actions are isomorphic if and only the actions are topologically conjugate. Since our observations will show that the actions of V on C and $2V$ on $C \times C$ cannot be topologically conjugate, we will have that V and $2V$ are not isomorphic.

9.1. THE THEOREM

We now give the definitions needed to state Rubin's result. If X is a topological space, if $H(X)$ is its group of self homeomorphisms, and if $F \subseteq H(X)$ is a subgroup of $H(X)$, then we say that F is *locally dense* if for every $x \in X$ and every open $U \subseteq X$ with $x \in U$, the closure of

$$\{f(x) \,|\, f \in F,\ f|_{(X-U)} = \mathbf{1}_{(X-U)}\}$$

contains some nonempty open set. In other words, for each open U, the subgroup of elements fixed off U has every orbit in U dense in some open set in U.

The following is essentially Theorem 3.1 of [12] where it is described as a combination of parts (a), (b) and (c) of Theorem 3.5 of [13]. The hypothesis that there be no isolated points was inadvertently omitted from [12] where it is needed. The terminology *locally dense* is not used in either [12] or [13]. However, in the absence of isolated points, it implies the notion of *locally moving* that is used in [12]. The absence of isolated points seems to correspond to the assumption of 'no atoms' in the Boolean algebras of [13].

THEOREM 2 (Rubin). *Let X and Y be locally compact, Hausdorff topological spaces without isolated points, let $H(X)$ and $H(Y)$ be the self homeomorphism groups of X and Y, respectively, and let $G \subseteq H(X)$ and $H \subseteq H(Y)$ be subgroups. If G and H are isomorphic and are both locally dense, then for each isomorphism $\phi : G \to H$ there is a unique homeomorphism $\tau : X \to Y$ so that for each $g \in G$, we have $\phi(g) = \tau g \tau^{-1}$.*

9.2. THE SCOPE OF RUBIN'S THEOREM

The Cantor set C is a locally compact, Hausdorff topological space with no isolated points as is the homeomorphic $C \times C$. That the action of $2V$ on $C \times C$ is locally dense follows immediately from the construction of the elements.

We will want to apply Rubin's theorem to other simple groups. Groups $V_{n,r}$ are introduced in [9] (where they are called $G_{n,r}$) that are generalizations of what we call $1V$. Patterns are created in r disjoint copies of the unit interval and these are used to create self homeomorphisms of r disjoint copies of the Cantor set. The first pattern consists of the r separate, original unit intervals and new patterns are created by subdividing intervals into n equal subintervals. Thus $1V = V_{2,1}$.

It is shown in [9] that the commutator subgroups $V'_{n,r}$ of all the $V_{n,r}$ are infinite, simple, and finitely presented. Further, in [14], subgroups $T^s_{n,r}$ of the $V_{n,r}$ (again, the $V_{n,r}$ are called $G_{n,r}$ in [14]) are given and are shown to be infinite, simple, and finitely presented. The groups $T^s_{n,r}$ are the second commutator subgroups of larger groups $T_{n,r}$ that are also studied in [14]. It is known that Proposition 3.2 applies to the $V_{n,r}$ and $T_{n,r}$ as well as a strengthening of Lemma 3.1 that says that if K and U of that lemma are contained in an open V, then the element h of that lemma can be chosen to have its support in V. It is now easy to argue that the given actions of the $V_{n,r}$ and $T_{n,r}$ are locally dense as well as the restriction of that action to any member of the derived series. Thus, Rubin's theorem applies to all of the $V'_{n,r}$ and $T^s_{n,r}$.

10. The Dynamics of Elements of V

The purpose of this section is to prove the following.

PROPOSITION 10.1. *Let f be an element of V. Then there is an $n(f) \in \mathbf{N}$ so that any finite orbit of f has no more than $n(f)$ elements.*

Trivial modifications of the arguments give the same results for the $V_{n,r}$. Since all of the groups mentioned after the statement of Rubin's theorem are subgroups of the $V_{n,r}$, we get similar results for them. From Proposition 10.1, Corollary 8.1.1 and Theorem 2, we get the following theorem:

THEOREM 3. *The group $2V$ is isomorphic to none of the infinite, simple, finitely presented groups $V^+_{n,r}$ and $T^s_{n,r}$.*

10.1. THE SCOPE OF THEOREM 3

The papers [15, 16] give constructions of other infinite, simple, and finitely presented groups. We do not know if Theorem 3 can be extended to cover these groups. There are also groups with these properties constructed in [17]. The theorem does cover the groups constructed in [17] for trivial reasons: the groups in [17] are all torsion free.

10.2. STRATEGY

Elements of V can be determined by pairs of trees plus a permutation. The proof proceeds by modifying a tree-pair of an arbitrary element f of V until it is possible to read all of the dynamics of f from the tree-pair.

We note that the modified tree-pair will not be the pair that is usually thought of as giving a 'normal form' for f. The normal form is the smallest in size (see [10]), while the pair that reveals the dynamics is usually not the smallest.

10.3. ELEMENTS AS PAIRS OF TREES

The statement $V = 1V$ should enable the reader to describe the elements of V: they are given by pairs of numbered patterns of the unit interval $I = [0, 1]$. The discussion is identical to that in Section 2. We will alter the description to pairs of trees with a permutation for two reasons. It is traditional (see [1] and [14]) and we will find it useful.

Let T be the set of finite words (including the empty word) on $\{0, 1\}$. It is a monoid under concatenation (and in fact the free monoid on two generators) with the empty word ϕ as the identity. We also think of it as the infinite binary tree (and we refer to the elements of T as *nodes* when we do) since we can think of $v0$ and $v1$ as the (respectively, left and right) child nodes of the node $v \in T$. The empty word ϕ is the root node of T.

Each node in T corresponds to an interval in a pattern on I. Recursively, ϕ corresponds to I itself and if v corresponds to $[a, b]$, then $v0$ corresponds to $[a, c]$ and $v1$ corresponds to $[c, b]$ where $c = (a + b)/2$.

For us, a finite tree will be a finite subset D of T so that (1) every prefix of a node in D is also in D, and (2) $v0$ is in D if and only if $v1$ is in D. The *leaves* of such a D will be the nodes in D whose children are not in D. Nodes of a tree that are not leaves are

called *interior nodes* of a tree. Below is a picture of a finite tree with five leaves and four interior nodes.

Note that the root of every tree is the empty word ϕ.

The leaves of a finite tree D give a pattern in I by taking the intervals in I corresponding to the leaves of D. Two trees D and R (for Domain and Range) with the same number n of leaves define two patterns in I with the same number of intervals. If we are now given a one-to-one correspondence between the intervals obtained from D to the intervals obtained from R, then we can build a homeomorphism from the Cantor set C to itself in a manner analogous to that in Section 2.3.

We now think of elements of V as triples (D, σ, R) where D and R are finite trees with the same number n of leaves and where σ is a bijection from the leaves of D to the leaves of R.

Note that σ can be replaced by numberings of the leaves of D and R (as is done in [1]), but the triple notation (as used in [14]) will be more convenient for us.

10.4. TREE OPERATIONS AND CARETS

Insisting that every finite tree be a subset of a single infinite tree T will have its advantages. Given a triple (D, σ, R), we will have reason to refer to $D \cap R$, to $D - R$, and to $R - D$ which are now nicely defined. We improve on the niceness by introducing carets. A caret is any triple $(v, v0, v1)$ in T. Every finite tree is a finite union of carets if we sloppily declare that the trivial tree $\{\phi\}$ is the union of zero carets.

We will insist on this view when we write down $D - R$ and $R - D$ and will say that $D - R$ is the set of carets in D that are not in R. It is seen that $D - R$ breaks up into a union of pairwise disjoint 'trees' (called the *components* of $D - R$) whose roots are not the empty word ϕ, but are leaves of R. Similar remarks apply to $R - D$.

In the picture below are two trees with five leaves, one D with solid lines and the other R with dashed lines. They are drawn slightly offset so they can both be seen. In the picture, $D - R$ has one component with two carets, and $R - D$ has two components with one caret each. It is important to keep in mind that $D - R$ and $R - D$ are differences of sets of carets.

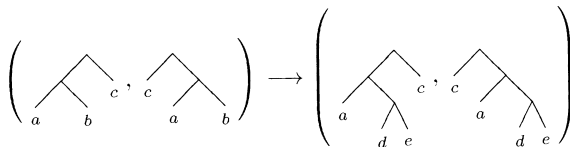
An important triviality is that the number of leaves of a finite tree is one more than the number of carets in the tree. Since number of carets is clearly the number of interior nodes, the number of leaves is one more than the number of interior nodes.

10.5. ALTERING TREE PAIRS

If (D, σ, R) represents an element of V, then we can modify the triple to create another representing the same element. Let u be one of the leaves of D and let U be a binary tree with k leaves. We can create a new triple (D', σ', R') from (D, σ, R) and the pair (u, U) as follows.

We attach a copy of U to D to form D' by identifying the root of U to u. Note that this is simply forming the tree $D' = D \cup uU$ where uU is just the product (concatenation) of u with the nodes of U. We get R' as $R \cup \sigma(u)U$. Note that the leaves of D' that are not leaves of D are of the form uv as v runs over the leaves of U. The leaves of R' that are not leaves of R are $\sigma(u)v$ as v runs over the leaves of U.

We define σ' so that it agrees with σ on leaves that D' shares with D and so that it takes each leaf of the form uv with v a leaf of U to $\sigma(u)v$. We call the triple (D', σ', R') an *augmentation* of (D, σ, R) at leaf u by U. The next figure gives an element where the bijection is illustrated by the labels on the leaves and the result of the modification is the pair $(b, \quad)$.



It is elementary that if (D', σ', R') is obtained from (D, σ, R) by an augmentation, then (D', σ', R') represents the same element of V as (D, σ, R).

10.6. ITERATED AUGMENTATIONS

Augmentations can be iterated under certain conditions. Assume we have a sequence of leaves $u_1, \ldots, u_n$ of D so that all of $u_1, \ldots, u_n$ and $\sigma(u_n)$ are different and so that $u_{i+1} = \sigma(u_i)$ for $1 \leqslant i < n$. Here we take advantage of the fact that all trees are rooted subtrees of the complete binary tree T. Now given a tree U, we can add, for each $1 \leqslant i \leqslant n$, a copy of U to D at u_i and a copy of U to R at $\sigma(u_i)$. There is a resulting one-to-one correspondence that makes the resulting triple a representative of the same element as represented by (D, σ, R). We call the alterations to R and D just described as an *iterated augmentation by U along $u_1, \ldots, u_n$* and we call the sequence of leaves $u_1, \ldots u_n$ that it is based on an *iterated augmentation chain*.

10.7. THE ARGUMENT

We now look at particular properties of the triple (D, σ, R) that represents the element f of V.

The number of carets of D equals the number of carets of R since the number of leaves of D equals the number of leaves of R. Let p be the number of carets of D and of R, and let m be the number of carets of $D \cap R$. Recall that $D - R$ refers to carets of D that are not in R. Then each of $D - R$ and $R - D$ has $p - m$ carets.

We call $p - m$ the *imbalance* of the representative (D, σ, R) of f. We assume from this point on that (D, σ, R) has the least imbalance of all the representatives of f.

We now look at the number of components of $D - R$ and $R - D$. Among the representatives with minimal imbalance, we take a representative that has the smallest number of components of $D - R$. Among all such representatives, we take a representative that has the smallest number of components of $R - D$. We now prove facts about any representative triple (D, σ, R) chosen in this manner.

LEMMA 10.2. *It is impossible to have an iterated augmentation chain $u_1, \ldots, u_n$ so that u_1 is an interior node of R and $\sigma(u_n)$ is an interior node of D.*

Proof. If false, then with U the component of $D - R$ with root at $\sigma(u_n)$, we can perform an iterated augmentation by U along $u_1, \ldots, u_n$. This removes the copy of U with root at $\sigma(u_n)$ from $D - R$. There are copies of U added to both D and R at each u_i with $2 \leqslant i \leqslant n$ so these do not contribute to $D - R$. Now if k is the number of carets in U, then the number of carets from the copy of U added to D at u_1 that are added to $D - R$ is strictly fewer than k since u_1 is an interior node of R. This would lower the imbalance which is not possible by choice. □

LEMMA 10.3. *It is impossible to have an iterated augmentation chain $u_1, \ldots, u_n$ so that u_1 is not a node of R, so that $\sigma(u_n)$ is an interior node of D and so that the component of $D - R$ containing u_1 is not the component of $D - R$ whose root is at $\sigma(u_n)$.*

Proof. Let U be the component of $D - R$ whose root is at $\sigma(u_n)$, and let V be the component of $D - R$ that contains the leaf u_1 of D. An iterated augmentation by U along $u_1, \ldots, u_n$ would remove U as a component of $D - R$, would introduce no new components of $D - R$ and would add a copy of U to the component V of $D - R$. This iterated augmentation would leave the imbalance unchanged, and would reduce the number of components of $D - R$. This is not possible by choice. □

LEMMA 10.4. *It is impossible to have an iterated augmentation chain $u_1, \ldots, u_n$ so that $\sigma(u_n)$ is not a node of D, so that u_1 is an interior node of R and so that the component of $R - D$ containing $\sigma(u_n)$ is not the component of $R - D$ whose root is at u_1.*

Proof. The proof is dual to that of the previous lemma. If V is the component of $R - D$ whose root is at u_1 and U is the component of $R - D$ that contains the leaf $\sigma(u_n)$ of R, then an iterated augmentation by V along $u_1, \ldots, u_n$ would leave both the imbalance and the number of components of $D - R$ unchanged, and would reduce the number of components of $R - D$. □

LEMMA 10.5. *For each nontrivial component U of $D - R$ there is a unique leaf $\lambda(U)$ of U so that if $r(U)$ is the root of U, then there is an iterated augmentation chain $\lambda(U) = u_1, \ldots, u_n$ with $\sigma(u_n) = r(U)$.*

For each nontrivial component V of $R - D$ there is a unique leaf $\lambda(V)$ of V so that if $r(V)$ is the root of V, then there is an iterated augmentation chain $r(V) = u_1, \ldots, u_n$ with $\sigma(u_n) = \lambda(V)$.

Proof. We consider the first claim. The argument for the second is obtained from the first by making a few mechanical substitutions.

Consider $u_1 = \sigma^{-1}(r(U))$. From previous lemmas, u_1 can only be a leaf of D that is also a leaf of R, or it is a leaf of U.

Now, assume that we have an iterated augmentation chain $u_1, \ldots, u_n$ with $r(U) = \sigma(u_n)$ and with u_1 either a leaf of U or a leaf of R. From the previous paragraph, we know that there is at least one such chain. In the case that u_1 is a leaf of U, we are done.

Assume that u_1 is a leaf of R. Increase the subscript of every element of the chain by 1, so the chain now starts with u_2 and consider $u_1 = \sigma^{-1}(u_2)$. From previous lemmas, u_1 can only be a leaf of D that is also a leaf of R, or it is a leaf of U. Thus, we have created a chain that is one longer than the original.

Since the element represented by (D, σ, R) is a homeomorphism, the chain cannot extend to an infinite loop. Since there are finitely many leaves of D, this process must stop. □

We now make some definitions. A leaf u of D is called:

(1) *neutral* if it is a leaf of R,
(2) a *repeller* if it is $\lambda(U)$ of some component U of $D - R$, and
(3) a *source* if it is a leaf other than $\lambda(U)$ of some component U of $D - R$,
(4) a *domain of attraction* if it is $r(V)$ of some component V of $R - D$.

From what we have shown, these cases are exhaustive and mutually exclusive.

A leaf v of R is called:

(1) *neutral* if it is a leaf of D,
(2) an *attractor* if it is $\lambda(V)$ of some component V of $R - D$, and
(3) a *sink* if it is a leaf other than $\lambda(V)$ of some component V of $R - D$,
(4) a *range of repulsion* if it is $r(U)$ of some component U of $D - R$.

From what we have shown, these cases are exhaustive and mutually exclusive.

LEMMA 10.6. *The imbalance is the number of sinks and is also the number of sources.*

Proof. The imbalance is the number of carets summed over all components of $D - R$. If a component U of $D - R$ has n carets, then it has $n + 1$ leaves, of which exactly one is $\lambda(U)$ and n are sources. Thus the imbalance is the number of sources. Similarly, it is the number of sinks. □

Proof of Proposition 10.1. We are now ready to describe the dynamics of an f whose representative (D, σ, R) has been chosen as described before Lemma 10.2. From Lemmas 10.2 through 10.6, we know the following since σ is a bijection from the leaves of D to the leaves of R.

The first nonneutral leaf in the forward orbit of a repeller $\lambda(U)$ of D is the range of repulstion $r(U)$ of R. The first nonneutral leaf in the forward orbit of a source of D is a sink of R. The first nonneutral leaf in the forward orbit of a domain of attraction $r(V)$ of D is the attractor $\lambda(V)$ of R. A neutral leaf of D that is not in the forward orbit of a repeller, source or domain of attraction is in a finite cyclic orbit of neutral leaves.

The interval I in the Cantor set represented by a repeller $\lambda(U)$ is properly contained in the interval J represented by $r(U)$ and is taken affinely to it by an iterate of f. Thus, I contains a unique periodic point of f which is represented by $r(U)$ followed by infinite repetitions of the path from $r(U)$ to $\lambda(U)$. The period of the periodic point is the number of iterations of f required to take $\lambda(U)$ to $r(U)$.

The interval I in the Cantor set represented by a domain of attraction $r(V)$ properly contains the interval J represented by $\lambda(V)$ and is taken affinely to it by an iterate of f. Thus, I contains a unique periodic point of f which is represented by $r(V)$ followed by infinite repetitions of the path from $r(V)$ to $\lambda(V)$. The period of the periodic point is the number of iterations of f required to take $r(V)$ to $\lambda(V)$.

A sink of R is a leaf other than $\lambda(V)$ of a component V of $R - D$, and the points in the interval I corresponding to the sink are contained in the larger interval J corresponding to $r(V)$ of D. A finite iteration of f takes all of J into the interval K corresponding to the attractor $\lambda(V)$ of R. Since I and K are disjoint, no point in I is periodic. Since every source has a sink in its forward orbit, no point in an interval corresponding to a source of D is periodic.

The neutral leaves not involved in any of the above will be organized into a finite number of circuits, each of finite period.

The number of periods in the above discussion is finite which completes the proof. □

11. More on the Baker's Map

In the next section, we give a simple proof of the simplicity of V. It is much simpler than our proof the simplicity of $2V$. As an obstruction to an analogous proof for $2V$ stands the baker's map.

Since $2V$ is simple, the baker's map must be a product of commutators. Before doing the calculations showing that $2V$ equals its commutator subgroup, the author did not believe that it was. A calculation that is an extension of the proof of Proposition 7.1 yields the following expression of C_0 (the baker's map) as a product of commutators:

$$\begin{aligned} C_0 = {} & \bar{\pi}_1(K_6K_1K_2K_3K_4K_5K_1K_2K_8K_4^{-1})\bar{\pi}_1^{-1}K_2^{-1}K_1^{-1}K_6K_6K_1K_2\times \\ & \times K_3K_4K_1K_2K_8K_7(K_1K_2K_3K_4K_5K_1K_2)^{-1}. \end{aligned} \tag{7}$$

In that expression

$$K_1 = [A_0^{-1}, A_1], \qquad K_5 = [A_1^{-1}, \pi_0^{-1}],$$
$$K_2 = A_1[\pi_1^{-1}, A_0^{-1}]A_1^{-1}, \qquad K_6 = [A_0^{-1}, C_1^{-1}],$$
$$K_3 = [\pi_1^{-1}, \bar{\pi}_1^{-1}], \qquad K_7 = [B_1^{-1}, \pi_0^{-1}],$$
$$K_4 = [\bar{\pi}_1^{-1}, A_0^{-1}], \qquad K_8 = [\pi_1^{-1}, A_0^{-1}].$$

The author has no idea if there is a shorter word than (7) expressing the baker's map as a product of commutators. The author found (7) quite useless as an aid to understanding why the baker's map was in the commutator subgroup.

12. The simplicity of V and the Problem of the Baker's Map

We give a short proof of the simplicity of V. It is along the lines of the proof in [14] but seems somewhat shorter. It is much shorter than the proof in [1] where the proof of simplicity is intimately tied to the calculation of a finite presentation for V. The proof below is due to Mati Rubin and is included here with his permission. The basic idea is that V is mostly a permutation group and all the permutations can be made even by augmentation.

We say that an element of V is a *permutation* if it is represented by a triple of the form (D, σ, D). We say that a caret $(u, u0, u1)$ in a finite tree D is *exposed* if both $u0$ and $u1$ are leaves of D. It is clear that every finite tree has at least one exposed caret.

LEMMA 12.1. *The permutations generate V.*

Proof. Let (D, σ, R) represent an arbitrary element f of V. There is a permutation (D, τ, D) so that $(D, \tau, D)(D, \sigma, R) = (D, \sigma', R)$ in which σ' takes the leaves in an exposed caret of D in left–right order to the leaves in an exposed caret of R. Now the reverse of an augmentation can remove this exposed caret from both D and R and we are done by induction since we will ultimately reduce D and R to a tree with no carets. □

We say that a nonidentity permutation (D, σ, D) is a *transposition* if σ fixes all but two leaves of D. The transposition is *proper* if D has at least three leaves.

LEMMA 12.2. *The proper transpositions generate V.*

Proof. By augmenting, every permutation can be represented as (D, σ, D) where D has at least three leaves. The result is clear. □

It is immediate that all of the proper transpositions are conjugate. Thus a normal subgroup of V is all of V if it contains a proper transposition. We now show that the normal closure of any nontrivial element contains a proper transposition.

A nontrivial f with normal closure N moves intervals in patterns, but we can think of this as moving nodes in the infinite binary tree T. Since f must move some interval off itself, it must move some caret $(u, u0, u1)$ to some caret $(v, v0, v1)$ so that neither u nor v is in a subtree of T with the other as root. These can be chosen so that

u and v are at least distance three from the root of T. Let g be a transposition interchanging $u0$ with $u1$. Now $h = [g,f]$ interchanges $u0$ with $u1$ and $v0$ with $v1$. Let j be a transposition interchanging $u0$ with $v0$. Now $k = [j,h]$ interchanges $u0$ with $v0$ and $u1$ with $v1$. But this just interchanges u with v and is a proper transposition since u and v are far enough from the root. However, h and thus k are in N. We have finished the proof of the following proposition:

PROPOSITION 12.3 *The group V is simple.*

The outline above breaks down at the first step for $2V$. The presence of the baker's map invalidates the proof of Lemma 12.1 for $2V$. It would be interesting to know if there is a proof of the simplicity of $2V$ that is shorter than the one in this paper.

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