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On the Structure and Symmetry Properties of Almost $\mathcal{S}$-manifolds

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Abstract. We prove that any simply connected $\mathcal{S}$-manifold of CR-codimension $s \geqslant 2$ is noncompact by showing that the complete, simply connected $\mathcal{S}$-manifolds are all the CR products $N \times \mathbb{R}^{s-1}$ with N Sasakian, endowed with a suitable product metric. N is a Sasakian φ-symmetric space if and only if M is CR-symmetric. The locally CR-symmetric $\mathcal{S}$-manifolds are characterized by $\tilde{\nabla}\tilde{R}=0$ where $\tilde{\nabla}$ is the Tanaka–Webster connection. This characterization is showed to be nonvalid for nonnormal almost $\mathcal{S}$-manifolds.

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1. Introduction

In this paper, we investigate some geometric features of *almost $\mathcal{S}$-manifolds*. These are the Hermitian CR-spaces (M, DM, J, g) such that $DM^{\perp}$ is trivial, and a global orthonormal frame $\xi_1, \dots, \xi_s$ of $DM^{\perp}$ can be chosen in such a way that the dual 1-forms η^i with respect to the metric g satisfy

$$\mathrm{d}\eta^i = \Phi, \quad i = 1, \dots, s,$$

where Φ is the 2-form with $\mathrm{Ker}(\Phi) = DM^{\perp}$ determined by

$$\forall X, Y \in \Gamma DM, \quad \Phi(X, Y) = g(X, JY).$$

Such a frame $\{\xi_i\}$ is called characteristic. An almost $\mathcal{S}$-manifold is called an $\mathcal{S}$-manifold if, in addition, the tensor field

$$N = [\varphi, \varphi] + 2\mathrm{d}\eta^i \otimes \xi_i$$

vanishes. Here, φ is the f-structure in the sense of Yano, which extends the partial complex structure J on the whole tangent bundle by the requirement that $\mathrm{Ker}(\varphi) = DM^{\perp}$.

This notion was studied first by Blair in [3] where various kinds of $\mathsf{U}(k) \times \mathsf{O}(s)$ structures are considered. More recently, a systematic study of nonnormal almost

$\mathcal{S}$-manifolds has been developed in [7]. We collect general information on (almost) $\mathcal{S}$-manifolds in Section 2.

The main motivation in studying this class of manifolds comes from the fact that they provide a higher CR-codimensional version of Sasakian manifolds. Indeed, the $\mathcal{S}$-manifolds having CR-codimension 1 are the Sasakian manifolds. Almost CR-manifolds (M, DM, J) whose analytic bundle DM admits a trivial complementary subbundle $D'M$ in TM were also studied in a very general fashion by Mizner in [15]. In particular, Mizner showed that the choice of a nondegenerate frame for $D'M$ gives rise to a linear connection on M which is well-adapted to the CR-structure. It turns out that, when specialized to almost $\mathcal{S}$-manifolds, Mizner's theory leads to their characterization by means of the existence of special adapted linear connections. This was proved in [13] by a direct approach. The connection canonically associated to a characteristic frame of a CR-integrable almost $\mathcal{S}$-manifold is called its *Tanaka–Webster connection* (cf. Theorem 2.3). See Section 2 for a review of this topic.

In [3], Blair obtained some interesting results about the Riemannian geometry of $\mathcal{S}$-manifolds having higher CR-codimension, showing that the structure of these manifolds can be very different from Sasakian manifolds. For instance, he proved that there are no $\mathcal{S}$-manifolds of constant curvature $K > 0$ and CR-codimension $s \geqslant 2$.

The aim of this paper is to develop further the understanding of the structure or $\mathcal{S}$-manifolds whose CR-codimension is at least 2. Our first result is the following:

THEOREM 1.1. *A simply connected $\mathcal{S}$-manifold of CR-codimension $s \geqslant 2$ is noncompact.*

Of course, the situation is different in the Sasakian category, since each odd-dimensional sphere carries a Sasakian structure.

We also remark that the above result is false without assuming the manifold is simply connected: indeed, each product $S^{2k+1} \times \mathbb{T}^s$, $k, s \geqslant 1$ can be endowed with an $\mathcal{S}$-structure of CR-codimension $s+1$. See Proposition 4.1.

Actually, we prove that each $\mathcal{S}$-manifold is locally a product $N \times \mathbb{R}^{s-1}$, where N is a Hermitian CR submanifold having CR-codimension 1 (Theorem 3.2). This leads to a complete classification of the simply connected *complete* $\mathcal{S}$-manifolds: up to CR-isometry, they are all the products $M = N \times \mathbb{R}^{s-1}$, where $(N, \varphi, \xi, \eta, g)$ is a complete Sasakian space. M is endowed with the natural CR structure of CR-codimension s and with the product metric $h = g_1 \oplus g_2$, where g_2 is the standard metric of $\mathbb{R}^{s-1}$, while g_1 is the modified metric on N such that

$$\forall X, Y \in DN, \quad g_1(X, Y) = g(X, Y), \quad g_1(X, \xi) = 0, \quad g_1(\xi, \xi) = s. \tag{1}$$

This is the content of Theorem 4.2.

In the last part of the paper we investigate symmetry properties of almost $\mathcal{S}$-manifolds from the point of view of Hermitian CR geometry. Recently, W. Kaup and D. Zaitsev [10] introduced the concept of *symmetric CR-space*. This is a Hermitian CR-space (M, DM, J, g) each point x of which admits a CR-symmetry σ_x. Such a symmetry is an isometric CR-diffeomorphism having x as a fixed point and whose differential $d\sigma_x$ equals –Id on the subspace $D_x M \oplus (\mathcal{D}_\infty(x))^\perp$ of the tangent space $T_x M$, where $\mathcal{D}_\infty$ is the Lie algebra of smooth vector fields generated by the sections of the analytic subbundle $DM \subset TM$. See Section 5 for more details.

It is interesting to remark that, when applied to Sasakian manifolds, the definition of Kaup and Zaitsev coincides with Takahashi's notion of φ-symmetric space (cf. [17] or [9]). We verify this in Section 6.

With this terminology, the above classification specializes as follows:

THEOREM 1.2. *Up to CR-isometry, the simply connected CR-symmetric $\mathcal{S}$-manifolds of CR-codimension $s>1$ are all the CR products $N \times \mathbb{R}^{s-1}$, where (N, DN, J, g) is a simply connected Sasakian φ-symmetric space, endowed with the product metric of the standard flat metric on $\mathbb{R}^{s-1}$ and the metric g_1 on N defined by (1).*

We conclude by proving that the *locally* CR-symmetric $\mathcal{S}$-manifolds of *any* CR-codimension are characterized by the condition $\tilde{\nabla}\tilde{R}=0$, where $\tilde{\nabla}$ is the Tanaka–Webster connection associated to some characteristic frame. The local version of the definition of Kaup and Zaitsev is introduced in Section 5.

Our characterization is a generalization of a theorem in [17] according to which the Sasakian locally φ-symmetric spaces are characterized by $\nabla' R' = 0$, where ∇' is the Okumura connection (cf. [16]). We think that, in the context of CR geometry, it is more convenient to have a characterization involving Tanaka–Webster connections.

It is worthwhile to remark that this result is false for *nonnormal* almost $\mathcal{S}$-manifolds. Indeed in the last section we show by examples that, in the nonnormal case, the property $\tilde{\nabla}\tilde{R}=0$ and the property to be locally CR-symmetric are independent.

2. Almost $\mathcal{S}$-manifolds

An almost CR-manifold M is a real smooth manifold with an almost CR-structure. This structure consists of a smooth real subbundle DM of the tangent bundle TM together with a smooth bundle isomorphism $J: DM \to DM$, such that $J^2 = -Id$. The linear subspace $D_x M \subset T_x M$, endowed with the complex structure J_x, is called the holomorphic tangent space to M at x. The complex dimension k of $D_x M$ is called the CR-dimension of M, while the real dimension s of the quotient vector space $T_x M / D_x M$ is the CR-codimension of M. The pair of integers (k, s) is called the *type* of the almost CR-manifold M.

We shall denote by $\mathcal{D} \subset \mathfrak{X}(M)$ the module of differentiable sections of DM. An almost CR-structure is said to be integrable if

$$[X, Y] - [JX, JY] \in \mathcal{D} \quad \forall X, Y \in \mathcal{D}$$

and the Nijenhuis torsion of J vanishes on DM, that is

$$[J, J](X, Y) = [JX, JY] - [X, Y] - J([JX, Y] + [X, JY]) = 0$$

for any $X, Y \in \mathcal{D}$. In this case we say that M is a CR-manifold.

Define inductively

$$\begin{aligned} &\mathcal{D}^1 = \mathcal{D}, \\ &\mathcal{D}^k = \mathcal{D}^{k-1} + [\mathcal{D}^1, \mathcal{D}^{k-1}], \quad k > 1, \\ &\mathcal{D}^j = 0, \quad j \leqslant 0. \end{aligned} \tag{2}$$

Thus we have an increasing sequence

$$\mathcal{D}^1 \subset \mathcal{D}^2 \subset \cdots \subset \mathcal{X}(M),$$

such that $[\mathcal{D}^r, \mathcal{D}^s] \subset \mathcal{D}^{r+s}$ for all integers r, s and $\mathcal{D}_\infty := \bigcup_k \mathcal{D}^k$ is the Lie subalgebra of $\mathfrak{X}(M)$ generated by $\mathcal{D}$. An almost CR manifold is called contact regular in the sense of Tanaka if, for each $j \geqslant 1$, $D^j := \{\mathcal{D}^j(x) \mid x \in M\}$ is a distribution of constant rank. Here by definition $\mathcal{D}^j(x) = \{X_x \in T_x M \mid X \in \mathcal{D}^j\}$. In this case, the smallest integer $\mu \geqslant 1$ such that $D^\mu = D^j$ for all $j \geqslant \mu$ is called the *kind* of M. We remark that in this case, D^μ is the smallest Frobenius integrable distribution containing DM (cf. [14]).

A *Hermitian CR-space* is an almost CR-manifold M with a Riemannian metric g compatible with the CR-structure in the sense that

$$g(Jv, Jw) = g(v, w)$$

for any $x \in M$, $v, w \in D_x M$. If the almost CR-structure is integrable we say that M is a Hermitian CR-manifold.

A smooth map $\varphi\colon M \to N$ between almost CR manifolds is called a CR-map if, for every $x \in M$, the differential $d_x\varphi\colon T_x M \to T_{\varphi(x)} N$ maps $D_x M$ into $D_{\varphi(x)} N$ and interchanges the partial complex structures. Two Hermitian CR-spaces are called *isomorphic* if there exists a diffeomorphism between them which is both a CR-map and an isometry.

Let (M, DM, J, g) be a Hermitian CR-space. We shall denote by $T_x^k \subset T_x M$ be the orthogonal complement of $\mathcal{D}_x^{k-1}$ in $\mathcal{D}_x^k$. Then $T_x^0 = 0$ and $T_x^1 = D_x M$. Let T_x^{-1} denote the orthogonal complement of $\mathcal{D}_\infty(x)$ in $T_x M$. We obtain the following orthogonal decomposition

$$T_x M = \bigoplus_{k \geqslant -1} T_x^k. \tag{3}$$

The partial complex structure J extends canonically to a tensor field φ of type $(1,1)$ such that $\mathrm{Ker}(\varphi)=DM^{\perp}$. This is an f-structure of rank $2k$ in the sense of Yano. The 2-form Φ defined by

$$\forall X, Y \in \mathfrak{X}(M) \quad \Phi(X,Y):=g(X,\varphi Y)$$

will be called the fundamental 2-form of the Hermitian CR-space M. In all that follows, when we consider a Hermitian CR-space (M, DM, J, g), the symbols φ and Φ denote these two tensors canonically associated to the triple (DM, J, g).

DEFINITION 2.1. A Hermitian CR-space (M, DM, J, g) is called an almost $\mathcal{S}$-manifold if the bundle $DM^{\perp}$ is trivial, and there exists a global orthonormal frame $\{\xi_1, \ldots, \xi_s\}$ for $DM^{\perp}$, whose dual frame $\{\eta^1, \ldots, \eta^s\}$ with respect to g satisfies

$$d\eta^i = \Phi \quad i=1,\ldots,s. \tag{4}$$

Such a frame $\{\xi_1, \ldots, \xi_s\}$ is called characteristic.

If, moreover, the tensor field N defined by

$$N=[\varphi,\varphi]+2d\eta^i\otimes\xi_i,\ [\varphi,\varphi]= \text{ Nijenhuis torsion of } \varphi \tag{5}$$

vanishes, (M, DM, J, g) is called an $\mathcal{S}$-manifold.

The term $\mathcal{S}$-manifold was introduced by D.E. Blair in [3] where more emphasis is given on the associated f-structure rather than to the underlying almost CR-structure. Since in this paper we are interested in these manifolds as a particular class of Hermitian CR-spaces, we prefer to adopt the above definition. Almost $\mathcal{S}$-structures were studied in [7]. If (M, DM, J, g) is an (almost) $\mathcal{S}$-manifold and a characteristic frame $\{\xi_i\}$ for $DM^{\perp}$ is chosen, then we refer to $(\varphi, \xi_i, \eta^i, g)$ as an associated (almost) $\mathcal{S}$-structure. It is proved in [7] that a necessary and sufficient condition for M to be CR-integrable is that $N(X,Y)=0$ for all $X, Y \in \mathcal{D}$, where N is the tensor field in (5). The condition $N=0$ which characterizes the $\mathcal{S}$-manifolds will be referred to as the *normality* condition.

Note that an almost $\mathcal{S}$-manifold with CR-codimension $s=1$ and a fixed associated structure (φ, ξ, η, g) is a contact metric manifold according to [1]. An $\mathcal{S}$-manifold with CR-codimension $s=1$ is a Sasakian manifold.

Let (M, DM, J, g) and (M', DM', J', g') be almost $\mathcal{S}$-manifolds of the same type (k,s). Choose a characteristic frame $\{\xi_1, \ldots, \xi_s\}$ for $DM^{\perp}$ and a characteristic fame $\{\xi'_1, \ldots, \xi'_s\}$ for $DM'^{\perp}$. We explicitly remark that an isomorphism $f\colon M \to M'$ of Hermitian CR-spaces does not need to map these characteristic frames one onto the other. By a result of S.Ianus and A.M. Pastore, this is true if $s=1$, i.e. M and M' are contact metric manifolds [8]. In the general case the following result holds ([7]):

THEOREM 2.2. *Let* (M, DM, J, g) *and* (M', DM', J', g') *be almost* $\mathcal{S}$- *manifolds of type* (k, s), *with characteristic frames* $\{\xi_1, \ldots, \xi_s\}$ *and* $\{\xi'_1, \ldots, \xi'_s\}$. *Let* $f: M \to M'$ *be an isomorphism. Then we have* $f_*(\bar{\xi}) = \bar{\xi}'$ *where* $\bar{\xi} = \sum_{i=1}^{s} \xi_i$, $\bar{\xi}' = \sum_{i=1}^{s} \xi'_i$.

We now recall some useful properties of almost $\mathcal{S}$-structures (for a proof, see [7]). Let M be an almost $\mathcal{S}$-manifold with structure $(\varphi, \xi_i, \eta^i, g)$. Define $h_i = \frac{1}{2}\mathcal{L}_{\xi_i}\varphi$ for any $i = 1, \ldots, s$, where $\mathcal{L}$ denotes a Lie derivative. Then each operator h_i is self-adjoint and anticommutes with φ; it vanishes on $Ker(\varphi)$, takes values in $\mathcal{D}$ and satisfies

$$\varphi(h_i X) = -\tfrac{1}{2} N(X, \xi_i)$$

for any $X \in \mathcal{X}(M)$. We also have

$$[\xi_i, \mathcal{D}] \subset \mathcal{D}, \ N(X, Y) \in \mathcal{D} \text{ for any } X, Y \in \mathcal{D}.$$

Denoting by ∇ the Levi-Civita connection of the metric g, we have

$$\nabla_X \xi_i = -\varphi X - \varphi(h_i X) \tag{6}$$

for any $X \in \mathcal{X}(M)$. If the almost $\mathcal{S}$-structure is CR-integrable, then

$$(\nabla_X \varphi)Y = -\Phi(X, \varphi Y)\bar{\xi} + \bar{\eta}(Y)\varphi^2(X) + \sum_j g(h_j(X), Y)\xi_j - \eta^j(Y) h_j(X) \tag{7}$$

where $\bar{\xi} = \sum_j \xi_j$ and $\bar{\eta} = \sum_j \eta^j$.

If M is an $\mathcal{S}$-manifold, then each operator h_i vanishes. Thus, (6) and (7) simplify as follows:

$$\nabla_X \xi_i = -\varphi X,$$
$$(\nabla_X \varphi)Y = g(X, Y)\bar{\xi} - \bar{\eta}(Y)X - \sum_j \eta^j(X)\eta^j(Y)\bar{\xi} + \bar{\eta}(Y)\eta^j(X)\xi_j$$

for any $X, Y \in \mathcal{X}(M)$. Furthermore, in this case each ξ_i is a Killing vector field.

The following result [13] provides a geometric characterization of the *CR-integrable* almost $\mathcal{S}$-manifolds.

THEOREM 2.3. *Let* (M, DM, J, g) *be a Hermitian* CR-*space. Assume that the bundle* $DM^{\perp}$ *is trivial and fix an orthonormal frame* $\{\xi_1, \ldots, \xi_s\}$ *of* $DM^{\perp}$. *Set* $\bar{\xi} := \sum_i \xi_i$. *Then* M *is a* CR-*integrable almost* $\mathcal{S}$-*manifold with characteristic frame* $\{\xi_1, \ldots, \xi_s\}$ *if and only if there exists a linear connection* $\tilde{\nabla}$ *with the following properties:*

(1) $\tilde{\nabla}\varphi = 0$, $\tilde{\nabla}\xi_i = 0$, $\tilde{\nabla}g = 0$;

(2) *the torsion* $\tilde{T}$ *of* $\tilde{\nabla}$ *satisfies*

$$\tilde{T}(X,Y)=2\Phi(X,Y)\bar{\xi} \quad \text{for any } X,Y\in\mathcal{D}, \tag{8}$$

$$\tilde{T}(\xi_i,\varphi X)=-\varphi\tilde{T}(\xi_i,X) \quad \text{for any } X\in\mathcal{X}(M), \tag{9}$$

$$\tilde{T}(\xi_i,\xi_j)=0. \tag{10}$$

Such a linear connection $\tilde{\nabla}$ *is uniquely determined and it is given by* $\tilde{\nabla}=\nabla+H$, *where*

$$H(X,Y)=\Phi(X,Y)\bar{\xi}+\bar{\eta}(Y)\varphi(X)+\bar{\eta}(X)\varphi(Y)+ \\ +\Phi(h_jX,Y)\xi_j\eta^j(Y)\varphi h_j(X) \tag{11}$$

for any $X,Y\in\mathcal{X}(M)$.

Remark 2.4. The linear connection $\tilde{\nabla}$ is called in [13] the *Tanaka–Webster connection* of M. This terminology is justified by the fact that in the case where the CR-codimension is 1, the contact form η is a pseudo Hermitian structure and $\tilde{\nabla}$ is the corresponding Tanaka–Webster connection (see, e.g., [18]).

In the general case, if M is a CR-integrable almost $\mathcal{S}$-manifold with a fixed characteristic frame $\{\xi_1,\ldots,\xi_s\}$, then the dual forms $\{\eta^1,\ldots,\eta^s\}$ make up a nondegenerate frame of type $\{1,\ldots,s\}$ according to the terminology of Mizner ([15], p. 1341). This means that at each point $p\in M$, and for each $j\in\{1,\ldots,s\}$, $\eta^j\circ\mathcal{L}_p$ is a nondegenerate Hermitian form on $\mathcal{H}_p=\{X-iJX \mid X\in D_pM\}\subset T_p^{\mathbb{C}}M$, where

$$\mathcal{L}_p:\mathcal{H}_p\times\mathcal{H}_p\to T_pM^{\mathbf{C}}/\mathcal{H}_p^{\mathbf{C}}$$

is the Levi form (cf. e.g. [15], p. 1340). Then $\tilde{\nabla}$ coincides with the connection canonically associated with $\{\eta^1,\ldots,\eta^s\}$ according to Theorem 1, p. 1355 in [15]. For a proof of this, see [13].

PROPOSITION 2.5 ([13]). *Let* M *be a* CR*-integrable almost* $\mathcal{S}$*-manifold with structure* (φ,ξ_i,η^i,g). *Let* $\tilde{\nabla}$ *be its Tanaka–Webster connection. Then we have:*

(1) $\tilde{\nabla}_Z X\in\mathcal{D}$ *for any* $X\in\mathcal{D}$, $Z\in\mathcal{X}(M)$;

(2) $\tilde{T}(\xi_i,X)=-\varphi(h_iX)$ *for any* $X\in\mathcal{X}(M)$.

Moreover, the almost $\mathcal{S}$*-structure is normal if and only if* $\tilde{T}(\xi_i,X)=0$ *for each* $X\in\mathcal{D}$ *and* $i=1,\ldots,s$.

PROPOSITION 2.6. *Let* M *and* M' *be* $\mathcal{S}$*-manifolds. Choose characteristic frames for* M *and* M' *and let* $\tilde{\nabla}$ *and* $\tilde{\nabla}'$ *be the corresponding Tanaka–Webster connections. Then every* CR*-isomorphism* $\sigma: M\to M'$ *is an affine map with respect to* $\tilde{\nabla}$ *and* $\tilde{\nabla}'$

Proof. Since the $\mathcal{S}$-structure of M is normal, according to (11) we have $\tilde{\nabla}=\nabla+H$ with

$$H(X,Y)=\Phi(X,Y)\bar{\xi}+\bar{\eta}(Y)\varphi(X)+\bar{\eta}(X)\varphi(Y).$$

An analogous formula holds for $\tilde{\nabla}'$. Hence, the assertion follows from the fact that $\sigma_* \bar{\xi} = \bar{\xi}'$ (cf. Theorem 2.2). □

PROPOSITION 2.7. *Let M be an $\mathcal{S}$-manifold with structure $(\varphi, \xi_i, \eta^i, g)$ and CR codimension $s \geqslant 1$. Let R be the curvature tensor of the Levi-Civita connection ∇; then*

$$R(X, \xi_i)Y = -(\nabla_X \varphi)Y \tag{12}$$

for any $X, Y \in \mathcal{X}(M)$ and $i = 1, \ldots, s$.

Proof. Since each ξ_i is a Killing vector field, according to Proposition 2.6 of Chapter VI in [11], we obtain:

$$R(X, \xi_i)Y = \nabla_X \nabla_Y \xi_i - \nabla_{\nabla_X Y} \xi_i,$$

and, hence,

$$R(X, \xi_i)Y = -\nabla_X \varphi Y + \varphi(\nabla_X Y) = -(\nabla_X \varphi)Y.$$ □

PROPOSITION 2.8. *Let M be an $\mathcal{S}$-manifold with structure $(\varphi, \xi_i, \eta^i, g)$, $i = 1, \ldots, s$. Let ∇ be the Levi-Civita connection and $\tilde{\nabla}$ the Tanaka–Webster connection of M. Let R and $\tilde{R}$ denote the curvature tensor fields of ∇ and $\tilde{\nabla}$ respectively. Then, we have*

$$\tilde{R}(X, Y)Z = R(X, Y)Z + B(X, Y)Z, \tag{13}$$

where

$$\begin{aligned} B(X, Y)Z = {} & 2s\Phi(X, Y)\varphi(Z) + s\Phi(X, Z)\varphi(Y) - s\Phi(Y, Z)\varphi(X) - \\ & - \bar{\eta}(X)\bar{\eta}(Z)\varphi^2(Y) + \bar{\eta}(Y)\bar{\eta}(Z)\varphi^2(X) + \\ & + \{\bar{\eta}(X)\Phi(Y, \varphi Z) - \bar{\eta}(Y)\Phi(X, \varphi Z)\}\bar{\xi} \end{aligned} \tag{14}$$

for any $X, Y, Z \in \mathcal{X}(M)$. Consequently, we get

$$\tilde{R}(X, \xi_i)Y = 0 \tag{15}$$

for any $X, Y \in \mathcal{X}(M)$ and $i = 1, \ldots, s$.

Proof. Straightforward computation. □

3. On the Structure of $\mathcal{S}$-manifolds

PROPOSITION 3.1. *Let (M, DM, J, g) be an almost $\mathcal{S}$-manifold of type (k, s), with associated structure $(\varphi, \xi_i, \eta^i, g)$. Then M is contact regular of kind 2. The distribution D_∞ has constant rank $2k + 1$ and is given by $DM \oplus [\bar{\xi}]$. It is the smallest*

Frobenius integrable distribution containing DM. The orthogonal decomposition (3) *becomes*

$$T_x M = T_x^{-1} \oplus T_x^1 \oplus T_x^2$$

where

$$T_x^1 = D_x M, \quad T_x^2 = [\bar{\xi}_x], \quad T_x^{-1} = \{X_x \mid X = \sum_i \eta^i(X)\xi_i,\ \bar{\eta}(X) = 0\}.$$

Here as usual $\bar{\xi} = \sum_i \xi_i$ *and* $\bar{\eta} = \sum_i \eta^i$.

Proof. We consider the submodules $\mathcal{D}^k$, $k \geqslant 1$, of $\mathfrak{X}(M)$ defined by (2). For every $X, Y \in \mathcal{D} = \mathcal{D}^1$, we can write $[X, Y] = Z + \eta^k([X, Y])\xi_k$, with $Z \in \mathcal{D}$; hence

$$[X, Y] = Z - 2d\eta^k(X, Y)\xi_k = Z - 2\Phi(X, Y)\bar{\xi}. \tag{16}$$

From this it follows that

$$\mathcal{D}^2 \subset \bar{\mathcal{D}} + \mathcal{D}^1$$

where $\bar{\mathcal{D}}$ denotes the module on $\mathcal{C}^\infty(M)$ generated by $\bar{\xi}$. On the other hand, choosing a unit vector field $X \in \mathcal{D}$ and setting $Y = JX$ in (16), we obtain

$$[X, JX] - 2\bar{\xi} \in \mathcal{D}$$

which implies that $\bar{\xi} \in \mathcal{D}^2$. Hence,

$$\mathcal{D}^2 = \bar{\mathcal{D}} + \mathcal{D}^1. \tag{17}$$

On the other hand, we know that $[\bar{\xi}, \mathcal{D}] \subset \mathcal{D}$. Applying this property and (17), we conclude that $[\mathcal{D}^2, \mathcal{D}^2] \subset \mathcal{D}^2$. This implies that $\mathcal{D}^k = \mathcal{D}^2$ for every $k > 2$, so that $\mathcal{D}_\infty = \mathcal{D}^2$. For each point $x \in M$

$$\mathcal{D}_x^2 = D_x M \oplus [\bar{\xi}_x].$$

This yields the conclusion. □

THEOREM 3.2. *Let (M, DM, J, g) be a CR-integrable almost $\mathcal{S}$-manifold of CR-codimension $s \geqslant 2$ with characteristic frame $\{\xi_i\}$. Denote by $\tilde{\nabla}$ the Tanaka–Webster connection associated with $\{\xi_i\}$. Set $\bar{\xi} = \sum_i \xi_i$.*

(1) *The distributions $T' := D_\infty = DM \oplus [\bar{\xi}]$ and $T'' := D_\infty^\perp$ are both involutive and $\tilde{\nabla}$-parallel. Each integral manifold N of T' is a CR-integrable almost $\mathcal{S}$-manifold with respect to the induced CR-structure of CR-codimension 1 and the Hermitian metric g' such that*

$$\forall X, Y \in DN \;\; g'(X, Y) = g(X, Y), \quad g'(X, \bar{\xi}) = 0, \quad g'(\bar{\xi}, \bar{\xi}) = 1. \tag{18}$$

The restriction to N of the vector field $\bar{\xi}$ yields a characteristic frame and the corresponding Tanaka–Webster connection is the connection canonically induced on N by $\tilde{\nabla}$. If M is normal, also N is, i.e. N is a Sasakian manifold with respect to the metric g'.

(2) *Assume that all the operators $h_i = \frac{1}{2}\mathcal{L}_{\xi_i}\varphi$ coincide. Then T' and T'' are both parallel for the Levi-Civita connection. If furthermore M is normal, then each integral manifold P of T'' is a flat Riemannian submanifold.*

Proof. (1) We remark that $D_\infty^\perp$ is spanned by $\zeta_1, \dots, \zeta_{s-1}$ where $\zeta_j := \xi_j - \xi_s$, $j = 1, \dots, s-1$. Hence it is involutive because $[\zeta_i, \zeta_j] = 0$. The fact that T' and T'' are $\tilde{\nabla}$-parallel is clear. Consider an integral manifold N of T', endowed with the almost CR structure induced by M and the Hermitian metric g' defined by (18). We remark that the connection $\tilde{\nabla}^N$ induced on N by $\tilde{\nabla}$ is metric with respect to g'. It is immediate to verify that $\tilde{\nabla}^N$ satisfies (1 and 2) in Theorem 2.3, so that N is an almost $\mathcal{S}$-manifold with characteristic frame $\{\bar{\xi}\}$ whose Tanaka–Webster connection is $\tilde{\nabla}^N$. If M is normal, also N is according to Prop. 2.5.

(2) To show that T' and T'' are parallel with respect to the Levi-Civita connection observe that, assuming $h_1 = \cdots = h_s$, the formula for the covariant derivative of φ simplifies as follows:

$$(\nabla_X\varphi)Y = -\Phi(X, \varphi Y)\bar{\xi} + \bar{\eta}(Y)\varphi^2 X + g(h(X), Y)\bar{\xi} - \bar{\eta}(Y)h(X)$$

for all $X, Y \in \mathcal{X}(M)$, where we have set $h = h_1 = \cdots = h_s$. It follows that

$$\nabla_X \mathcal{D} \subset T'.$$

Moreover, since $\nabla_X \xi_i = -\varphi X - \varphi h_i(X) = -\varphi X - \varphi h(X)$ for all $i \in \{1, \dots, s\}$, we get

$$\nabla_X \bar{\xi} = -s\varphi(X + hX) \in \mathcal{D}, \text{ and } \nabla_X \zeta_i = 0$$

and this proves the first assertion. If M is normal, since P is a totally geodesic submanifold in M, its flatness follows from (12) and the Gauss equation. □

4. Classification of Simply Connected $\mathcal{S}$-manifolds

The following proposition gives a simple way to construct examples of $\mathcal{S}$-manifolds starting from Sasakian manifolds.

PROPOSITION 4.1. *Let (N, DN, J, g) be a Sasakian manifold. Denote by g_1 the Hermitian metric on N such that*

$$\forall X, Y \in DN \quad g_1(X, Y) = g(X, Y), \quad g_1(X, \xi) = 0, \quad g_1(\xi, \xi) = s,$$

where ξ is the characteristic vector field on N.

Let G be an Abelian real Lie group of dimension $s-1$, $s \geqslant 2$. Then the CR manifold $M = N \times G$ of CR-codimension s is an $\mathcal{S}$-manifold with respect to any product metric $h = g_1 \oplus g_2$, where g_2 is a left-invariant metric on G. The maximal integral manifolds of the distribution $D_\infty(M)$ are $N \times \{a\}$, $a \in G$.

Proof. We fix a global g_2-orthonormal frame $\zeta_1, \dots, \zeta_{s-1}$ of G consisting of left-invariant vector fields. Let $V \subset \mathfrak{X}(M)$ be the s-dimensional Abelian Lie subalgebra of $\mathfrak{X}(M)$ generated by $\{\xi, \zeta_i\}$. Then, since $h(\xi, \xi) = s$, there exist $\xi_1, \dots, \xi_s \in V$ such that

$$\xi = \sum_{i=1}^{s} \xi_i, \; h(\xi_i, \xi_j) = \delta^i_j.$$

Hence, it is straightforward to verify that M is an $\mathcal{S}$ -manifold with characteristic frame $\{\xi_1, \dots \xi_s\}$. □

The main result of this section states that the simply connected, complete $\mathcal{S}$-manifolds of CR-codimension $s \geqslant 2$ are exactly the ones constructed with the above procedure by choosing as G an Euclidean space. More precisely,

THEOREM 4.2. *For a given Sasakian manifold N denote by N_s the $\mathcal{S}$-manifold $N \times \mathbb{R}^{s-1}$ of CR-codimension $s \geqslant 2$ obtained according to Proposition 4.1 choosing $G = \mathbb{R}^{s-1}$ and g_2 = standard flat metric.*

For each $s \geqslant 2$, the mapping $\Phi \colon N \mapsto N_s$ induces a bijection between the isomorphism classes of simply connected complete Sasakian manifolds and the isomorphism classes of simply connected complete $\mathcal{S}$-manifolds of CR-codimension s.

Proof. We remark that, if (N, DN, J, g) and (N', DN', J', g') are CR-isometric Sasakian manifolds, then every CR-isometry $f \colon N \to N'$ satisfies $f_* \xi = \xi'$. This implies that f is also an isometry between the Riemannian manifolds (N, g_1) and (N', g'_1) where g_1 and g'_1 are defined according to Proposition 4.1. If follows that $f \times id$ is a CR-isometry between $N \times \mathbb{R}^{s-1}$ and $N' \times \mathbb{R}^{s-1}$. Hence, Φ actually determines a map

$$[N] \to [N \times \mathbb{R}^{s-1}]$$

where the symbol [ ] denotes a CR-isometry class. To show that this map is a bijection, we observe that, according to Theorem 3.2, a simply connected, complete $\mathcal{S}$-manifold M is CR-isometric to a product $N \times \mathbb{R}^{s-1}$ where N is a maximal integral manifold of the distribution D_∞, considered as an Hermitian CR-submanifold of M. We also know that with respect to the metric g' in (18), N is a Sasakian manifold. Denote by N' this Sasakian manifold. Then the $\mathcal{S}$-manifold $\Phi(N')$ coincides with $N \times \mathbb{R}^{s-1}$ and, hence, it coincides with M up to CR-isometry. This proves that Φ is surjective. On the other hand, it follows from (1) in Theorem 3.2 that if N'' is any Sasakian manifold such that $[\Phi(N'')] = [M]$, then N'' embeds in M as an integral manifold of D_∞. Thus $[N'] = [N'']$. This completes the proof. □

As a consequence, we get

COROLLARY 4.3. *A simply connected $\mathcal{S}$-manifold of CR-codimension $s > 1$ is noncompact.*

In the following (see Corollary 6.4) we shall also verify that the above correspondence Φ preserves the property of being CR-symmetric in the sense of [10]. The next session is devoted to a preliminary discussion of the notion of CR-symmetry for general Hermitian CR-spaces. We come back to the context of almost $\mathcal{S}$-manifolds in Section 6.

5. Symmetric *CR*-manifolds and φ-symmetric Spaces

In literature there are two definitions of symmetries on almost CR-manifolds: the CR-symmetries on *Hermitian CR-spaces*, introduced by W. Kaup and D. Zaitsev [10], and the φ-geodesic symmetries on *Sasakian manifolds*, according to T. Takahashi [17]. It turns out that a Sasakian manifold is (locally) φ-symmetric if and only if it is (locally) CR-symmetric (cf. Corollary 6.1 below). In this section we recall the two definitions.

We keep the notation in Section 2.

DEFINITION 5.1 ([10]). Let M be a Hermitian CR-space and let $\sigma: M \to M$ be an isometric CR-diffeomorphism. Then σ is called a *symmetry* at the point $x \in M$ if x is a fixed point of σ and the differential of σ at x coincides with the negative identity on the subspace $T_x^{-1} \oplus T_x^1$ of $T_x M$. A connected Hermitian CR-space M is called a (globally) *CR-symmetric space* if for each point $x \in M$ there exists a symmetry σ_x at x.

We remark that a symmetric CR manifold of CR-codimension 0 is a Hermitian symmetric space, while a symmetric CR space of CR-dimension 0 is a Riemannian symmetric space.

It is proved in [10] that a CR-symmetric space M is *CR-homogeneous*: the group of isometric CR automorphisms of M acts transitively. In particular, M is a complete Riemannian manifold.

THEOREM 5.2 ([10]). *Let φ, ψ be isometric CR-diffeomorphisms of a Hermitian CR-space M. Assume that $\varphi(x) = \psi(x)$ for some $x \in M$. If the differentials $(\mathrm{d}\varphi)_x$ and $(\mathrm{d}\psi)_x$ coincide on the subspace $T_x^{-1} \oplus T_x^1$ of $T_x M$, then $\varphi = \psi$. Hence, if M is a CR-symmetric space, the symmetry σ_x at x is unique; its differential is given by*

$$(d\sigma_x)_x = \sum_{k \geqslant -1} (-1)^k \pi_x^k, \tag{19}$$

where $\pi_x^k: T_x M \to T_x^k$ is the orthogonal projection.

This result allows us to give the following definition:

DEFINITION 5.3. A Hermitian almost CR manifold M will be called *locally CR-symmetric* if for every point $x \in M$ the mapping

$$s_x = \exp_x \circ L_x \circ \exp_x^{-1}, \quad L_x := \sum_{k \geqslant -1} (-1)^k \pi_x^k \tag{20}$$

defined on a normal neighborhood of x, is a local isometric CR-diffeomorphism.

As a consequence of Theorem 5.2, Definitions 5.1 and 5.3 are well-related.

PROPOSITION 5.4. *A connected Hermitian almost CR-manifold M is globally CR-symmetric if and only if each s_x extends to a global symmetry in the sense of Definition 5.1.*

We end this section by recalling the definition of a φ-symmetric space (cf. [17] and [6]). Let M be a contact metric manifold with associated structure (φ, ξ, η, g). A geodesic $\gamma = \gamma(s)$ in M is said to be *φ-geodesic* if its tangent vectors are horizontal, that is $\eta(\dot{\gamma}(s)) = 0$ for each s. Let σ_x be a local diffeomorphism defined in a neighborhood U of $x \in M$. Then σ_x is called a (local) *φ-geodesic symmetry* if for each point $y \in U$ which lies on the integral curve of ξ through x, and for each φ-geodesic γ of M such that $\gamma(0) = y$, we have $\sigma_x(\gamma(s)) = \gamma(-s)$, for all s with $\gamma(\pm s) \in U$.

A contact metric manifold M with associated structure (φ, ξ, η, g) is called a *locally φ-symmetric space* if it admits at every point $x \in M$ a φ-geodesic symmetry, which is a local automorphism, i.e. a local diffeomorphism leaving all structure tensor fields invariant. A *Sasakian φ-symmetric space* is a complete Sasakian manifold all of which φ-geodesic symmetries extend to global automorphisms of the Sasakian structure.

6. *CR*-symmetries on Almost $\mathcal{S}$-manifolds

In this section, we establish some general properties of CR-symmetric almost $\mathcal{S}$-manifolds, and we prove that for Sasakian manifolds the concepts of CR-symmetry and φ-symmetry coincide. We keep the notations of the above sections.

PROPOSITION 6.1. *A Sasakian manifold is a (locally) φ-symmetric space if and only if it is (locally) CR-symmetric in the sense of Definition 5.3.*

Proof. Let M be a Sasakian manifold with associated structure (φ, ξ, η, g). Hence, M is an $\mathcal{S}$-manifold of CR-codimension 1 and characteristic frame $\{\xi\}$. For each $x \in M$ the φ-geodesic symmetry σ_x on a normal neighborhood is given by

$$\sigma_x = \exp_x \circ S_x \circ \exp_x^{-1},$$

where $S_x := -I + 2\eta \otimes \xi$ (cf. [5]). On the other hand, specializing Proposition 3.1 to the case where the CR-codimension $s = 1$, we get $S_x = L_x$ where L_x is defined in (20). Thus, keeping the notation of Definition 5.3, we have $\sigma_x = s_x$ and the assertion follows. □

PROPOSITION 6.2. *Assume that M is a locally CR-symmetric almost $\mathcal{S}$-manifold of type (k, s) with associated structure $(\varphi, \xi_i, \eta^i, g)$. Then M is CR-integrable and the operators $h_i = \frac{1}{2}\mathcal{L}_{\xi_i}\varphi, i = 1, \ldots s$, coincide.*

Proof. Fix a point $x \in M$. Since

$$N = [\varphi, \varphi] + 2\Phi \otimes \bar{\xi}$$

we have that N is preserved by σ_x since φ, g and $\bar{\xi}$ are preserved. Now, assume $s > 1$ and set $\zeta_j := \xi_j - \xi_s$ for each $j = 1, \ldots, s-1$. For each $X \in \mathcal{D}$ we have $N(X, \zeta_i) \in \mathcal{D}$. It follows that

$$-N_x(X, \zeta_i) = (\sigma_x)_* N_x(X, \zeta_i) = N_x(-X, -\zeta_i)$$

whence $N_x(X, \zeta_i) = 0$. We have thus proved that $N(X, \zeta_i) = 0$ for all $X \in \mathcal{D}$ and this implies that $h_k = h_s$ for all $k = 1, \ldots, s-1$. Turning to the general case $s \geqslant 1$, observe that, since $N(X, Y) \in \mathcal{D}$ for all $X, Y \in \mathcal{D}$, the same argument applies to show that $N(X, Y) = 0$ for $X, Y \in \mathcal{D}$ which means that M is CR-integrable. □

PROPOSITION 6.3. *Let (M, DM, J, g) be a CR-integrable almost $\mathcal{S}$-manifold of CR-codimension $s \geqslant 2$ with characteristic frame $\{\xi_i\}$. Denote by $\tilde{\nabla}$ the Tanaka–Webster connection associated with $\{\xi_i\}$ and set $\bar{\xi} = \sum_{i=1}^{s} \xi_i$ and $\zeta_j = \xi_j - \xi_s$, $j = 1, \ldots, s-1$. If M is (locally) CR-symmetric, then each maximal integral manifold N of $T' = D_\infty$ is (locally) CR-symmetric with respect to the metric* (18)*, while each maximal integral manifold P of $T'' = D_\infty^\perp$ is a Riemannian (locally) symmetric space for the metric induced by g.*

Proof. To prove that N and P are locally CR-symmetric, we show that the CR-symmetry σ_x at each point $x \in M$ leaves the maximal integral submanifolds N and P through x invariant. Indeed, observe that T' and T'' are σ_x-invariant because they are ∇-parallel and $(\sigma_x)_* T'_x \subset T'_x, (\sigma_x)_* T''_x \subset T''_x$. Let $y \in N$ and take a piecewise smooth curve γ in N joining x and y. Then T' being (σ_x)-invariant, we have that the tangent vectors of $\sigma_x \circ \gamma$ are tangent to N. According to a general property of involutive distributions (cf. [11], p. 86) it follows that $\sigma_x \circ \gamma$ lies in N. Hence $\sigma_x(y) \in N$. An analogous argument applies to P. Since, according to Theorem 2.2, $(\sigma_x)_* \bar{\xi} = \bar{\xi}$, σ_x restricts to an isometry of (N, g'), which is a local CR-symmetry at x. It follows that N is locally CR-symmetric as claimed. Moreover, σ_x restricts to a geodesic reflection of P, so that P is Riemannian locally symmetric. □

COROLLARY 6.4. *Fix an integer $s \geqslant 2$. The mapping Φ in Theorem 4.2 induces a bijection between the isomorphism classes of simply connected Sasakian φ-symmetric spaces and the isomorphism classes of simply connected CR-symmetric $\mathcal{S}$-manifolds.*

In particular, any simply connected CR-symmetric $\mathcal{S}$-manifold of CR-codimension $s \geqslant 2$ is noncompact.

Proof. We recall that CR-symmetric spaces are always complete since they are Riemannian homogeneous.

Let N be a Sasakian φ-symmetric space with characteristic vector field ξ. If σ_x is the CR-symmetry at the point $x \in N$, then $\sigma_* \xi = \xi$, so that σ is an isometry for the metric g_1 defined in 1) of Theorem 4.2. Hence, $\sigma_x \times (-\mathrm{Id})$ is a CR-isometry on $\Phi(N) = M \times \mathbb{R}^{s-1}$ which is clearly a CR-symmetry at (x, p) for each $p \in \mathbb{R}^{s-1}$. It follows that $\Phi(N)$ is CR-symmetric.

Assume now that M is a CR-symmetric space; let $N \in \Phi^{-1}([M])$. Then, by construction, N is CR-isometric to a maximal integral manifold of the distribution D_∞ on M, endowed with the Sasakian structure induced on it according to Theorem 3.2. Hence, N is Sasakian φ-symmetric according to Prop. 6.3. □

7. A Characterization of Locally *CR*-symmetric $\mathcal{S}$-manifolds

Takahashi ([17]) proved that a necessary and sufficient condition for a Sasakian manifold to be a locally φ-symmetric space is that $\bar{\nabla}\bar{R} = 0$, where $\bar{\nabla}$ is a special linear connection, called the Okumura's linear connection, and $\bar{R}$ is its curvature tensor field. If M is a Sasakian manifold with structure (φ, ξ, η, g), the Okumura's connection $\bar{\nabla}$ is given by

$$\bar{\nabla}_X Y = \nabla_X Y + T(X, Y),$$

where

$$T(X, Y) = d\eta(X, Y)\xi - \eta(X)\varphi(Y) + \eta(Y)\varphi(X). \tag{21}$$

The tensors φ, ξ, η, T are parallel with respect to $\bar{\nabla}$. This connection was considered first in [16].

Taking into account Theorem 2.3 and the subsequent Remark, it is natural to ask for a similar characterization involving the Tanaka–Webster connection. We shall prove the following theorem:

THEOREM 7.1. *Let (M, DM, J, g) be an $\mathcal{S}$-manifold with characteristic frame $\{\xi_1, \dots, \xi_s\}$, $s \geqslant 1$, and corresponding Tanaka–Webster connection $\tilde{\nabla}$. A necessary and sufficient condition for M to be locally CR-symmetric is that $\tilde{\nabla}\tilde{R} = 0$.*

Proof. We shall consider first the case where $s = 1$. Hence, M is a Sasakian manifold. We shall reduce to the Theorem of Takahashi by showing that $\bar{\nabla}\bar{R} = \tilde{\nabla}\tilde{R}$. Indeed, the two connections are defined by

$$\bar{\nabla} = \nabla + T, \quad \tilde{\nabla} = \nabla + H,$$

where T and H are given by (21) and (11), respectively. Now we set

$$Q(X,Y)=(H-T)(X,Y)=2\eta(X)\varphi(Y)$$

for any $X,Y\in\mathcal{X}(M)$. By computation we get

$$(\bar\nabla_V\bar R)(X,Y)Z=(\nabla_V R)(X,Y)Z+T(V,R(X,Y)Z)-R(T(V,X),Y)Z-\\ -R(X,T(V,Y))Z-R(X,Y)T(V,Z)$$

and similarly,

$$(\tilde\nabla_V\tilde R)(X,Y)Z=(\nabla_V R)(X,Y)Z+H(V,R(X,Y)Z)-R(H(V,X),Y)Z-\\ -R(X,H(V,Y))Z-R(X,Y)H(V,Z)$$

for any $X,Y,Z,V\in\mathcal{X}(M)$. Hence,

$$\begin{aligned}&(\tilde\nabla_V\tilde R)(X,Y)Z-(\bar\nabla_V\bar R)(X,Y)Z\\ &=Q(V,R(X,Y)Z)-R(Q(V,X),Y)Z-R(X,Q(V,Y))Z-R(X,Y)Q(V,Z)\\ &=2\eta(V)[\varphi(R(X,Y)Z)-R(\varphi X,Y)Z-R(X,\varphi Y)Z-R(X,Y)\varphi Z]\\ &=0,\end{aligned}$$

where the last equality follows applying Lemmas 2.8 and 5.1 in [17].

Now let M be an $\mathcal{S}$-manifold with CR-codimension $s\geqslant 2$. Fix a point $p\in M$; according to Theorem 3.2, we can choose a cubic coordinate system $(x^1,\dots,x^{2k+1},y^1,\dots y^{s-1})$ defined on an open neighborhood V of p, such that $V'=\{q\in V\mid y^i(q)=0\}$ and $V''=\{q\in V\mid x^j(q)=0\}$ are open neighborhoods of p in the maximal integral manifolds N and P of the distributions $T'=D_\infty=DM\oplus[\bar\xi]$ and $T''=D_\infty^\perp$ respectively. We know that the CR-submanifold N is a Sasakian manifold with respect to the Hermitian metric g' defined by (18). Moreover, P is a flat Riemannian submanifold of M. Since the statement to be proved is of local nature, we are reduced to show that V is locally CR-symmetric if and only if $\tilde\nabla\tilde R$ vanishes on V. We remark that the natural diffeomorphism $V\cong V'\times V''$ is a CR-map and an isometry, provided we put on V' and V'' the product metric of the metrics induced by V.

We first prove that a necessary and sufficient for V to be locally CR-symmetric is that (V',DV',J,g') be locally CR-symmetric. Indeed, the necessity follows from Prop. 6.3. For the sufficiency, let $p=(x,z)\in V\cong V'\times V''$ and let σ' be a local CR-symmetry of V' at x; denote also by σ'' be the geodesic reflection of V'' at z, which is a local isometry of V'' because V'' is flat. We remark that, according to the definition of g', since $(\sigma')_*\bar\xi=\bar\xi$, then σ is also an isometry with respect to the metric g restricted to V', so that $\sigma:=\sigma'\times\sigma''$ is a local isometry of $V\cong V'\times V''$. It is clear that σ is a local CR transformation of V. Since $(\sigma)_{*p}=-Id$ on $D_pM\oplus(D_\infty^\perp)_p$ we have that σ is a local CR-symmetry at p. This implies that V is locally CR-symmetric proving our claim.

Now, we know that, N being Sasakian, (V', DV', J, g') is locally CR-symmetric if and only if on V' it holds $\tilde{\nabla}^N \tilde{R}^N = 0$, where $\tilde{\nabla}^N$ is the Tanaka–Webster connection of N. On the other hand, $\tilde{\nabla}^N$ coincides with the connection induced by $\tilde{\nabla}$ on N (Theorem 3.2, 1). Moreover, according to (15), we have that $\tilde{R} = 0$ on V''. To conclude the proof, we observe that according to Proposition 2.5 we have $\tilde{T}(X, Y) = 0$ for any $X \in T', Y \in T''$. This implies that

$$\tilde{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial y^j} = \tilde{\nabla}_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial x^i} = 0,$$

whence, being $\tilde{R} = 0$ on V'', we conclude that $\tilde{\nabla}^N \tilde{R}^N = 0$ on V' if and only if $\tilde{\nabla}\tilde{R} = 0$ on V. This completes the proof of Theorem 7.1. □

COROLLARY 7.2. *A complete and simply connected locally CR–symmetric $\mathcal{S}$-manifold is globally CR-symmetric.*

Proof. This can be proved by a standard argument using Corollary 7.9 in [11], Ch. VI, page 265. Indeed, we remark that $\tilde{\nabla}$, being a metric connection, is complete by assumption. Since $\tilde{\nabla}\tilde{R} = \tilde{\nabla}\tilde{T} = 0$, for each $x \in M$ the linear transformation $L_x : T_xM \to T_xM$ in (20) extends to a global affine transformation $f : M \to M$, such that $df_x = L_x$. This transformation is actually an isometric CR-diffeomorphism because g and φ are $\tilde{\nabla}$-parallel. Thus f is a CR-symmetry at x. □

8. Failure of Theorem 7.1 in the Nonnormal Case

In this section we provide explicit examples to show that Theorem 7.1 is false without the assumption of normality.

First, we consider a class of CR-integrable almost $\mathcal{S}$-manifolds which are locally CR-symmetric but $\tilde{\nabla}\tilde{R} \neq 0$.

Let M be a contact metric manifold with structure (φ, ξ, η, g). We suppose that the characteristic vector field ξ belongs to the (k, μ)-nullity distribution for some real numbers k and μ. This means that the curvature tensor R satisfies

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \tag{22}$$

for any $X, Y \in \mathcal{X}(M)$, where $h = \frac{1}{2}\mathcal{L}_\xi\varphi$. In [4] it is proved that, for such manifolds, $k \leqslant 1$. If $k = 1$, then $h = 0$ and M is Sasakian. If $k < 1$, then the contact metric structure is not Sasakian and M admits three mutually orthogonal integrable distributions: $D(0) = \mathbb{R}\xi$, $D(\lambda)$ and $D(-\lambda)$ defined by the eigenspaces of h, where $\lambda = \sqrt{1-k}$.

Note that

$$(\nabla_X\varphi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX)$$

for any $X, Y \in \mathcal{X}(M)$. This is a necessary and sufficient condition for M to be CR-integrable [18], and this allows us to consider the Tanaka–Webster connection.

In the following we consider the non-Sasakian case. In [6] it is proved that the contact metric manifolds satisfying (22), with $k<1$, are all locally φ-symmetric. We shall prove the following:

PROPOSITION 8.1. *Let M be a contact metric manifold whose characteristic vector field ξ belongs to the $(k,0)$-nullity distribution, $k<1$. Suppose that* $\dim_{\mathbf{R}} M>3$. *Let $\tilde{\nabla}$ be the Tanaka–Webster connection and $\tilde{R}$ its curvature tensor field. Then $\tilde{\nabla}\tilde{R}\neq 0$.*

For instance, the tangent sphere bundle T_1M of a flat Riemannian manifold satisfies the above conditions with $k=0$.

For the proof, we shall make use of the following proposition:

PROPOSITION 8.2 ([6]). *Let M be a contact metric manifold with structure (φ,ξ,η,g), such that ξ belongs to the (k,μ)-nullity distribution. Then, there exists a homogeneous structure on M, that is a tensor field T of type* $(1,2)$ *satisfying*

$$\nabla' g=0, \quad \nabla' R=0, \quad \nabla' T=0,$$

where $\nabla'=\nabla-T$. T is given by

$$T(X,Y)=g(\varphi X+\varphi(hX),Y)\xi-\eta(Y)(\varphi X+\varphi(hX))-\frac{\mu}{2}\eta(X)\varphi(Y) \tag{23}$$

for any $X,Y\in\mathcal{X}(M)$. Moreover, the tensor fields ξ, R, φ, h are all parallel with respect to ∇'.

Proof of 8.1 The Tanaka–Webster connection of M is given by $\tilde{\nabla}=\nabla+H$, where

$$H(X,Y)=-g(\varphi X+\varphi(hX),Y)\xi+\eta(Y)(\varphi X+\varphi(hX))+\eta(X)\varphi(Y).$$

Now we consider the tensor T in (23) and we put

$$Q(X,Y)=(T+H)(X,Y)=\eta(X)\varphi(Y)$$

for any $X,Y\in\mathcal{X}(M)$. Then $\tilde{\nabla}=\nabla'+Q$. By a direct calculation we see that the curvature tensor $\tilde{R}$ is given by

$$\tilde{R}(X,Y)Z=R(X,Y)Z+B(X,Y)Z,$$

where

$$\begin{aligned}B(X,Y)Z=k[&g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi-\\&-\eta(Y)\eta(Z)X+\eta(X)\eta(Z)Y]-\\&-\Phi(Y+hY,Z)(\varphi X+\varphi hX)+\\&+\Phi(X+hX,Z)(\varphi Y+\varphi hY)+2\Phi(X,Y)\varphi Z.\end{aligned} \tag{24}$$

Now we compute $(\tilde{\nabla}_\xi \tilde{R})(X, Y)Y$, choosing $X \in D(\lambda)$ and $Y \in D(-\lambda)$, such that $g(Y, Y) = 1$ and $g(\varphi X, Y) = 0$; this is possible since we assumed $\dim_{\mathbf{R}} M > 3$. We know that $\nabla' R = 0$. Moreover, B is built from the tensors g, φ, h, η, ξ, which are ∇'-parallel; hence, $\nabla' B = 0$ and this implies $\nabla' \tilde{R} = 0$. Making use of Theorem 1 in [4], we get

$$\begin{aligned} &Q(\xi, R(X, Y)Y) - R(Q(\xi, X), Y)Y - R(X, Q(\xi, Y))Y - R(X, Y)Q(\xi, Y) \\ &\quad = \varphi(R(X, Y)Y) - R(\varphi X, Y)Y - R(X, \varphi Y)Y - R(X, Y)\varphi Y \\ &\quad = 2(\lambda - 1 + k)\varphi X. \end{aligned}$$

Applying (24), we obtain

$$\begin{aligned} &Q(\xi, B(X, Y)Y) - B(Q(\xi, X), Y)Y - B(X, Q(\xi, Y))Y - B(X, Y)Q(\xi, Y) \\ &\quad = \varphi(B(X, Y)Y) - B(\varphi X, Y)Y - B(X, \varphi Y)Y - B(X, Y)\varphi Y \\ &\quad = 2\lambda(1 + \lambda)\varphi X. \end{aligned}$$

Hence,

$$(\tilde{\nabla}_\xi \tilde{R})(X, Y)Y = 2(\lambda - 1 + k + \lambda + \lambda^2)\varphi X = 4\lambda\varphi X \neq 0. \qquad \square$$

Next we exhibit an example of a nonnormal almost $\mathcal{S}$-manifold which has vanishing Tanaka–Webster curvature without being locally CR-symmetric.

Set

$$\mathfrak{m} = \mathbb{R}^{2k} \oplus \mathbb{R}^s = V_1 \oplus V_2, \ s \geqslant 2$$

and denote by $\{X_1, \dots, X_k, JX_1, \dots, JX_k\}$ the standard basis of $\mathbb{R}^{2k}$ endowed with the complex structure J associated with the matrix $\begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix}$.

Moreover let $\{\xi_1, \dots, \xi_s\}$ denote the natural basis of V_2 and let g be the inner product on $\mathfrak{m}$ obtained by declaring the basis $\{X_i, JX_i, \xi_j\}$ to be orthonormal. Let $\varphi : \mathfrak{m} \to \mathfrak{m}$ be the natural f-structure on $\mathfrak{m}$, i.e. φ is the endomorphism which coincides with J on V_1 and vanishes on V_2.

We also denote by U the endomorphism of $\mathfrak{m}$ which is associated to the matrix

$$\begin{pmatrix} I_k & 0 & 0 \\ 0 & -I_k & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Notice that $U\varphi = -\varphi U$.

We denote by $\mathfrak{h}$ the Lie subalgebra of $\mathrm{End}(\mathfrak{m})$ consisting of all endomorphisms which vanish on V_2 and annihilate the tensors φ, g and U when extended to the tensor algebra of $\mathfrak{m}$ as derivations. We remark that

$$A \in \mathfrak{so}(k) \mapsto \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

provides a Lie-algebra isomorphism $\mathfrak{so}(k) \cong \mathfrak{h}$. In particular, $\mathfrak{h}$ is compact semisimple provided $k \geqslant 3$.

Now we define a Lie algebra structure on $\mathfrak{g} := \mathfrak{h} \oplus \mathfrak{m}$ as follows:

$$[X, Y] := -2g(X, JY)e, \quad [v, X] := a(v)UX = -[X, v],$$
$$[A, X] = A \cdot X - [X, A], \quad [A, v] := 0, \quad [v, w] := 0, \quad [A, B] := AB - BA$$

for each $X, Y \in V_1$, $v, w \in V_2$, $A \in \mathfrak{h}$. Here $e := \sum_i \xi_i \in V_2$, and $a \colon V_2 \to \mathbb{R}$ is a fixed nonnull linear functional such that $a(e) = 0$.

Let G be the simply connected Lie group with Lie algebra $\mathfrak{g}$ and let H denote the analytic subgroup corresponding to the subalgebra $\mathfrak{h}$. Assuming $k \geqslant 3$, we have that H is compact, so that $M = G/H$ is a reductive homogeneous space. In [13] it is verified that G/H carries a G-invariant CR-integrable almost $\mathcal{S}$-structure which is *not* normal; moreover, the vectors ξ_i extend in a G-invariant fashion to a characteristic frame and the associated Tanaka–Webster connection is the canonical G-invariant linear connection. It follows that $\tilde{R} = 0$. On the other hand, M is not CR-symmetric provided $a(\xi_i) \neq a(\xi_j)$ for some $i, j \in \{1, \ldots, s\}$. Indeed, under the natural identification $T_oM \equiv \mathfrak{m}$, $o = H$, we have $(h_i)_o = a(\xi_i)\,\varphi U$. Hence, $h_i \neq h_j$ while a necessary condition for M to be CR-symmetric is that the operators h_i are all equal (cf. Prop. 6.2).

References

1. Blair D. E.: *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Math. 509, Springer-Verlag, Berlin, 1976.
2. Blair D. E.: *Riemannian Geometry of Contact and Symplectic Manifolds*, Progr. in Math. 203, Birkhäuser, Boston, 2002.
3. Blair D. E.: Geometry of manifolds with structural group $\mathsf{U}(n) \times \mathsf{O}(s)$, *J. Differential Geom.* **4** (1976), 155–167.
4. Blair D. E., Koufogiorgos T. and Papantoniou B. J.: Contact metric manifolds satisfying a nullity condition, *Israel J. Math.* **91** (1995), 189–214.
5. Blair D. E. and Vanhecke L.: Symmetries and φ-symmetric spaces, *Tohoku Math. Journ.* **39** (1987), 373–383.
6. Boeckx E. : Locally φ-symmetric contact metric spaces, *Arch. Math.* **72** (1999), 466–472.
7. Duggal K. L., Ianus S. and Pastore A. M.: Maps interchanging f-structures and their harmonicity, *Acta Appl. Math.* **67** (2001), 91–115.
8. Ianus S. and Pastore A. M.: Harmonic maps on contact metric manifolds, *Ann. Math. Blaise Pascal* **2** (1995), 43–53.
9. Jiménez J. A. and Kowalski O.: The classification of φ-symmetric Sasakian manifolds, *Mh. Math.* **115** (1993), 83–98.
10. Kaup W. and Zaitsev D.: On symmetric Cauchy–Riemann manifolds, *Adv. Math.* **149** (2000), 145–181.
11. Kobayashi S. and Nomizu K.: *Foundations of Differential Geometry*, Vol. I, Interscience Publishers, New York, 1963.
12. Kowalski O.: *Generalized Symmetric Spaces*, Lecture. Notes in Math. 805, Springer-Verlag, Berlin, 1980.

13. Lotta A. and Pastore A. M.: The Tanaka–Webster connection for almost $\mathcal{S}$-manifolds and Cartan geometry, to appear on *Arch. Math. (Brno)*.
14. Medori C. and Nacinovich M.: Levi–Tanaka algebras and homogeneous CR manifolds, *Compositio Math.* **109** (1997), 195–250.
15. Mizner R. I.: Almost CR structures, f-structures, almost product structures and associated connections, *Rocky Mount. J. Math.* **23** (1993), 1337–1359.
16. Okumura M.: Some remarks on spaces with a certain contact structure, *Tohoku Math. J.* **14** (1962), 135–145.
17. Takahashi T.: Sasakian φ-symmetric spaces, *Tohoku Math. J.* **29** (1977), 91–113.
18. Tanno S.: Variational problems on contact Riemannian manifolds, *Trans. Amer. Math. Soc.* **314** (1989), 349–379.