GeoInformatica 9:2, 139–158, 2005

How to Get 3-D for the Price of 2-D—Topology and Consistency of 3-D Urban GIS

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Received April 4, 2004; Revised January 5, 2005; Accepted January 28, 2005

Abstract

This article deals with topological concepts and models which are necessary to represent three-dimensional urban objects in a geographical information system (GIS). Depending on the shape and the representation of features, several classes with increasing topological complexity are identified and described. This complexity has strong impacts on the models and tools which are required to represent, manage and edit the data. One specific model we call '2.8-D map' is identified, which covers many 3-D applications in GIS. It is a slight extension of a 2-D or 2.5-D model and preserves the algorithmic and conceptual simplicity of the 2-D case as much as possible. The model is described in a formal way. Integrity axioms are given, which detect errors in corresponding data sets safely and guarantee the consistency of 2.8-D maps in a mathematically sound and provable way. These axioms are effectively and efficiently checkable by automatic procedures. The model extends digital terrain models (2.5-D) by allowing for vertical walls and projections like balconies or ledges. The conceptual simplicity is due to the two-dimensional topology of the model. Thus bridges and tunnels are special cases; it is shown how to detect and handle these cases efficiently. Based on this model, thematic objects and their aggregation structures are defined in a consistent way.

Keywords: GIS, 2.5-D, 2.8-D, 3-D, topology, consistency, axioms, maps, correctness, completeness

1. Introduction

The demand for three-dimensional applications of geographical information systems (GIS) increases. Examples are urban and telecommunication planning [16], [26] or disaster management [28]. Most commercial GIS currently available are, however, limited to two or two and a half dimensions, and are not able to cope with these applications. On the other hand, three-dimensional systems originating from computer aided design (CAD) [19] or computer graphics [7] are limited, since they do not handle topology and semantic properties of objects adequately and do not provide the required GIS functionality. The corresponding models, basically boundary representation (B-Rep) or constructive solid geometry (CSG) [7], [19], are complex and difficult to handle from an algorithmic perspective.

This paper answers the question, which types of three-dimensional objects may be represented by models and topological concepts from 2-D, or by slightly adopting these concepts without requiring the algorithmic complexity of 'real' 3-D models. A model is presented, which covers many important 3-D applications while preserving the algorithmic and conceptual simplicity of the 2-D case as much as possible. The model is an extension of *digital terrain models* [21], often called 2.5-D, which are restricted since each (x,y)-location has at most one height value. This restriction is omitted, allowing to represent vertical walls and projections like balconies, ledges or roof overhangs, and thus most urban objects. Hence, the geometry is three-dimensional, while topology remains two-dimensional. From a mathematical point of view, the model is based on the notion of a *two-dimensional manifold* or *2-manifold* embedded in 3-D space [19]. Since the model extends 2.5-D, but is less than three-dimensional, we call it '2.8-D.' The 2-D topology prevents modeling of overpasses like bridges, viaducts, tunnels and pedestrian underpasses, which are all equivalent from a topological perspective. It is shown how to handle these cases efficiently.

Spatial data quality is an important issue in GIS [8], since it has impacts on the quality of maps and decisions generated based on GIS. Thus achieving and preserving consistency of spatial data is a problem of high practical significance. Therefore, the focus of the paper is on maintaining consistency of the 2.8-D model.

The main contribution of this paper is an axiomatic characterization of 2.8-D maps by a set of integrity constraints. These integrity constraints are correct and complete for maps. Correctness means that each violation of an integrity constraint identifies an error of a given database. Completeness means that each error leads to a violation of an integrity constraint. In addition, the axioms are effectively—and in fact efficiently—checkable by a computer.

Our model generalizes the formal data model of a 2-D map, which was presented in earlier papers [9], [10], [22]–[24]. A 2-D map provides an irregular tessellation, i.e., a partition of the plane with non-overlapping and covering faces. Corresponding integrity rules or axioms were given which are provably correct and complete for maps, guaranteeing consistency in an effective and efficient way. It is shown how consistency of a 2.8-D model can be guaranteed by slightly extending the axioms derived for the two-dimensional case.

Several extensions of the model are discussed, which enable the representation of more types of objects or more details. An example is to use curved surfaces instead of planar ones. It is illustrated how the integrity rules have to be modified to cope with these extensions. Another issue is the definition of spatial thematic objects, for example a building, by the aggregation of its bounding wall or roof faces. It is shown how the concept of nested maps developed to aggregate 2-D objects may be used to maintain consistency.

In the last decades, modeling of 2.5-D or 3-D surfaces and solids has been studied intensively. *Boundary representations* [7], [19] are employed to model surfaces and solids. In GIS, 2.5-D surface models are essentially *TINs* (*triangulated irregular networks*) [21], which are suitable for modeling the relief structure, but are limited since the deviation of the normal vector of each triangle must be non-negative. In [6] this approach is extended by allowing for a zero deviation, thus vertical walls can be represented. But projections like balconies are still not covered by this approach. Furthermore, integrity constraints for maintaining consistency of the model are not given. In modeling solids by the Boundary Representation, two approaches may be distinguished, which differ basically in the handling of *non-visible faces*. The computer graphics approach omits those faces, while in the GIS community, they are represented explicitly, and are used to glue objects together [14], [17], [20], [25].

Apart from boundary representations, the other prominent approach for modeling solids is the *constructive solid geometry (CSG)* [7], [19]. CSG defines solid objects by applying transformations and operations like intersection, union and difference to primitive solids. These primitives can be simple entities like cylinders or boxes [19], or complex objects like buildings [15], which may be parameterized [5]. Topology is not represented explicitly in those models.

Another way of modeling solids is the representation by three-dimensional *simplicial complexes* [3], [4], [12], which are adequate for geological applications. For modeling man-made objects, however, the partition in tetrahedrons introduces faces and edges without any semantic relevance. The representation is not unique and matching problems arise.

The rest of this paper is organized as follows: The second section presents topological concepts and models necessary to represent objects in the range of two to three dimensions. The third section recalls, after giving some basic mathematical definitions, the concept of 2-D maps and presents the corresponding consistency axioms. In the next section, maps are generalized to cope with three-dimensional applications. The corresponding axioms are adjusted and it is proven that they are correct and complete. The fifth section extends the model to handle spatial objects and their aggregation in a consistent way. In addition, the interpolation of faces is generalized. The paper ends with concluding remarks and a discussion of open questions.

2. From 2-D to 3-D: Topological concepts behind GIS

This section addresses the question, which topological concepts are necessary to model three-dimensional real world phenomena, and which GIS models are suitable to represent these phenomena. This issue is crucial for GIS tools to store, manipulate, visualize and these models, and for concepts to maintain consistency of this data, where a focus is on the cost of maintaining consistency in terms of running time. The discussion starts with the 2-D case and extends the model incrementally, covering more real-world phenomena and specifying the topological concepts, which are necessary for modeling.

2.1. The 2-D case

In the simple 2-D case, areal features are represented by the well-known two-dimensional node-edge-face topology [18], [24]. An example for a simple 2-D structure is given in

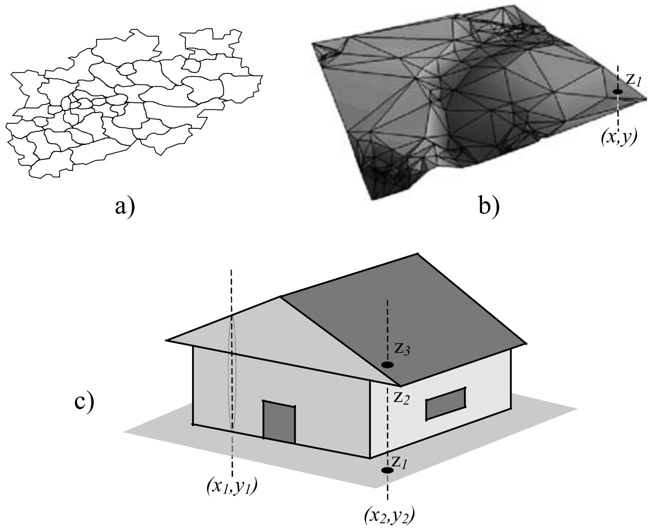


Figure 1. Examples for a plane 2-D structure (a), a 2.5-D TIN (Triangulated Irregular Network) (b), and a 2,8-D map (c). In a) and b), each (x, y)-value has at most one z-value representing its height, while in c, the point (x_2, y_2) has three z-values, and (x_1, y_1) has an infinite number of z-values. In all three cases, the scene is represented by a single 2-manifold, which is topologically equivalent to a two-dimensional disk.

Figure 1a. Most commercial GIS offer these structures; a prominent example is the *coverage* model used in ArcGIS/ArcInfo. As *simple topology* it has been part of the standardization accomplished by the International Standardization Organization (ISO) and the Open Geospatial Consortium (OGC) [14]. For aggregated areal objects, the concept of *nested maps* [23], [24] may be employed.

2.2. 2.5-D: Modeling the relief

To represent the relief structure of a scene, the 2-D model has to be extended to a so-called 2.5-D model [21]. The coordinates of this model are three-dimensional, but each x/y-position has at most one height value, i.e. the z-value is a function of the corresponding x- and y-values (see Figure 1b). The topology of a 2.5-D structure, however, is the same as in the 2-D case: it is a *2-manifold* from a mathematical point of view, which is embedded in 3-D space. A 2-manifold embedded in 3-D space is a 2-D point set (a 2-D topological space), where every point has a neighborhood which is topologically equivalent to an open disk of $\mathbb{R}^2$ [19]. Intuitively, a 2-manifold can be illustrated as a cloth, which is draped over the terrain. This cloth can be stretched arbitrarily, but it may not be folded or touch itself, and no hole may be torn in the cloth.

In commercial GIS, two representations of 2.5-D models are commonly used: Triangulated irregular networks (TINs) [21] and the raster or grid structure [27].

2.3. Extending 2.5-D: Vertical walls

In a 2.5-D model, each x/y-value has at most one height value. Thus the representation of vertical walls of artificial or natural objects like buildings or rocks is not feasible. An example is given by the building in Figure 1c, where the position (x_1,y_1) has an infinite number of height values. A model that extends 2.5-D to handle such features is discussed in [6]. It is topologically equivalent to 2-D and 2.5-D models, since it is still a 2-manifold from a mathematical point of view.

This extended 2.5-D model may be implemented by a TIN structure, but not by a grid or a raster, since the 2.5-D property is crucial for these representations. In addition, no orthogonal projection of an image onto a 2.5-D structure with vertical walls is feasible, since no patch of the image is mapped onto these walls.

2.4. Extending 2.5-D: Overhangs

The model proposed in [6] handles vertical walls, but does not cover overhangs like balconies or roof resp. rock overhangs. A model that copes with such phenomena has been called ‘2.8-D map’ [11]. An example for a 2.8-D map is depicted in Figure 1c. From a geometrical point of view, the cases of vertical walls and overhangs are different, but from a topological perspective, both are equal, since both are equivalent to 2-manifolds. In a 2.8-D map, however, the ‘cloth,’ which is draped over the terrain, includes buildings and other objects on the surface.

The topological equivalence between a 2.8-D and a 2-D model is illustrated in Figure 2. In Figure 2a, the saddle roof building with vertical walls and roof overhangs is represented by a 2.8-D model, whereas the corresponding flattened model in b is a 2-D map. Both models are topologically equivalent, since there is a one-to-one correspondence between faces in both models.

As a consequence, the orthogonal projection of an image onto a 2.8-D map yields empty patches at vertical walls and—even worse—an assignment, which is not unique at overhangs, where there is more than one height value for one x/y position.

2.5. Covering bridges and tunnels (‘handles’)

All phenomena discussed so far may be modeled by a single 2-manifold, which is topologically equivalent to a single two-dimensional disk, and can be handled by concepts of 2-D topology. Another cases, however, are bridges and overpasses (see Figure 3). Although scenarios with bridges may still be represented by a 2-manifold embedded in 3-D space, the topology is quite different. Such a 2-manifold is *locally* equivalent to a 2-D disk, but not *globally.* Intuitively, if one wants to map or drape a 2-D disk or an image onto a scene with a bridge, he has to cut holes in the disk or the image first and then has to stitch it together. However, the neighborhood at the cutting edge has changed after stitching.

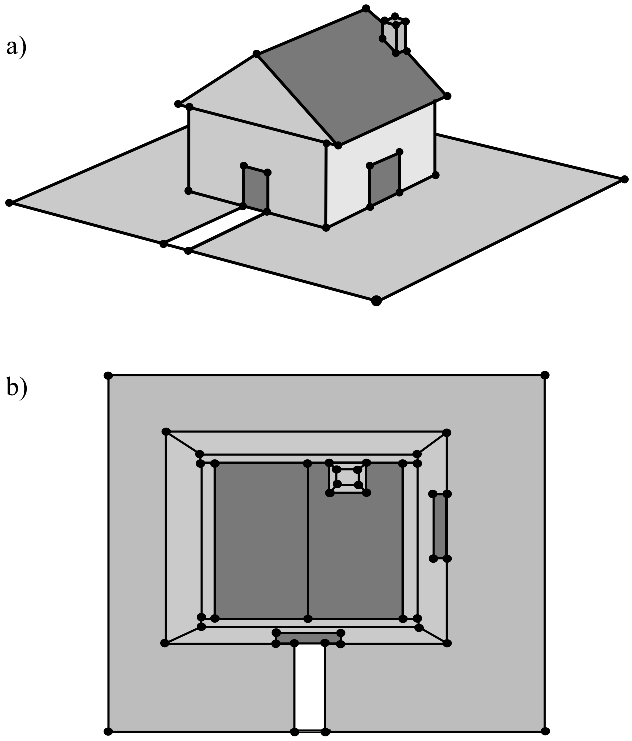


Figure 2. a) A saddle roof building as an example for a 2.8-D map. b) Shows the flattened two-dimensional model, which is topologically equivalent to the map in a).

From a mathematical point of view, bridges and overpasses are described by the concept of a *handle* [19]. Each simple bridge is a handle (see Figure 3), but more complex bridges with several ramps and piers consist of more than one handle.

Another cases are tunnels and pedestrian underpasses. Although different from a geometrical perspective, these features are topologically equivalent to bridges. It is not difficult to imagine a transformation from a simple bridge (see Figure 3) to a tunnel with one tube. This transformation maintains the topology of the scene. Therefore, both cases

Figure 3. A terrain with a bridge. From a mathematical point of view, it is a 2-manifold with one handle.

can be described by the concept of a handle and be treated identically. But there is a difference between both cases. A bridge itself is a handle. On the other hand, suppose a tunnel is constructed passing through a mountain. In this case, the mountain above the tunnel is the handle, not the tunnel itself.

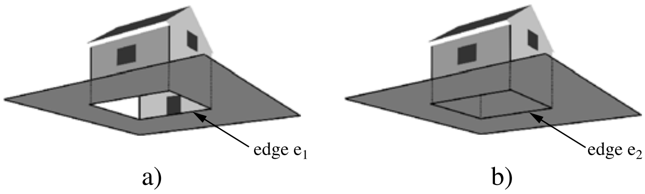
Efficient methods to detect the occurrence of handles are discussed in Section 4.2 in the context of consistency for 2.8-D maps, based on the well-known Euler equation [19].

2.6. *Modeling buildings as solids: 3-D topology*

The concepts discussed so far model a real-world scene by a single 2-manifold. This is sufficient for many analysis and visualization purposes—at least if one looks at buildings from outside. On the other hand, many relevant questions may not be answered by these models. In this model, for example, it is not feasible to compute the volume of a building or to determine conflicts between buildings and underground pipes. To deal with these tasks, single 2-manifolds are not sufficient, since buildings must be closed from below by using solids. As an example, Figure 4a shows a building modeled by a single 2-manifold, while in b, the same building is represented by a solid. The topology of solids is totally different from the ones discussed in the previous cases, since closed solids are completely bounded by a single 2-manifold. But the complete scene may not be represented by a single 2-manifold, since it contains edges whose neighborhood is not equivalent to an open disk: An edge is in the boundary of three faces. Consider for example the edge e_2 in Figure 4b, which is in the boundary of the wall face, the ground face and the face surrounding the building. In contrast, the corresponding edge e_1 in Figure 4a is in the boundary of only two faces, of a wall face and a face surrounding the building.

2.7. *Modeling invisible faces: Aggregation of solids*

The models discussed in the last paragraph consider the outer hull of a building, seen from outside or below. Often the internal structure of a building is of interest, too. For example, if a house is located close to a garage, some applications require the computation of the volume of the garage and the house separately. The face separating both objects



a) b)

Figure 4. Two different representations of the same building, depicted from below. In a), it is modeled by a single 2-manifold. The ground face of the building is not represented. The building in b) is given by a solid, which is completely surrounded by faces. The topology in a) is two-dimensional, while in b), it is not.

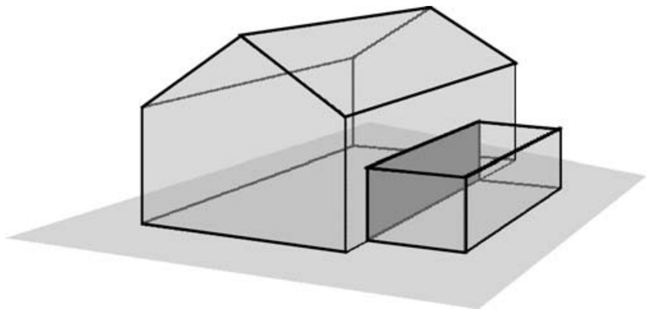


Figure 5. Aggregation of two solids—a building and a garage. Both solids share a common wall (dark gray face), which is not visible from outside.

must be modeled explicitly, although it is not visible from outside. Figure 5 shows an invisible face (dark gray) separating a building and a neighboring garage as an example.

Another important GIS application, which requires invisible faces, is efficient visualization. In the visualization of an urban scene, often parts of buildings are omitted for generalization reasons. Inconsistencies occur, since remaining objects may be drawn incompletely and get holes. For example, if the garage in Figure 5 is omitted in the visualization process, the remaining building gets a hole when the invisible face shared with the garage (depicted dark gray in Figure 5) is not represented. Thus invisible faces separating neighboring features are essential for fast, consistent visualization of 3-D models.

The topology which is necessary to model solids with invisible faces is inherently three-dimensional. In the models discussed in the last section, a face may be in the boundary of at most one solid. An invisible face, in contrast, has exactly two neighboring solids. A 3-D topology is required, which has to manage the neighborhood between solids.

In computer graphics, invisible faces are not considered as part of a Boundary Representation [7], [19]. In GIS, however, they are modeled explicitly [20], [14], extending pure boundary representation. The most important standard for geometric and topological models in GIS, ISO 19107 ‘Spatial Schema’ [14], which was issued by the ISO and the Open Geospatial Consortium recently, allows to aggregate solids to complexes and to represent the invisible faces and its 3-D topology explicitly. However, tools to manage such 3-D models are very complex and maintaining consistency is a difficult problem.

2.8. Summary of Section 2

Features in the range from 2-D to 3-D can be classified according to geometrical and topological criteria. Seven classes have been identified: 2-D, 2.5-D, 2.5-D with vertical walls, 2.5-D with vertical walls and overhangs (2.8-D), 2.8-D with handles, solids without invisible faces, and aggregation of solids with invisible faces. The geometric complexity increases in each step, but the topological complexity remains constant from 2-D to 2.8-D. But a 2.8-D model is sufficient to cover the most features in 3-D urban GIS applications Thus the rest of this paper focuses on 2.8-D maps and methods to maintain consistency of these models.

3. Maintaining consistency of 2-D maps

In the last section, the concept of a 2.8-D map was identified and described. It is suitable to model many 3-D urban objects, but preserves the simplicity of the 2-D case as much as possible. To deal with the problem of maintaining consistency for 2.8-D maps, the 2-D case is discussed first.

Two-dimensional *maps* generalize common data structures in commercial GIS. Maps provide a tessellation of the plane with covering and non-overlapping faces, being the base for many application-specific integrity constraints.

To define concept of a map in a formal way, some basic notions of topology and graph theory are given. The reader interested in more details is referred to textbooks on graph theory or topology [13], [1]. Graphs play a central role in the definition of spatial models in GIS. Basically, a *graph* $G(V,E)$ is specified by a set V of *vertices* and a set E of *edges*. An edge e is specified by a set $\{v_1,v_2\}$ of two end vertices; e is called *incident* to both v_1 and v_2 and vice versa, while v_1 is *adjacent* to v_2. The *degree* of a vertex is the number of its incident edges. Note that in our context all edges are undirected.

Subsequent edges form a *path*. If for each pair of vertices there is a connecting path, the graph is called *connected*, otherwise *unconnected*. A *cycle* is a path where the start and the end vertex are the same, and a cycle is *simple* if this start resp. end vertex is the only vertex which occurs more than once, while all others occur only once. A graph may be embedded in the plane. Vertices correspond to points and edges correspond to straight line segments. A more general shape of the edges may be obtained by a homeomorphic embedding [7]. If nothing else is stated explicitly graphs are identified with their embedding in the plane. In a *plane* graph, edges never cross. More precisely, if two edges have a common point, it is an end vertex of both edges.

A graph embedded in the Euclidean plane defines faces as atomic areal entities. A *face* f is mathematically defined by the following property: f is a maximal part of the plane such that for any two points X and Y in it there is a continuous (not necessarily straight) line from X to Y, which does neither cross nor touch any edge of the graph.

Now we are in a position to define a 2-D map formally: a 2-D map is a graph embedded in the two-dimensional Euclidean plane, together with the set of faces defined by the embedding. Each face is bounded by exactly one simple cycle of the graph. An example for a map was already shown in Figure 1a. Note that there is a special unbounded face called OUT, which is bounded by one interior simple cycle.

A data structure representing the mathematical concept of a map is depicted in Figure 6, using the graphical modeling language UML [2].

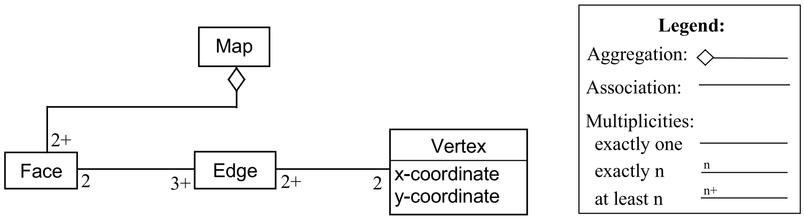


Figure 6. UML class diagram for 2-D maps. A map consists of faces, which are bounded by edges, and an edge is defined by two vertices.

Spatial data structured according to this model, however, may contain errors. There are well-known error cases, like over- and undershoots, violating the map or tessellation properties [18], [24]. To detect this, *axioms* are provided being provably correct and complete. These axioms are given in Table 1 and are discussed in detail in [9] or [24].

These axioms are the base for maintaining consistency of 2.8-D maps, which is topic of the next section.

4. Maintaining consistency of 2.8-D maps

The concept of a map and the consistency rules given in the last section are restricted to the modeling of two-dimensional objects. In this section, both are generalized to cope with three-dimensional phenomena.

4.1. 2.8-D maps: Formal definition and error cases

In Section 2.4, the model of a 2.8-D map was introduced informally as a single 2-D surface embedded in 3-D space, which in particular allows the modeling of vertical walls and overhangs. From a mathematical point of view, it is characterized by a 2-manifold. Now this concept is described more formally, based on the model of 2-D maps (Section 3

Table 1. Correct and complete axioms for 2-D maps [9], [24].

Vertices:	1. No two different vertices have the same coordinates.
	2. Each vertex has at least two incident edges (*vertex degree* ≥ 2).
Edges:	3. For every edge, there are exactly two distinct vertices as endpoints.
	4. Edges correspond geometrically to straight line segments.
	5. Any common point of two segments is a common end vertex of both segments (intersection-free edges).
	6. Each edge has exactly two distinct incident faces.
Faces:	7. Each face is bounded by exactly one simple cycle.
	8. No midpoint of an edge lies inside the interior of a face.

and Figure 6). A 2.8-D map is a 2-D map, where the 2-D coordinates of the nodes are replaced by 3-D coordinates, and which is topologically equivalent to a single 2-D disk. This implies that two edges touch only at common vertices and that two faces touch only at common edges.

If the nodes are embedded in 3-D space, then so are the edges. In our case, edges are represented by straight line segments. If the edges are embedded in 3-D space, then so are the faces. Faces are restricted to be planar, i.e. all vertices on its bounding cycle are located in the same 2-D plane. However, the location of this 2-D surface in 3-D space is arbitrary. In Section 5.2, the consequences of considering non-planar faces will be discussed.

The simple saddle roof building in Figure 2 was already discussed as an example for a 2.8-D map. To illustrate the formal definition of 2.8-D maps, it is considered again. In the figure, the house is bounded by six wall faces, two of them being triangles, two door faces, four roof faces, two of them being overhangs, and five faces bounding a chimney. In addition, two faces represent the parcel and a path. Note that each face is bounded by a single, simple cycle.

For completeness reasons, a 2.8-D map has an unbounded face *OUT*, the geometry of which is undefined. In contrast to all other faces, its boundary must not be planar.

Not all structures, which are constructed according to the extended UML diagram in Figure 6, satisfy the mathematical definition of a 2.8-D map. Figure 7 depicts some error cases for 2.8-D maps. In Figure 7a, the neighborhoods of the points on edge e_2 are not topologically equivalent to an open disk. In this case, the 'cloth' is folded at face f_3, which is incident to edge e_2. A similar error appears at edge e_1. All points on this edge have a neighborhood, which is not topologically equivalent to a 2-D disk, too. The structure shown in Figure 7b is not a 2.8-D map, since it consists of two 2-manifolds: a closed 2-manifold representing the building, and another 2-manifold which models the relief. This is an error, since a 2.8-D map must correspond to exactly one 2-manifold.

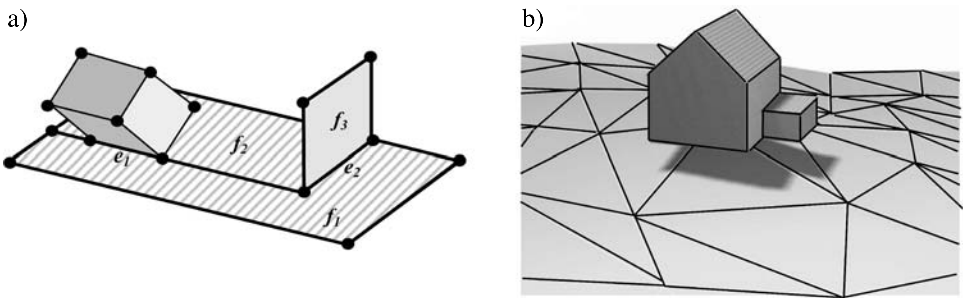


Figure 7. Examples of error cases for 2.8-D maps. The structure in a) is not a 2-manifold, since the neighborhoods of all points on the edges e_1 and e_2 are not topologically equivalent to a 2-D disk. In b), the structure consists of two 2-manifolds: a triangulated irregular network (TIN) and a building, which is not connected to the TIN. A 2.8-D map must correspond to exactly one 2-manifold.

4.2. Preserving 2-D topology—Detection of handles

Each 2.8-D map is a 2-manifold, but the converse of this statement is not true: There are 2-manifolds which are not accepted as valid 2.8-D maps. As discussed in Section 2.5, a 2-manifold may contain handles, which violate the 2-D topology of the surface. Figure 3 depicts a bridge as a well-known example for a handle. Due to its topology, this case is difficult to handle from an algorithmic point of view.

To check the consistency of 2.8-D maps, a procedure to detect handles is required. A simple method for counting handles in surfaces is given by the Euler characteristic, which can be computed by the well-known Euler formula [1], [19]. If N, E and F are the numbers of nodes, edges and faces, then the Euler characteristic C is given by the equation:

$$C = N - E + F$$

The Euler characteristic of a 2.8-D map is *two*. From this equation, the number H of handles may be derived by the formula [19]:

$$H = (2 - C)/2 = (2 - N + E - F)/2$$

It can easily be computed whether the number H of handles is *zero*, just by counting the numbers of vertices, edges and faces. Note that this version of the Euler formula is valid for surfaces without boundary (there is an additional face *OUT*, so a surface has no boundary) and with exactly one component. From a graph theoretical point of view, this corresponds to connected graphs.

It is important to note that Euler's formula is only a necessary condition, not a sufficient one for the consistency of a map: The consistency of a surface implies the validity of the Euler formula, while from the validity of the formula, no statement about consistency can be derived. By adding nodes to an inconsistent map, the formula can be forced to be valid, although the map remains, of course, inconsistent. Thus the Euler formula in general is not appropriate for checking consistency.

Nevertheless, when the map is consistent, the Euler formula counts the number of handles in a correct way. Thus the consistency must be guaranteed by other axioms before the Euler formula can be applied for counting handles.

4.3. Axioms for 2.8-D maps

Now we are in a position to give axioms, which are correct and complete for 2.8-D maps. These axioms, which are a generalized version of the axioms for 2-D maps (Table 1), are given in Table 2.

The first seven axioms are identical to the corresponding axioms in Table 2, which cover the 2-D case. They only differ in the complexity of the methods which are required to implement the geometrical axioms: To check if axiom 5 (intersection-free edges)

Table 2. Axioms for 2.8-D maps, which are correct and complete. The axioms 1 to 7 are identical to the 2-D axioms in Table 1, while the others have been changed (axiom 8) or added (axioms 9 to 11).

Vertices:	1. No two different vertices have the same coordinates.
	2. Each vertex has at least two incident edges (*vertex degree* ≥ 2).
Edges:	3. For every edge there are exactly two distinct vertices as endpoints.
	4. Edges correspond geometrically to straight line segments.
	5. Any common point of two segments is a common end vertex of both segments (intersection-free edges).
	6. Each edge has exactly two distinct incident faces.
Faces:	7. Each face is bounded by exactly one simple cycle.
	8. No point of an edge touches the interior of a face.
	9. The vertices of the boundary of a face—apart from OUT—and its interior are planar.
Graph:	10. The underlying graph is connected.
	11. The number of handles is zero (Euler's formula).

holds, a 3-D algorithm for segment intersection [7] has to be employed, instead of a 2-D procedure.

Axiom 8 guarantees that the interiors of faces do not touch. Since all faces are planar (axiom 9), a touching of two face's interiors implies that a point on an edge touches an interior of a face. The example in Figure 8a illustrates this fact. The interiors of the faces f_1 and f_2 touch. Since both faces are planar, there must be an edge in the boundary of face f_1 which touches f_2. In the example, the edges e_1 and e_2 touch the interior of f_2. Thus axiom 8 is sufficient to prevent touching of the interiors of faces. The case of curved faces (Figure 8b) is dealt with in Section 5.2.

Axiom 10 states that the graph has to be connected. In the two-dimensional case, it is not necessary to state this property explicitly as an axiom, since it is implied by the tessellation property and by the fact that each face is bounded by exactly one cycle (axiom 7). In three dimensions, the situation is different. An example was already given by the structure in Figure 7b, which is obviously a tessellation. But the scene contains two surfaces, not exactly one as required for a 2.8-D map. The consequence in the example is a building floating above the terrain. Obviously, the underling graph is

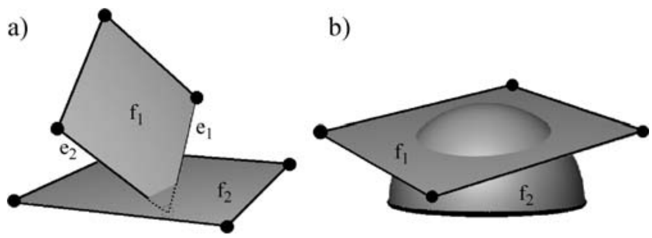


Figure 8. Two different cases of face–face intersections. In a), the interiors of the faces f_1 and f_2 intersect, but there is also an intersection of the boundary of face f_1, in particular the edges e_1 and e_2, and the interior of f_2. Both faces are planar. b) Shows an intersection of the interiors of faces f_1 and f_2, but the boundaries of both faces do not touch. While f_1 is planar, the interior of face f_2 is a hemisphere.

not connected. Thus axiom 10 forces that there is only one connected surface. Axiom 11 prevents handles by employing Euler's formula, as discussed above.

4.4. Correctness and completeness of the axioms

So far, the mathematical definition of a 2.8-D map was presented, which may not effectively be checked by automatic procedures, and axioms for 2.8-D maps (Table 2), which are checkable effectively and efficiently. To ensure that these axioms guarantee the consistency of 2.8-D maps, it has to be proven mathematically that the axioms are in fact equivalent to the mathematical notion. Thus the correctness and completeness of the axioms for 2.8-D maps must be shown. This assertion is stated by the following proposition:

Proposition: The set of axioms given in Table 2 is correct and complete for 2.8-D maps.

Proof: For the completeness of the axioms, it must be shown that for any structure, which satisfies all axioms, the following properties hold:

1. The structure is a tessellation of planar faces, i.e. it has no gaps, no overlaps and no touches of the interiors of faces. Faces are bounded by exactly one simple cycle.
2. From a mathematical point of view, the structure is equivalent to exactly one 2-manifold.
3. The surface has no handles.

The first property follows from axioms 7 and 8 together with their discussion above, which guarantee that no interiors of faces touch. Axiom 6, which forces each edge to be connected to two faces, prevents gaps between faces. Note that a 'gap' inside the map, which is filled by the face OUT, is detected by axiom 7. In this case, the face OUT has more than one bounding cycle, which contradicts axiom 7.

For the proof of the second property, it must be shown that the neighborhood of each point on the 2.8-D map is topologically equivalent to an open disk. This is implied by the first property, together with axiom 6, which states that each edge has no more than two incident faces. This argument is illustrated in Figure 7a, where all points on the edges e_1 and e_2 violate this property. But both edges are in the boundary of more than two faces; e_1 bounds four faces (f_1, f_2 and two faces on the lower sides of the box), and e_2 bounds three faces (f_1, f_2 and f_3). The fact that the structure consists of exactly one 2-manifold, is an immediate consequence of axiom 10.

From axiom 11 it can be deduced immediately that there are no handles; the third property is valid as well. Thus the axioms are complete.

To proof the correctness of the axioms, it must be shown that the axioms are implied by the mathematical definition. For the axioms 1 to 5 and 7, the proof is similar to the 2-D case [9], [24]. Axiom 6 ('Each edge has exactly two distinct incident faces'), however, follows from the tessellation property together with the second property by contradiction: Suppose an edge e has three or more incident faces. If two of these faces touch in an interior point, this violates the tessellation property. If the interiors of the faces are disjoint, the neighborhoods of points on e are not topologically equivalent to an open disk (see edges e_1 or e_2 in Figure 7a as examples), thus the structure is not a 2-manifold. This contradicts the second property. If an edge has one or zero incident faces, the contradiction may be derived in a similar way. Thus axiom 6 holds.

The correctness of axioms 8, 9 and 11 is obvious. The connectedness of the graph (axiom 10) is a consequence of the second property, which states that a 2.8-D map consists of exactly one 2-manifold (see Figure 7b for a counterexample). However, even a graph representing a single 2-manifold may be unconnected, since a face may have holes. But this is prevented by the first property, forcing the boundary of a face to be exactly one simple cycle. This completes the proof of the proposition. □

The most important implication of the proposition is, that it can safely be assumed that a given data set is a consistent 2.8-D map, if the axioms are valid for this data set.

5. Modeling thematic objects and curved surfaces

Based on the model of a 2.8-D map and the corresponding axioms given in the last section, now two extensions are discussed, which allow for representing more types of objects. The first extension enables the definition of thematic spatial objects like buildings and their structure, while the second extension generalizes the interpolation of faces to include curved faces. In both cases, the impacts on the axioms, which of course continue to be correct and complete, are discussed.

5.1. *Defining thematic spatial objects and their aggregation hierarchy*

The 2.8-D maps presented in the last section can be used to define the geometrical and topological properties of thematic spatial objects, like buildings, garages, balconies, windows or doors. Obviously, a thematic object consists of the (visible) faces, which bound the object. For example, a simple saddle roof building consists of four wall faces and two roof faces. The face representing the footprint is not visible and is neglected in the model. To guarantee consistency of such objects, it must be assured that the faces of an object do not overlap and that there are no gaps between faces. In addition, neighboring objects, for example a house and a garage, may not overlap. To guarantee these properties, the concept of nested maps is adopted from 2-D to 2.8-D.

Nested maps have been developed to represent the aggregation hierarchy of 2-D objects [23], [24]. An example is the hierarchical structure of administrative entities: Municipalities can be aggregated to counties, counties to states, and states to countries. A nested map consists of a map representing the lowest level with the finest granularity. This level is the base level of the aggregation tree. Each administrative level is represented as a level of the aggregation tree, and each face on one level is the aggregation of the corresponding faces on the next lower level. The crucial property of this structure is that

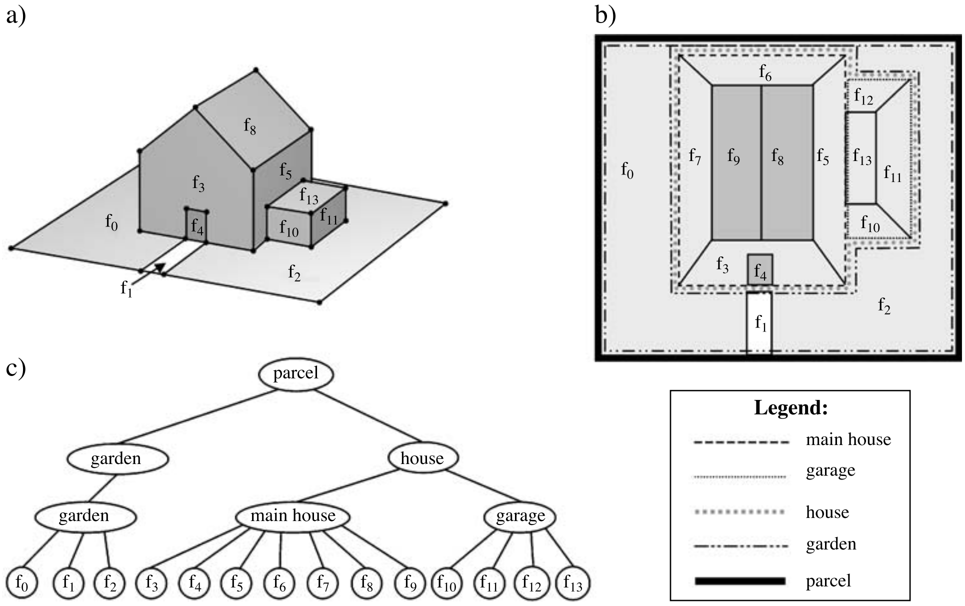


Figure 9. Example of a nested map, representing thematic objects of a 2.8-D map. The scene consists of a house and a garden, where the house is the aggregation of a garage and a main house a). The flattened model is depicted in b), where the simple cycles bounding aggregates are distinguished using different signatures. The aggregation tree of the nested map is shown in c).

each level of the hierarchy is a map, i.e. it consists of non-overlapping and covering faces. This is guaranteed by the tree structure and one additional axiom, the *Simple Cycle Axiom: Each face on each level is bounded by only one simple cycle*.

Now this concept can be adapted to the definition of thematic spatial objects in 2.8-D maps and their aggregation. An example is depicted in Figure 9, where the leaf nodes of the tree (Figure 9c) represent all faces of the scene. On the next level, there are three aggregates: a garden, a main house and a garage. The next level consists of two objects, the (complex) house and the garden, and the highest level represents the whole scene.

Similar to the 2-D case, the consistency of a scene can be asserted by the simple cycle axiom, which has to be added to the axioms in Table 2:

	12. Each face on each level is bounded by only one simple cycle.

Together with the tree property of the aggregation and the consistency of lowest level of the hierarchy, this guarantees that the objects on each level are non-overlapping. Otherwise, one face would have two or more predecessor in the tree. Holes or gaps are prevented by the simple cycle axiom: if a face has a gap, it would have more than one cycle in its boundary.

The depth of the hierarchy is not restricted; it is straightforward to include more details like chimneys, windows, or doors.

In addition to a consistent definition of spatial objects, the concept of nested maps supplies with useful information about the exact geometries where two neighboring objects are glued together. This is related to the problem of invisible faces discussed in Section 2.7. Since each object is defined by a simple cycle, the edges belonging to two cycles are edges used by two objects. In the example in Figure 9, the edges common to the main house and the garage are exactly the edges common to both corresponding simple cycles. Since 2.8-D maps represent only the visible faces of objects, it is not obvious how to deal with objects as solids. In the context of nested maps, the geometry of the simple cycles that represent objects supplies the missing faces to close the object. For instance, the house given in Figure 9 can be closed by the 'face', which is given by the cycle of the node labeled 'house'. This face is not part of the model, since it violates the 2-D topology: the neighborhoods of all points located on this cycle are topologically not equivalent to an open disk. But this face can be calculated temporarily and be used to compute the desired result, which requires closed solids.

5.2. Modeling curved surfaces

The faces of 2.8-D maps defined in Section 4 are restricted to be planar. Many buildings, like churches, towers, or mosques, are bounded by curved faces, for example by hemispheres. The interpolation of such faces has been studied intensively in computer graphics [7], examples are *bezier-surfaces*, *B-spline-surfaces* and *NURBS* (*non-uniform rational B-splines*). To extend our model to these types of faces, additional information must be assigned to faces, since the shape of a face is no longer determined uniquely by the location of its bounding cycle. To guarantee consistency of this extended model, the axioms given in Table 2 are not sufficient. The interiors of curved faces may touch, although no edge in the boundary of a face touches an interior of a face (see Figure 8b). Axiom 8 in Table 2 does not detect touching of curved faces and is replaced by a new version of axiom 8:

Faces:	8. For each pair of faces, both interiors do not touch.

This stronger version of the axiom detects error cases involving curved faces. The implementation of this axiom is more complex and less efficient. It can be based on algorithms from computational geometry [7].

6. Conclusions

Most commercial GIS currently available are able to cope with two or 2.5 dimensional data only. Those models are not sufficient for the representation of urban objects. 'Real' 3D-models like the Boundary Representation or the Constructive Solid Geometry, which are usually employed for this purpose, are complex from an algorithmic perspective and difficult to handle. We define the concept of a 2.8-D map, which extends 2-D or 2.5-D models by allowing vertical walls and overhangs, and demonstrate that this model can handle most urban phenomena. The simplicity of the 2-D or 2.5-D case is maintained as much as possible. This simplicity mainly stems from the two-dimensionality of its topology—a 2.8-D map is topologically equivalent to a two-dimensional disk. The paper defines the limits of the simplicity and the two-dimensionality exactly: tunnels and bridges are outside the scope of a 2-D topology. Those 'handles' are difficult to cope with and need further investigation. We show how to detect these cases efficiently. Based on 2.8-D maps, we present concepts to define thematic objects and their aggregation structures in a consistent way.

The conceptual simplicity of 2.8-D maps allows to adopt 2-D concepts for guaranteeing consistency, enabling users to detect errors and check consistency effectively and efficiently. In fact, we give a set of axioms, which are provably correct and complete for 2.8-D maps. Since correctness and completeness are proved with mathematical rigor, users can rely on the results. The costs to check these axioms increase negligibly compared to the 2-D case.

Most commercial GIS tools can handle 2-D or 2.5-D applications. From a tool perspective, the step from 2.5-D to 2.8-D maps is not huge. It can be realized with little effort, but yields a large profit due to the modeling power of 2.8-D maps. This implies a good cost-benefit ratio between adoption of algorithms and added value for 3-D applications.

Further work remains to be done. One important question is the incorporation of unconnected graphs, which are necessary to model objects like windows or chimneys, which do not touch other edges of surrounding faces. This work can be done based on [9], which covers the 2-D case. The extension of the model to cope with handles and invisible faces is the following step to be taken. The procedure to check consistency of such models should first separate handles and invisible faces, check all these parts separately and finally check the consistency of the interfaces between the parts.

Another challenge is the specification of transaction rules for 2.8-D maps, which maintain consistency once given, when updates are performed. Here, the 2-D case was dealt with in [10]. In addition, an important question is how to incorporate other efficient data structures like the winged edge structure [27]. The redundancy introduced by this structure requires additional efforts to check consistency.

Acknowledgments

The ideas described in this article partially emerged from a joint report prepared by both authors together with Ingo Petzold and Prof. Dr.-Ing. Wolfgang Förstner, Director of the Institute for Photogrammetry of the University of Bonn, funded by the Land Surveying Agency of North Rhine Westphalia. Discussions with Ingo Petzold and Thomas H. Kolbe helped to generate and clarify the ideas described here. We thank Daniela Schulz and Viktor Stroh for assistance in preparing the illustrations.

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