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Geometry of Log-concave Functions and Measures

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Abstract. We present a view of log-concave measures, which enables one to build an isomorphic theory for high dimensional log-concave measures, analogous to the corresponding theory for convex bodies. Concepts such as duality and the Minkowski sum are described for log-concave functions. In this context, we interpret the Brunn–Minkowski and the Blaschke–Santaló inequalities and prove the two corresponding reverse inequalities. We also prove an analog of Milman's quotient of subspace theorem, and present a functional version of the Urysohn inequality.

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1. Introduction

A measure μ on $\mathbb{R}^n$ is log-concave if for any measurable $A, B \subset \mathbb{R}^n$ and $0 < \lambda < 1$,

$$\mu(\lambda A + (1-\lambda)B) \geqslant \mu(A)^{\lambda}\mu(B)^{1-\lambda}$$

whenever $\lambda A + (1-\lambda)B$ is measurable, where $A + B = \{a + b; a \in A, b \in B\}$ and $\lambda A = \{\lambda a; a \in A\}$. Such measures naturally appear in convex geometry, since the Brunn–Minkowski inequality states that uniform measures on convex sets are log-concave measures (including the Lebesgue measure on $\mathbb{R}^n$). The Brunn–Minkowski inequality also implies that lower-dimensional marginals of uniform measures on convex bodies are log-concave. In fact, marginals of uniform measures on convex bodies are essentially the only source for log-concave measures, as these marginals form a dense subset in the class of log-concave measures. A function $f: \mathbb{R}^n \to [0, \infty)$ is log-concave if $\log f$ is concave. As was shown in [7], a measure μ on $\mathbb{R}^n$ whose support is not contained in any affine hyperplane is a log-concave measure if and only if it is absolutely continuous with respect to the Lebesgue measure, and its density is a log-concave function.

As log-concave measures retain some features of uniform measures on convex bodies, many results on uniform measures on convex bodies may be generalized to log-concave measures (two samples among many are [5] and [6]). However, it has recently become clear to the authors that such a generalization may shed new light on uniform measures on convex bodies, and may help clarify the difficult open problems regarding these measures. Such an approach is demonstrated in [9], and has led there to some progress regarding the slicing problem. Therefore we believe that a systematic study of the geometry of log-concave measures is essential in order to understand uniform measures on convex bodies.

In this paper we present some steps in this direction, and we recover most of the isomorphic results for convex bodies in the context of log-concave functions. When trying to generalize the geometry of convex bodies to log-concave measures, the first problem we encounter is that of duality. For a convex body $K \subset \mathbb{R}^n$ which is centrally-symmetric (i.e., $K = -K$), its polar is defined by $K^\circ = \{x \in \mathbb{R}^n; \forall y \in K, \langle x, y\rangle \leqslant 1\}$. The polar body is a fundamental tool in convex geometry. We show that a suitable variation of the Legendre transform may constitute a proper replacement in the context of functions. Given a function $f\colon \mathbb{R}^n \to \mathbb{R}$, its Legendre transform is defined by

$$\mathcal{L}f(x) = \sup_{y\in\mathbb{R}^n} [\langle x, y\rangle - f(y)].$$

The function $\mathcal{L}(f)$ is convex. If f is convex as well as continuous, then $\mathcal{L}(\mathcal{L}(f)) = f$. The Legendre transform is a classical operation, which was used, for example, in the derivation of Hamilton equations in classical mechanics (e.g., [1]). Since the most natural domain for the Legendre transform is convex functions, we define the dual of a log-concave function $f\colon \mathbb{R}^n \to [0, \infty)$ by

$$f^\circ = e^{-\mathcal{L}(-\log f)}.$$

This definition is closely related to the duality of convex bodies. Let $\|\cdot\|, \|\cdot\|_*$ be the norms that K, K° are their unit balls, correspondingly. Then the dual functions to $1_K, \mathrm{e}^{-\|x\|}, \mathrm{e}^{-\frac{1}{2}\|x\|^2}$ are exactly the functions $\mathrm{e}^{\|x\|_*}, 1_{K^\circ}$, and $\mathrm{e}^{-\frac{1}{2}\|x\|_*^2}$, respectively. Let us demonstrate the usefulness of this definition with the Blaschke–Santaló inequality and its converse. These inequalities state that there exists a numerical constant $c > 0$ such that for any centrally symmetric convex body $K \subset \mathbb{R}^n$,

$$c < \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(D)}\frac{\mathrm{Vol}(K^\circ)}{\mathrm{Vol}(D)}\right)^{\frac{1}{n}} \leqslant 1 \tag{1}$$

where D is the standard Euclidean unit ball in $\mathbb{R}^n$. The right-most inequality is due to Santaló (see, e.g., [11] for a clear presentation), and the left-most one was proved by Bourgain and Milman [8]. Log-concave functions satisfy corresponding inequalities, which are functional analogs of Santaló and reverse Santaló inequalities:

THEOREM 1.1. *There exist universal constants $c, C > 0$ such that for any dimension n and for any $f\colon \mathbb{R}^n \to [0, \infty)$, an even log-concave function with $0 < \int f < \infty$, we have*

$$c < \left(\int_{\mathbb{R}^n} f \int_{\mathbb{R}^n} f^\circ\right)^{\frac{1}{n}} \leqslant C.$$

The optimal value of the constant C from Theorem 1.1 is known to be 2π (see [2] and [3]). Regarding the equality case in the right-most inequality in Theorem 1.1; in the case of convex bodies, it is known that

$$\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(D)}\frac{\mathrm{Vol}(K^\circ)}{\mathrm{Vol}(D)}=1$$

if and only if K is an ellipsoid. In the functional version of the Santaló inequality, the role of ellipsoids is replaced by Gaussian functions (functions of the form $c\mathrm{e}^{-\langle Ax,x\rangle}$ for a positive-definite matrix A and a positive $c>0$). Note that the standard Gaussian $\mathrm{e}^{-\frac{|x|^2}{2}}$ is the only function which is dual to itself. As is proved in [3] (the equality case appears in [2]):

THEOREM 1.2. *Let* $f\colon \mathbb{R}^n \to [0,\infty)$ *be an even function such that* $0<\int f<\infty$. *Then,*

$$\int_{\mathbb{R}^n} f \int_{\mathbb{R}^n} f^\circ \leqslant (2\pi)^n$$

where equality holds exactly for Gaussians.

An operation related to Legendre transform is the Asplund sum. In a completely analogous way to the definition of the Asplund sum, we define the Asplund product of two functions $f,g\colon\ \mathbb{R}^n\to[0,\infty)$ as

$$f\star g(x)=\sup_{x_1+x_2=x} f(x_1)f(x_2).$$

The Asplund product preserves log-concavity. Also, $(f\star g)^\circ=f^\circ g^\circ$, i.e., the dual to the Asplund product is simply the usual product of the dual functions. The Asplund product of log-concave functions is analogous to the Minkowski sum of convex bodies. Indeed, $1_A\star 1_B=1_{A+B}$ for any $A,B\subset\mathbb{R}^n$. A central inequality connected with the Minkowski sum of two bodies $A,B\subset\mathbb{R}^n$ and a parameter $0\leqslant\lambda\leqslant 1$ is the Brunn–Minkowski inequality:

$$\mathrm{Vol}(\lambda A+(1-\lambda)B)\geqslant \mathrm{Vol}(A)^\lambda \mathrm{Vol}(B)^{1-\lambda}.$$

The Brunn–Minkowski inequality is also known in the following equivalent formulation,

$$\mathrm{Vol}(A+B)^{\frac{1}{n}}\geqslant \mathrm{Vol}(A)^{\frac{1}{n}}+\mathrm{Vol}(B)^{\frac{1}{n}}$$

for any $A,B\subset\mathbb{R}^n$. Define $\lambda\cdot f=f^\lambda(x/\lambda)$. Note that $f\star f=2\cdot f$ for a log-concave f, and that $(\lambda\cdot f)^\circ=(f^\circ)^\lambda$. The function $\lambda\cdot f$ is the analog of a λ-homothety of a convex body. The functional analog of the Brunn–Minkowski inequality is the Prekopa–Leindler inequality (e.g., [15]). In our terminology, it states that given $f,g\colon\mathbb{R}^n\to[0,\infty)$ and $0\leqslant\lambda\leqslant 1$,

$$\int [\lambda\cdot f]\star[(1-\lambda)\cdot g]\geqslant \left(\int f\right)^\lambda \left(\int g\right)^{1-\lambda}. \tag{2}$$

Therefore, the Prekopa–Leindler inequality constitutes a complete analog to the Brunn–Minkowski inequality for bodies, where the Minkowski sum of bodies is replaced by the Asplund product of functions (see also [2]). Here we prove the analog of the inverse Brunn–Minkowski inequality (see [13] or the book [15]), as follows. We denote $(f\circ T)(x)=f(Tx)$.

THEOREM 1.3. *Let $f,g:\mathbb{R}^n\to[0,\infty)$ be even log-concave functions with $f(0)=g(0)=1$. Then there exist $T_f,T_g\in \mathrm{SL}(n)$ such that $\tilde f=f\circ T_f$ and $\tilde g=g\circ T_g$ satisfy*

$$\left(\int \tilde f\star\tilde g\right)^{\frac{1}{n}}<C\left[\left(\int\tilde f\right)^{\frac{1}{n}}+\left(\int\tilde g\right)^{\frac{1}{n}}\right]$$

where $C>0$ is a universal constant, T_f depends solely on f, and T_g depends solely on g.

Further inequalities which are the analogs to the quotient of subspace theorem and related results are formulated and proved in Section 2. In Section 2 we also prove Theorem 1.1 and Theorem 1.3. The main tool in the proofs of these isomorphic results is a method of attaching a convex body to any log-concave function, which is due to Ball [4].

In Section 3, a functional analog of the mean width is introduced. We show that an analog of the Urysohn inequality holds in this setting. Throughout this paper, the letters c,C,c_1,c', etc., denote positive universal constants whose value is not necessarily the same in various appearances. $A\approx B$ means that $cA<B<CA$ for some universal constants c and C. A convex body is a convex set with a non-empty interior in $\mathbb{R}^n$.

2. Convex Bodies

Let $f:\mathbb{R}^n\to\mathbb{R}$ be an even log-concave function. For $x\in\mathbb{R}^n$ define

$$\|x\|_f=\left(\int_0^\infty f(rx)r^{n-1}\,\mathrm{d}r\right)^{-\frac{1}{n}}.$$

In [4] it is proven that $\|\cdot\|_f$ is a norm on $\mathbb{R}^n$. Denote its unit ball by K_f, the convex body that is associated with f. Then K_f is convex, centrally symmetric, and

$$\mathrm{Vol}(K_f)=\frac{1}{n}\int_{S^{n-1}}\int_0^\infty f(rx)r^{n-1}\,\mathrm{d}r\,\mathrm{d}x=\frac{1}{n}\int_{\mathbb{R}^n}f$$

where $S^{n-1}=\partial D$. Next, we shall elaborate on some connections between the body K_f and the log-concave function f. We start with a one-dimensional lemma, in the spirit of the Laplace method. Recall (e.g., [16]) that if $g:\mathbb{R}\to[0,\infty]$ is convex, then its left and right derivatives, denoted here as g_{L} and g_{R}, exist whenever g is finite. The function $\varphi(t)=g(t)-n\log t$ is convex in $(0,\infty)$, and if $g\not\equiv\mathrm{Const}$ then

$\varphi(t) \xrightarrow{t\to 0,\infty} \infty$. By strict convexity, there exists a unique critical point t_0 of φ such that $g(t) - n\log t$ is nonincreasing for $t < t_0$ and nondecreasing for $t > t_0$. Note that it is possible that $g(t) = \infty$ for $t \geqslant t_0$, however, in that case,

$$\lim_{t\to t_0^-} [g(t) - n\log t] = \inf_{t\in\mathbb{R}} [g(t) - n\log t].$$

LEMMA 2.1. *Let* $g : [0,\infty) \to [0,\infty]$ *be a nondecreasing convex function such that* $g(0) = 0$ *and* $g \not\equiv 0$. *Denote* $M = \sup_{t>0} \mathrm{e}^{-g(t)} t^n$, *and let* t_0 *be the corresponding (unique) critical point. Then,*

$$M\frac{t_0}{n+1} \leqslant \int_0^\infty \mathrm{e}^{-g(t)} t^n \,\mathrm{d}t < cM\frac{t_0}{\sqrt{n}}, \quad g(t_0) \leqslant n, \quad g(2t_0) \geqslant n$$

and $g(ln) \geqslant (l-1)n$ *for any* $l > 1$. *In addition,*

$$\int_{5t_0}^\infty \mathrm{e}^{-g(t)} t^n \,\mathrm{d}t < \mathrm{e}^{-2n} \int_0^\infty \mathrm{e}^{-g(t)} t^n \,\mathrm{d}t.$$

Proof. The left-most inequality is straightforward: since $g(t)$ is nondecreasing,

$$\int_0^\infty \mathrm{e}^{-g(t)} t^n \,\mathrm{d}t \geqslant \mathrm{e}^{-\lim_{t\to t_0^-} g(t)} \int_0^{t_0} t^n \,\mathrm{d}t = M\frac{t_0}{n+1}.$$

To prove the right-most inequality, recall that t_0 is a critical point of the convex function $\varphi(t) = g(t) - n\log t$. Hence $\varphi_{\mathrm{L}}(t_0) \leqslant 0 \leqslant \varphi_{\mathrm{R}}(t_0)$. We conclude that $g_{\mathrm{L}}(t_0) \leqslant n/t_0 \leqslant g_{\mathrm{R}}(t_0)$ and $g(t_0) + n/t_0(t - t_0)$ is a supporting line to g at t_0. Since g is convex, $g(t) \geqslant g(t_0) + n/t_0(t - t_0)$ for every t, and

$$\int_0^\infty t^n \mathrm{e}^{-g(t)} \,\mathrm{d}t \leqslant \mathrm{e}^{n-g(t_0)} \int_0^\infty t^n \mathrm{e}^{-\frac{nt}{t_0}} \,\mathrm{d}t = \mathrm{e}^{n-g(t_0)} \left(\frac{t_0}{n}\right)^{n+1} \int_0^\infty t^n \mathrm{e}^{-t} \,\mathrm{d}t$$
$$= \mathrm{e}^{-g(t_0)} t_0^n \frac{\mathrm{e}^n n!}{n^n} \frac{t_0}{n} \approx M\frac{t_0}{\sqrt{n}}.$$

Additionally, for $t < t_0$, we have $g_{\mathrm{R}}(t) \leqslant n/t_0$ and, hence, $g(t_0) \leqslant g(0) + \int_0^{t_0} n/t_0 = n$. Also, $g(2t_0) \geqslant g(t_0) + n/t_0(2t_0 - t_0) \geqslant n$. The estimate for $g(lt_0)$ follows the same argument. The last assertion follows from

$$\int_{5t_0}^\infty \mathrm{e}^{-g(t)} t^n \,\mathrm{d}t \leqslant \mathrm{e}^{n-g(t_0)} \int_{5t_0}^\infty t^n \mathrm{e}^{-\frac{tn}{t_0}} \,\mathrm{d}t << \mathrm{e}^{-2n} M t_0. \qquad \square$$

A convex function is differentiable almost everywhere (e.g., [17]). Yet, we still need a notion of a gradient for the relatively few nonsmooth points. For a convex function g we define its gradient in a nonsmooth point x (see, e.g., [17]) as $\nabla g(x) = \{y \in \mathbb{R}^n;$ for all z, $g(z) \geqslant g(x_0) + \langle y, z - x\rangle\}$. For an even log-concave function f define

$$\bar{K}_f = \{x \in \mathbb{R}^n; \langle \nabla(-\log f)(x), x\rangle \leqslant n - 1\},$$

where for a nonsmooth point x, the condition $\langle \nabla(-\log f)(x), x\rangle \leqslant n-1$ should be understood as $\exists y \in \nabla(-\log f)(x), \langle y, x\rangle \leqslant n-1$. Define also

$$\bar{\bar{K}}_f = \{x \in \mathbb{R}^n;\ f(x) > \mathrm{e}^{-n}\}.$$

Then $\bar{\bar{K}}_f$ is clearly convex, but $\bar{K}_f$ is not necessarily convex. Nevertheless, we show that K_f, $\bar{K}_f$ and $\bar{\bar{K}}_f$ are close to each other. The radial function of a convex body K in direction θ is

$$r(K,\theta) = \sup\{r > 0;\ r\theta \in K\}.$$

LEMMA 2.2. *Assume that $f : \mathbb{R}^n \to [0,\infty)$ is an even log-concave function with $f(0)=1$. Then, $K_f \subset \bar{K}_f \subset \bar{\bar{K}}_f \subset cK_f$ for some universal constant $c > 0$.*

Proof. Fix $\theta \in S^{n-1}$ and let $g(r) = -\log f(r\theta)$. If $g \equiv 0$ then $r(K_f,\theta) = r(\bar{K}_f,\theta) = r(\bar{\bar{K}}_f,\theta) = \infty$. Otherwise, denote $M = \sup_{t>0} \mathrm{e}^{-g(t)} t^{n-1}$, and let t_0 be the corresponding critical point. By Lemma 2.1,

$$r(K_f,\theta) \approx (Mt_0)^{1/n} = \left(\mathrm{e}^{-g(t_0)} t_0^n\right)^{1/n} \approx t_0$$

and actually, $r(K_f,\theta) < t_0$. On the other hand, since $g_L(t_0)t_0 \leqslant n-1 \leqslant g_R(t_0)t_0$, we have $r(\bar{K}_f,\theta) = t_0$, and since $t_0 \leqslant g^{-1}(n) \leqslant 2t_0$, then also $t_0 \leqslant r(\bar{\bar{K}}_f,\theta) \leqslant 2t_0$. □

COROLLARY 2.3. *Let $f : \mathbb{R}^n \to [0,\infty)$ be an even log-concave function with $f(0) = 1$. Let $E \subset \mathbb{R}^n$ be a λn-dimensional subspace, for some $0 < \lambda < 1$. Then*

$$c_1 \lambda K_f \cap E \subset K_{f|_E} \subset c_2 K_f \cap E$$

where $f|_E$ is the restriction of f to the subspace E and $c_1, c_2 > 0$ are universal constants.

Proof. By the log-concavity of f,

$$\bar{\bar{K}}_f \cap E = \{x \in E;\ f(x) > \mathrm{e}^{-n}\} \subset \frac{c}{\lambda}\{x \in E;\ f(x) > \mathrm{e}^{-\lambda n}\} = \frac{c}{\lambda}\bar{\bar{K}}_{f|_E}.$$

According to Lemma 2.2,

$$K_{f|_E} \cap E \subset c\bar{\bar{K}}_{f|_E} \subset c\bar{\bar{K}}_f \cap E \subset \frac{c'}{\lambda}\bar{\bar{K}}_{f|_E} \subset \frac{c''}{\lambda} K_{f|_E}$$

and since $\bar{\bar{K}}_f \subset K_f \subset \bar{\bar{K}}_f$, the corollary follows. □

By a polar integration of the last inequality in Lemma 2.1, we obtain the following:

COROLLARY 2.4. *Let $f:\mathbb{R}^n\to[0,\infty)$ be an even log-concave function with $f(0)=1$ and a finite integral. Then*

$$\int_{cK_f} f \geqslant \left(1-\mathrm{e}^{-2n}\right)\int_{\mathbb{R}^n} f,$$

where $c>0$ is a universal constant.

Next, we should exhibit a connection between K_{f° and K_f°. □

PROPOSITION 2.5. *Assume that $f:\mathbb{R}^n\to\mathbb{R}$ is even and log-concave. Then,*

$$c_1 n K_f^\circ \subset K_{f^\circ} \subset c_2 n K_f^\circ$$

where $c_1, c_2>0$ are universal constants.

Proof. Since $(cf)^\circ=(1/c)f^\circ$ and $K_{cf}=c^{1/n}K_f$, multiplying f by a scalar if needed, we may assume that $f(0)=1$. Denote $g=-\log f$. Assume first that g is smooth and strictly convex. A crucial simple observation is that $\nabla\mathcal{L}(g)(x)=(\nabla g)^{-1}(x)$ (e.g., [16]). Hence,

$$(\nabla g)\bar{K}_f=\{(\nabla g)x;\ \langle x,(\nabla g)x\rangle\leqslant n-1\}=\{x;\ \langle(\nabla g)^{-1}x,x\rangle\leqslant n-1\}=\bar{K}_{f^\circ}.$$

Let $x_0\in\partial\bar{K}_f$. Then, $\langle x_0,\nabla g(x_0)\rangle=n-1$. Denote by $\|y\|=\sup_{x\in\bar{K}_f}\langle x,y\rangle$, the norm that $\left(\bar{K}_f\right)^\circ$ is its unit ball. Then,

$$x_0\in\partial\bar{K}_f \quad\Rightarrow\quad \|\nabla g(x_0)\|\geqslant n-1$$

and, hence, $(n-1)\left(\bar{K}_f\right)^\circ\subset(\nabla g)\bar{K}_f=\bar{K}_{f^\circ}$. By Lemma 2.2,

$$cnK_f^\circ\subset(n-1)\left(\bar{K}_f\right)^\circ\subset\bar{K}_{f^\circ}\subset c'K_{f^\circ}.$$

Regarding the opposite inclusion, since g is convex, for any $y\in\mathbb{R}^n$,

$$g(y)\geqslant g(x_0)+\langle\nabla g(x_0),y-x_0\rangle=\langle\nabla g(x_0),y\rangle+g(x_0)+1-n.$$

If, furthermore, $y\in\bar{\bar{K}}_f$ then $g(y)\leqslant n$ and

$$\langle\nabla g(x_0),y\rangle\leqslant g(y)+n-1-g(x_0)<2n-1.$$

Hence $\nabla g(x_0)\in 2n\left(\bar{\bar{K}}_f\right)^\circ$ and $\bar{K}_{f^\circ}\subset 2n\left(\bar{\bar{K}}_f\right)^\circ$. An application of Lemma 2.2 concludes the proof under the assumption that g is smooth and strictly convex. For an arbitrary function, an approximation argument is needed. For instance, we may define $f_\varepsilon=\left(f\star\mathrm{e}^{-\varepsilon|x|^2}\right)\mathrm{e}^{-\varepsilon|x|^2}$. Then f_ε is smooth and strictly log-concave for any $\varepsilon>0$. If ε is small enough, the bodies $K_{f_\varepsilon}, K_{f_\varepsilon^\circ}$ are close to K_f, K_{f°, and the proposition follows. □

Proof of Theorem 1.1. By Proposition 2.5,

$$\left(\int_{\mathbb{R}^n} f \int_{\mathbb{R}^n} f^\circ\right)^{\frac{1}{n}} = \left(n^2 \mathrm{Vol}(K_f)\mathrm{Vol}(K_{f^\circ})\right)^{\frac{1}{n}} \approx \left(\mathrm{Vol}(K_f)\mathrm{Vol}(cnK_f^\circ)\right)^{\frac{1}{n}}$$

and by Santaló's inequality and its converse (e.g., [8] and [11]),

$$\left(\int f \int f^\circ\right)^{\frac{1}{n}} \approx n\mathrm{Vol}(D)^{\frac{2}{n}} \approx c. \qquad \square$$

The reverse Brunn–Minkowski inequality for convex bodies (see [13]) is the following theorem:

THEOREM 2.6. *Let $K, P \subset \mathbb{R}^n$ be centrally symmetric convex bodies. Then there exist invertible linear transformations T_K, T_P of determinant one, such that $\tilde{K} = T_K(K)$, $\tilde{P} = T_P(P)$ satisfy*

$$\mathrm{Vol}(\tilde{K} + \tilde{P})^{\frac{1}{n}} < C\left[\mathrm{Vol}(\tilde{K})^{\frac{1}{n}} + \mathrm{Vol}(\tilde{P})^{\frac{1}{n}}\right]$$

where $C > 0$ is a numerical constant, T_K depends solely on K and T_P depends solely on P.

LEMMA 2.7. *Let f, g be even log-concave functions with $f(0) = g(0) = 1$. Then,*

$$c_1 K_{f\star g} \subset K_f + K_g \subset c_2 K_{f\star g}$$

where $c_1, c_2 > 0$ are numerical constants.

Proof. $x \in \bar{\bar{K}}_{f\star g}$ implies that there exists $x_1 + x_2 = x$ with $f(x_1)g(x_2) \geqslant e^{-n}$. Since the functions are not larger than one, necessarily $x_1 \in \bar{\bar{K}}_f$ and $x_2 \in \bar{\bar{K}}_g$, hence $\bar{\bar{K}}_{f\star g} \subset \bar{\bar{K}}_f + \bar{\bar{K}}_g$. Combining this with Lemma 2.2 we conclude the left-most inclusion. The other inclusion follows from the fact that for any $x \in \bar{\bar{K}}_f + \bar{\bar{K}}_g$ we have that $(f\star g)(x) \geqslant e^{-2n}$. From Lemma 2.1 it follows that $\bar{\bar{K}}_f + \bar{\bar{K}}_g \subset \bar{\bar{K}}_{f\star g}$. $\square$

Proof of Theorem 1.3. Note that for any T, a linear transformation, $K_{f\circ T} = T^{-1}(K_f)$. By Theorem 1.3 we may choose T_f, T_g, linear transformations of determinant one, such that $K_{\tilde{f}}$ and $K_{\tilde{g}}$ satisfy

$$\mathrm{Vol}\left(K_{\tilde{f}} + K_{\tilde{g}}\right)^{\frac{1}{n}} < C\left[\mathrm{Vol}\left(K_{\tilde{f}}\right)^{\frac{1}{n}} + \mathrm{Vol}\left(K_{\tilde{g}}\right)^{\frac{1}{n}}\right].$$

According to Lemma 2.7,

$$\left(\int \tilde{f}\star\tilde{g}\right)^{\frac{1}{n}} = n^{\frac{1}{n}}\mathrm{Vol}\left(K_{\tilde{f}\star\tilde{g}}\right)^{\frac{1}{n}} \approx \mathrm{Vol}\left(K_{\tilde{f}} + K_{\tilde{g}}\right)^{\frac{1}{n}}$$
$$< C\left[\mathrm{Vol}\left(K_{\tilde{f}}\right)^{\frac{1}{n}} + \mathrm{Vol}\left(K_{\tilde{g}}\right)^{\frac{1}{n}}\right] \leqslant C\left[\left(\int \tilde{f}\right)^{\frac{1}{n}} + \left(\int \tilde{g}\right)^{\frac{1}{n}}\right].$$

$\square$

Given two functions $f, g : \mathbb{R}^n \to [0, \infty)$ with finite mass, we say that $f \prec_\alpha g$ if there exists a set $A \subset \mathbb{R}^n$ such that for any $x \in A$,

$$f^{\frac{1}{n}}(x) \leqslant e g^{\frac{1}{n}}\left(\frac{x}{\alpha}\right)$$

and $\int_A f > \left(1 - \mathrm{e}^{-2n}\right) \int f$. We say that $f \sim_\alpha g$ if $f \prec_\alpha g$ and $g \prec_\alpha f$. If $f \sim_\alpha g$ for α being a numerical constant, we say that f and g are 'roughly isomorphic'.

LEMMA 2.8. *Let $f, g : \mathbb{R}^n \to [0, \infty)$ be even log-concave functions with $f(0) = g(0) = 1$ and finite, positive integrals. Then for any $\alpha > 1$,*

$$K_f \subset c_1 \alpha K_g \quad \Longrightarrow \quad f \prec_\alpha g \quad \Longrightarrow \quad K_f \subset c_2 \alpha K_g$$

where $c_1, c_2 > 0$ are universal constants.

Proof. Assume that $K_f \subset c\alpha K_g$. By Corollary 2.4,

$$\int_{\mathbb{R}^n \setminus c' \bar{\bar{K}}_f} f < \mathrm{e}^{-2n} \int_{\mathbb{R}^n} f.$$

Denote $A = c' \bar{\bar{K}}_f$. If $c, c' > 0$ are chosen properly, for any $x \in A$ we have that $x/\alpha \in \bar{\bar{K}}_g$, and hence $g\left(\frac{x}{\alpha}\right)^{\frac{1}{n}} \geqslant 1/e \geqslant (1/e) f(x)^{\frac{1}{n}}$. Therefore $f \prec_\alpha g$. Regarding the other statement, assume that $f \prec_\alpha g$ and let A be the corresponding witness set. If $x \in \frac{1}{\alpha}\left[\bar{\bar{K}}_f \cap A\right]$ then $g(x)^{\frac{1}{n}} \geqslant (1/e) f(\alpha x)^{1n} \geqslant 1/e^2$, and by Lemma 2.1 we get that $x \in 2\bar{\bar{K}}_g$. Since $\bar{\bar{K}}_g$ is a convex set, we conclude that $\mathrm{conv}\left(\bar{\bar{K}}_f \cap A\right) \subset 2\alpha \bar{\bar{K}}_g$. It remains to show that $c' \bar{\bar{K}}_f \subset \mathrm{conv}\left(\bar{\bar{K}}_f \cap A\right)$. This would follow if we prove that $\mathrm{Vol}\left(\bar{\bar{K}}_f \cap A\right) > \left(1 - \mathrm{e}^{-\frac{n}{2}}\right) \mathrm{Vol}\left(\bar{\bar{K}}_f\right)$ (e.g., Lemma 2.2 in [10]). Finally, note that $\int_A f > \left(1 - \mathrm{e}^{-2n}\right) \int f$ and that $f(x) > \mathrm{e}^{-n}$ for any $x \in \bar{\bar{K}}_f$. We conclude that

$$\mathrm{e}^{-n} \mathrm{Vol}\left(\bar{\bar{K}}_f \setminus A\right) < \int_{\mathbb{R}^n \setminus A} f < \mathrm{e}^{-2n} \int_{\mathbb{R}^n} f$$

and hence

$$\mathrm{Vol}\left(\bar{\bar{K}}_f \setminus A\right) < \mathrm{e}^{-n} \int f = n \mathrm{e}^{-n} \mathrm{Vol}(K_f) < \mathrm{e}^{-\frac{n}{2}} \mathrm{Vol}\left(\bar{\bar{K}}_f\right). \qquad \square$$

Lemma 2.8 implies that if $K_f = K_g$, then $f \sim_c g$ for some universal $c > 0$. In particular, if K_f is a Euclidean ball, then f is roughly isomorphic to a Gaussian. We may now formulate more analogs of isomorphic results from the asymptotic theory of convex bodies. It is known (see [14]) that given a centrally symmetric convex body $K \subset \mathbb{R}^n$, there exists $\tilde{K}$, a linear image of K, and two rotations $U_1, U_2 \in O(n)$ such that if we define $T = U_1(\tilde{K}) + \tilde{K}$ and $P = U_2(T) \cap T$, then $c_1 D \subset P \subset c_2 D$ for some universal $c_1, c_2 > 0$. The functional analog is presented below.

PROPOSITION 2.9. *Let* $f\colon \mathbb{R}^n \to [0,\infty)$ *be an even log-concave function with* $f(0)=1$. *Then there exists* $\tilde{f}(= f\circ T_f)$ *a linear image of* f *and two rotations* $U_1, U_2 \in O(n)$ *such that if* $g = \left(\tilde{f}\circ U_1\right)\star \tilde{f}$ *and* $h=(g\circ U_2)\cdot g$, *then* $h\sim_C G$ *where* $G(x)=\mathrm{e}^{-\frac{|x|^2}{2}}$ *is the standard Gaussian, and* $C>0$ *is a numerical constant.*

Proof. By Lemma 2.7, $K_{f\star g}$ is close to K_f+K_g. It is equally easy to realize that K_{fg} is close to $K_f\cap K_g$ in the same sense. Using the corresponding result for convex bodies, we conclude that $c_1D\subset K_h\subset c_2D$ for some universal constants $c_1,c_2>0$. The proposition follows by Lemma 2.8. □

Milman's quotient of subspace theorem [12] is the following statement.

THEOREM 2.10. *Let* $K\subset\mathbb{R}^n$ *be a convex, centrally symmetric body. Then there exist subspaces* $E\subset F\subset\mathbb{R}^n$ *with* $\dim(E)>n/2$ *and an ellipsoid* $\mathcal{E}\subset E$ *such that*

$$c_1\mathcal{E}\subset \mathrm{Proj}_E(K\cap F)\subset c_2\mathcal{E}$$

where $c_1,c_2>0$ *are universal constants.*

Let $E\subset\mathbb{R}^n$ be a subspace. Since $(f|_E)^\circ(x)=\sup_{y\in E^\perp} f^\circ(x+y)$, we naturally define $\mathrm{Proj}_E(f)=\sup_{y\in E^\perp} f(x+y)$. Assume that $\dim(E)=\lfloor\frac{n}{2}\rfloor$. Note that according to Corollary 2.3, $c_1K_{f|_E}\subset K_f\cap E\subset c_2K_{f|_E}$ for some universal constants $c_1,c_2>0$. We can now formulate the functional analog of the Quotient of subspace theorem. The proof is omitted, as it follows from Theorem 2.10 in a similar way to the previous proofs.

PROPOSITION 2.11. *Let* $f:\mathbb{R}^n\to[0,\infty)$ *be an even log-concave function with* $0<\int f<\infty$. *Then there exist two subspaces* $E\subset F\subset\mathbb{R}^n$ *such that* $\dim(E)>n/2$ *and*

$$\mathrm{Proj}_E(f|_F)\sim_c G$$

where G *is some Gaussian measure, and* $c>0$ *is a numerical constant.*

Remark. There is nothing special about the dimension $\frac{1}{2}n$. For any $0<\lambda<1$ one may find subspaces $E\subset F\subset\mathbb{R}^n$ such that $dim(E)>\lambda n$ and the conclusion of Proposition 2.11 holds with a constant $c(\lambda)$ that depends solely on λ. This follows from a corresponding sharpening of Theorem 2.10 (see [12]).

3. Urysohn Inequality

Let $K,T\subset\mathbb{R}^n$ be two convex, centrally symmetric bodies. A classical theorem due to Minkowski (e.g., [17]) states that $\mathrm{Vol}(K+\lambda T)$ is a polynomial in λ. In particular,

$$\mathrm{Vol}(K+\varepsilon T)=\mathrm{Vol}(K)+\varepsilon n V(T,1;K,n-1)+\mathrm{O}(\varepsilon^2)$$

where $V(T, 1; K, n-1)$ is the corresponding mixed volume (e.g., [17]). Let us define an analogous quantity for the log-concave case. If $f, H : \mathbb{R}^n \to [0, \infty)$ are even log-concave functions of finite positive masses, we define

$$V_H(f) = \lim_{\varepsilon \to 0^+} \frac{\int H \star [\varepsilon \cdot f] - \int H}{\varepsilon}.$$

This limit always exists, because by the Prekopa–Leindler inequality $\int H \star [\varepsilon \cdot f]$ is a log-concave function of ε. Denote $G(x) = \mathrm{e}^{-\frac{|x|^2}{2}}$, the standard Gaussian, and consider the case of $V_G(f)$, which may be viewed as the 'mean width' of a log-concave function, up to some normalization. We denote

$$M^*(f) = 2\frac{V_G(f)}{n \int G} = \frac{V_G(f)}{\frac{n}{2}(2\pi)^{\frac{n}{2}}}.$$

For a centrally symmetric convex body $K \subset \mathbb{R}^n$, the usual definition is $M^*(K) = \int_{S^{n-1}} \sup_{y \in K} \langle x, y \rangle \mathrm{d}\sigma(x)$ where σ is the unique rotation invariant probability measure on S^{n-1}. In the case $f = 1_K$, a straightforward calculation yields

$$V_G(1_K) = \lim_{\varepsilon \to 0^+} \frac{\int_{\mathbb{R}^n} \mathrm{e}^{-\frac{\mathrm{d}(x, \varepsilon K)^2}{2}} - \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{|x|^2}{2}}}{\varepsilon} = \frac{(2\pi)^{\frac{n-1}{2}} n \kappa_n}{\kappa_{n-1}} M^*(K),$$

where $\mathrm{d}(x, K) = \inf_{y \in K} |x - y|$ and κ_m is the volume of the unit ball in $\mathbb{R}^m$. We conclude that $M^*(K) = c_n M^*(1_K)$ for some normalization constant $c_n \approx \sqrt{n}$. Next we present an analytic formula for the mean width of a smooth log-concave function.

LEMMA 3.1. *Let $f : \mathbb{R}^n \to [0, \infty)$ be a log-concave function, strictly log-concave on its support, with continuous second derivatives, such that $0 < \int f < \infty$, and that satisfies* $\mathrm{Hess}(\mathcal{L}(-\log f))(x) \leqslant K \exp(K|x|)\mathrm{Id}$ *in the sense of positive matrices for some $K = K(f)$. Then,*

$$M^*(f) = \frac{2}{n(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{|y|^2}{2}} \left[\triangle(-\log f^\circ) + |y|^2 D_r \frac{\log f^\circ(y)}{|y|} \right] \mathrm{d}y.$$

Proof. Denote $g = -\log f$. Then g is strictly convex and smooth on its support, hence (e.g., [16]) $g(x) = \langle x, \nabla g(x) \rangle - \mathcal{L}g(\nabla g(x))$ and

$$\int_{\mathbb{R}^n} f(x)\, \mathrm{d}x = \int_{\mathbb{R}^n} \mathrm{e}^{-\langle x, \nabla g(x) \rangle + \mathcal{L}g(\nabla g(x))}\, \mathrm{d}x.$$

Substituting $y = (\nabla g)x$ and recalling that $\nabla \mathcal{L}g = (\nabla g)^{-1}$, we get

$$\int_{\mathbb{R}^n} f(x)\, \mathrm{d}x = \int_{\mathbb{R}^n} \mathrm{e}^{-\langle y, (\nabla \mathcal{L}g)y \rangle + \mathcal{L}g(y)} \det(\mathrm{Hess}(\mathcal{L}g))\, \mathrm{d}y.$$

Denote the radial derivative by D_r (i.e., $D_r(g)(x) = \langle \nabla g(x), x/|x| \rangle$). Since

$$D_r \frac{\mathcal{L}g(y)}{|y|} = \frac{\langle \nabla \mathcal{L}g(y), \frac{y}{|y|} \rangle |y| - \mathcal{L}g(y)}{|y|^2},$$

we obtain

$$\int_{\mathbb{R}^n} f(x)\mathrm{d}x = \int_{\mathbb{R}^n} \mathrm{e}^{-\left(|y|^2 D_r \frac{\mathcal{L}g(y)}{|y|}\right)} \det(\mathrm{Hess}(\mathcal{L}g))\,\mathrm{d}y. \tag{3}$$

This is true for any smooth and strictly log-concave function f. The function $G \star [\varepsilon \cdot f]$ is also smooth and strictly log-concave, and by Equation (3),

$$\int G \star [\varepsilon \cdot f] = \int_{\mathbb{R}^n} \mathrm{e}^{-\left(|y|^2 D_r \frac{\frac{1}{2}|y|^2 + \varepsilon \mathcal{L}g(y)}{|y|}\right)} \det(\mathrm{Id} + \varepsilon \mathrm{Hess}(\mathcal{L}g))\,\mathrm{d}y. \tag{4}$$

We would like to find an expression for $M^*(f)$. Since

$$\det(\mathrm{Id} + \varepsilon \mathrm{Hess}(\mathcal{L}g)) = 1 + \varepsilon \mathrm{Tr}(\mathrm{Hess}(\mathcal{L}g)) + \mathrm{O}(\varepsilon^2) = 1 + \varepsilon \triangle \mathcal{L}g + \mathrm{O}(\varepsilon^2),$$

$$\mathrm{e}^{-\left(|y|^2 D_r \frac{\frac{1}{2}|y|^2 + \varepsilon \mathcal{L}g(y)}{|y|}\right)} = \mathrm{e}^{-\frac{|y|^2}{2}} \left(1 - \varepsilon |y|^2 D_r \frac{\mathcal{L}g(y)}{|y|} + \mathrm{O}(\varepsilon^2)\right),$$

the integrand in Equation (4) is

$$\mathrm{e}^{-\frac{|y|^2}{2}} + \varepsilon \mathrm{e}^{-\frac{|y|^2}{2}} \left[\triangle \mathcal{L}g - |y|^2 D_r \frac{\mathcal{L}g(y)}{|y|}\right] + \mathrm{O}(\varepsilon^2).$$

Using our assumption on the growth of $\mathrm{Hess}(\mathcal{L}g)$, by the dominated convergence theorem,

$$V_G(f) = \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{|y|^2}{2}} \left(\triangle \mathcal{L}g - |y|^2 D_r \frac{\mathcal{L}g(y)}{|y|}\right) \mathrm{d}y. \qquad \square$$

From Lemma 3.1 it follows that $M^*(G) = 1$. It also follows that $M^*(f \star g) = M^*(f) + M^*(g)$ and $M^*(\lambda \cdot f) = \lambda M^*(f)$ for $\lambda > 0$ and f that satisfies the requirements of Lemma 3.1. However, glancing at Lemma 3.1, it is not obvious why $M^*(f)$ should be positive. This follows from our next proposition, which is the functional analog of Urysohn inequality.

PROPOSITION 3.2. *Let $f : \mathbb{R}^n \to [0, \infty)$ be any even log-concave function such that $\int f = \int G$. Then*

$$M^*(f) \geqslant M^*(G).$$

Proof. By the Prekopa–Leindler inequality,

$$\int G \star [\varepsilon \cdot f] \geqslant \left(\int \left[\frac{1}{1-\varepsilon} \cdot G\right]\right)^{1-\varepsilon} \left(\int f\right)^{\varepsilon} = \left(\int e^{-\frac{1-\varepsilon}{2}|x|^2}\,\mathrm{d}x\right)^{1-\varepsilon} \left(\int f\right)^{\varepsilon}$$

and computing the gaussian integral we obtain

$$\int G \star [\varepsilon \cdot f] \geqslant \left(\frac{2\pi}{1-\varepsilon}\right)^{\frac{n(1-\varepsilon)}{2}} \left(\int f\right)^{\varepsilon}.$$

Since $\int f = (2\pi)^{\frac{n}{2}}$ we conclude that

$$V_G(f) \geqslant \lim_{\varepsilon\to 0^+} \frac{\left(\frac{2\pi}{1-\varepsilon}\right)^{\frac{n(1-\varepsilon)}{2}} (2\pi)^{\frac{n\varepsilon}{2}} - (2\pi)^{\frac{n}{2}}}{\varepsilon} = (2\pi)^{\frac{n}{2}} \frac{n}{2}$$

and the proposition is proved. □

Remarks. (1) Note that if $f(0)=1$ and $\int f = \int G$, then $M^*(f) > cM^*(1_{K_f})$ for some universal constant $c>0$. Indeed, if $M^*(1_{K_f}) \approx M^*(G) = 1$ this follows from Proposition 3.2. Otherwise $M^*(1_{K_f}) > C$, and since $f \geqslant e^{-n} 1_{\bar{\bar{K}}_f}$ we conclude that

$$M^*(f) \geqslant M^*\left(e^{-n} 1_{\bar{\bar{K}}_f}\right) = M^*(1_{\bar{\bar{K}}_f}) - 2 > cM^*(1_{K_f}).$$

(2) Formally, the results in this paper are formulated and proved for even functions. Yet, the evenness is never essentially used, and in fact the results hold for an arbitrary log-concave function, provided that the origin is suitably chosen. If $f:\mathbb{R}^n \to [0,\infty)$ is log-concave and has a finite, positive mass, then it must be a bounded function, and its supremum is attained at some point. All of our results hold, with the same proofs, for log-concave functions that reach their maximum at the origin (note that the dual function also reaches its maximum at the origin).

References

1. Arnold, V. I.: Mathematical methods of classical mechanics. Translated from the Russian by K. Vogtmann and A. Weinstein, 2nd edn, Grad. Texts in Math. 60 Springer-Verlag, New York, 1989.
2. Artstein, S., Klartag, B. and Milman, V. D.: The Santaló point of a function, and a functional form of Santaló inequality, Preprint.
3. Ball, K.: PhD dissertation, Cambridge.
4. Ball, K.: Logarithmically concave functions and sections of convex sets in $\mathbb{R}^n$, *Studia Math.* **88**(1)(1988), 69–84.
5. Bobkov, S. G. and Nazarov, F. L.: On convex bodies and log-concave probability measures with unconditional basis, In: V. D. Milman *et al.* (eds), *Geometric Aspects of Functional Analysis (Israel seminar 2001–2002)*, Lecture Notes in Math. 1807, Springer, New York, 2003, pp. 53–69.
6. Borell, C.: Convex measures on locally convex spaces, *Ark. Mat.* **12** (1974), 239–252.
7. Borell, C.: Convex set functions in d-space, *Period. Math. Hungar.* **6**(2)(1975), 111–136.
8. Bourgain, J. and Milman, V. D.: New volume ratio properties for convex symmetric bodies in $\mathbb{R}^n$, *Invent. Math.* **88**(2)(1987), 319–340.
9. Klartag B.: An isomorphic version of the slicing problem, to appear in *J. Funct. Anal.*
10. Klartag, B.: A geometric inequality and a low M estimate, *Proc. Amer. Math. Soc.* **132**(9)(2004), 2619–2628.
11. Meyer, M. and Pajor, A.: On Santaló's inequality, In: *Geometric Aspects of Functional Analysis* (1987–88), Lecture Notes in Math. 1376, Springer, Berlin, 1989, pp. 261–263.
12. Milman, V. D.: Almost Euclidean quotient spaces of subspaces of a finite-dimensional normed space, *Proc. Amer. Math. Soc.* **94**(3)(1985), 445–449.

13. Milman V. D.: Inégalité de Brunn–Minkowski inverse et applications à le théorie locale des espaces normés, *C.R. Acad. Sci. Paris, Ser. I*, **302**(1986), 25–28.
14. Milman, V. D.: Isomorphic symmetrizations and geometric inequalities. *Geometric Aspects of Functional Analysis (1986/87)*, Lecture Notes in Math. 1317, Springer, Berlin, 1988, pp. 107–131.
15. Pisier, G.: *The Volume of Convex Bodies and Banach Space Geometry*, Cambridge Tracts in Math. 94, Cambridge University Press, Cambridge, 1989.
16. Rockafellar, R. T.: *Convex Analysis*, Princeton Math. Ser. 28, Princeton University Press, Princeton, N.J., 1970.
17. Schneider, R.: Convex bodies: *The Brunn–Minkowski Theory*, Encyclop. Math. Appl. 44, Cambridge University Press, Cambridge, 1993.