



The Equivalent Modulus of Elasticity of Soil Mediums for Designing Shallow Foundations

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Abstract As known, in a Winkler type of analysis the soil medium underneath the foundation is violently replaced by a row of parallel springs having constant k_s . For the effective calculation of the latter, which is called the modulus of subgrade reaction, the two elastic constants of the soil (the elastic modulus, E and the Poisson's ratio, ν) must be known. Although for homogenous soils this generally seems not to be a problem, the same does not stand for stratified mediums or mediums with linearly increasing modulus with depth. In addition, in a Winkler type of analysis, the proper pair of elastic constant values of soil should be selected. This refers to a Poisson's ratio value equal to zero corresponding to the deformation pattern of springs (compression with no lateral expansion) and the respective modulus. In the present paper a method for calculating the equivalent elastic constants for the above mentioned mediums is proposed based on the theory of elasticity combining the principle of superposition. Various cases are considered, since the equivalent modulus, E_{eq} , depends on the rigidity and the shape of the footing. As shown, the derived E_{eq} values not only return reliable settlement results, but also settlement profiles that are similar to those corresponding to the original soil mediums.

Keywords Equivalent modulus of elasticity · Mat foundations · Shallow foundations · Stratified soil · Winkler's spring method · Soil-structure interaction

1 Introduction

Since 1949, according to the best knowledge of the author (Pantelidis 2019), eighteen methods calculating the equivalent modulus of elasticity (E_{eq}) of stratified mediums have been proposed; indeed, eight of them after 2000 indicating that this research topic remains open. Parallely, it makes sense that these methods refer exclusively to stratified soil mediums, although soil mediums with linearly increasing modulus with depth are often met in practice. In brief, Odemark's (1949), de Barros's (1966), Sridharan et al.'s (1990), Hirai and Kamei's (2003, 2004), Hirai's (2008), Abu-Farsakh and Chen's (2012) methods are based on the concept of the flexural rigidity of thin slabs, Egorov and Nichiporovich (1961), Barden (1962), Gorbunov-Possadov and Malikova (1973), Sadrekarimi and Akbarzad (2009), Brahma and Mukherjee (2010), Budhu (2011) approached the problem from the mathematical mean point of view, Salamon (1968), Wardle and Gerrard (1972) and Gerrard (1982) suggested methods based on stress-strain relationships of the theory of elasticity while Ueshita and Meyerhof (1967), Fraser and Wardle (1985) and

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HariBharghav et al. (2017) proposed methods purely based on the theory of elastic settlement analysis. The comprehensive review of these methods offered recently by the author (Pantelidis 2019) clearly shows that there is still not a reliable method.

In the present paper a method for calculating both E_{eq} and ν_{eq} of stratified soil mediums or homogenous mediums with linearly increasing modulus with depth is proposed. This method is based on the theory of elasticity.

2 Derivation of the Proposed Equivalent Elastic Moduli

2.1 Derivation of the Proposed Equivalent Elastic Modulus for Flexible Rectangular Footings

According to Boussinesq (1885), the normal stress increase at any point in a homogenous, elastic and isotropic semi-infinite medium due to a concentrated load P on its surface is:

$$\Delta\sigma_x = \frac{P}{2\pi} \left(\frac{3x^2z}{L_1^3} - (1 - 2\nu) \left(\frac{x^2 - y^2}{L_1 r^2 (L_1 + z)} + \frac{y^2 z}{L_1^3 r^2} \right) \right) \tag{1}$$

$$\Delta\sigma_y = \frac{P}{2\pi} \left(\frac{3y^2z}{L_1^3} - (1 - 2\nu) \left(\frac{y^2 - x^2}{L_1 r^2 (L_1 + z)} + \frac{x^2 z}{L_1^3 r^2} \right) \right) \tag{2}$$

$$\Delta\sigma_z = \frac{P}{2\pi} \frac{z^3}{(r^2 + z^2)^{\frac{5}{2}}} \tag{3}$$

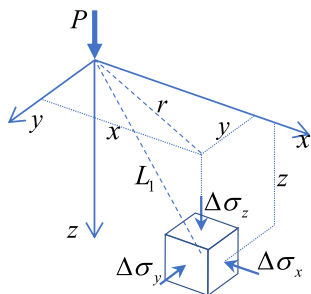


Fig. 1 Stresses in an elastic medium caused by a point load acting on the surface of a semi-infinite mass

where $L_1 = \sqrt{x^2 + y^2 + z^2}$, $r = \sqrt{x^2 + y^2}$ and ν is the Poisson’s ratio (see Das 2007). The notation followed is shown in Fig. 1.

Considering a rectangular, uniformly loaded area which extends from x_1 to x_2 and from y_1 to y_2 (in the x - and y - direction respectively) on the surface of the elastic medium, the increase in the normal stress at any point in the medium in question will be:

$$\Delta\sigma_{i,S} = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \Delta\sigma_i(x, y, z) \tag{4}$$

The differentials dx and dy of the variables x and y are included in $\Delta\sigma_i$ as $P = qdydx$ (recall Eqs. 1 to 3; q is the uniform loading). Consequently, from Hooke’s law, the unit vertical strain is:

$$\varepsilon_z = \frac{1}{E_s} [\Delta\sigma_{z,S} - \nu(\Delta\sigma_{x,S} + \Delta\sigma_{y,S})] \tag{5}$$

Using Eqs. 1 to 3, the latter can be rewritten as follows:

$$\varepsilon_z = \frac{q}{E_s} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \frac{z(1 + \nu)(3z^2 - 2(x^2 + y^2 + z^2)\nu)}{2\pi(x^2 + y^2 + z^2)^{5/2}} dx dy \tag{6}$$

The settlement, ρ , at the foundation level corresponding to a layer extending from $z = 0$ (level of foundation) to $z = H$, derives from the integration of Eq. 6 between these limits, that is:

$$\rho = \int_0^H \varepsilon_z dz \tag{7}$$

The problem has been solved by Steinbrenner (1934) for a general $B \times L$ footing. The solution can be found in any geotechnical engineering book, usually, combined with Fox’s (1948) embedment depth factor I_E . More specifically,

$$\rho = q(a'B') \frac{1 - \nu^2}{E_s} I_s I_E \tag{8}$$

where,

$$I_s = F_1 + \frac{1 - 2\nu}{1 - \nu} F_2 \tag{9}$$

$$F_1 = \frac{1}{\pi}(A_0 + A_1) \tag{10}$$

$$F_2 = \frac{n_1}{2\pi} \tan^{-1} A_2 \tag{11}$$

$$A_0 = m_1 \ln \frac{(1 + \sqrt{m_1^2 + 1}) \sqrt{m_1^2 + n_1^2}}{m_1 (1 + \sqrt{m_1^2 + n_1^2 + 1})} \tag{12}$$

$$A_1 = \ln \frac{(m_1 + \sqrt{m_1^2 + 1}) \sqrt{1 + n_1^2}}{m_1 + \sqrt{m_1^2 + n_1^2 + 1}} \tag{13}$$

$$A_2 = \frac{m_1}{n_1 \sqrt{m_1^2 + n_1^2 + 1}} \tag{14}$$

Also, $a' = 4$, $B' = B/2$, $m_1 = L/B$ and $n_1 = H/(B/2)$ for the settlement at the center of foundation and $a' = 1$, $B' = B$, $m_1 = L/B$ and $n_1 = H/B$ for the settlement at the corner of foundation.

Apparently, there is an equivalent homogenous soil medium with elastic constants (E_{eq}, ν_{eq}) producing under identical loading conditions the same settlement with the original soil medium. Equating these settlements (note: for the stratified medium the principle of superposition is necessary to be applied; see Barnes 2016)

$$\begin{aligned} & q(a'B') \frac{1 - \nu_{eq}^2}{E_{eq}} I_s(h_n, \nu_{eq}) I_E(\nu_{eq}) \\ &= q(a'B') \sum_{i=1}^n \frac{1 - \nu_i^2}{E_{s,i}} (I_s(h_i, \nu_i) - I_s(h_{i-1}, \nu_i)) I_E(\nu_i) \end{aligned} \tag{15}$$

and solving the latter as for E_{eq} , the following expression is obtained:

$$E_{eq} = \frac{I_s(h_n, \nu_{eq}) \cdot I_E(\nu_{eq}) \cdot (1 - \nu_{eq}^2)}{\sum_{i=1}^n \left[\frac{I_s(h_i, \nu_i) - I_s(h_{i-1}, \nu_i)}{E_{s,i}} I_E(\nu_i) \cdot (1 - \nu_i^2) \right]} \tag{16}$$

where, h_i is the vertical distance extending from the level of foundation to the bottom of the i -th layer (that is, it is not the thickness of the i -th layer); there are n such layers. Also, it is $I_s(h_0, \nu_1) = 0$. Equation 16 has two unknowns, namely, E_{eq} and ν_{eq} ; however as shown later, the two unknowns can easily be reduced to one, the equivalent modulus E_{eq} . Finally, from Eq. 16 it is inferred that, E_{eq} depends on the point

considered on the plan-view of footing, as the parameter n_1 in the I_s factor depends on the location of this point (recall Eqs. 12–14). By suitably applying the principle of superposition on the foundation level (see Barnes 2016), Eq. 16 can be used so that the E_{eq} value to be calculated at any point on the plan-view of footing.

It is noted that, Eq. 16 stands for any I_s factor derived from a settlement equation of the form:

$$\rho = q(\text{Width or Diameter of footing}) \frac{1 - \nu^2}{E_s} I_s I_E \tag{17}$$

2.2 Derivation of the Proposed E_{eq} for Circular Footings

2.2.1 Circular Flexible Footings

In a similar way, the settlement at the center of a flexible circular footing is:

$$\rho = q(2r_1) \frac{1 - \nu^2}{E_s} I_s I_E \tag{18}$$

where

$$\begin{aligned} I_s = & \left(1 - \frac{r_1}{\sqrt{r_1^2 + H^2}} \right) \\ & + \frac{1 - 2\nu}{2(1 - \nu)} \left(1 - \frac{H}{\sqrt{r_1^2 + H^2}} \right) \frac{H}{r_1} \end{aligned} \tag{19}$$

and r_1 is the radius of footing and H is the thickness of the compressible medium measured from the foundation level.

Equations 18 and 19 (ignoring the I_E factor) derives from the following single integral:

$$\varepsilon_z = \frac{q}{E_s} \int_0^{r_1} \frac{z(1 + \nu)(3z^2 - 2(r^2 + z^2)\nu)}{2\pi(r^2 + z^2)^{5/2}} (2\pi r_1) dr \tag{20}$$

where, the term $x^2 + y^2$ in Eq. 6 has been replaced by r^2 . E_{eq} is then obtained from Eq. 16 using the I_s factor of Eq. 19.

2.2.2 Circular rigid footings

The settlement of a rigid circular footing is (see Kézdi and Rétháti 1988):

$$\rho = (1 + \nu) \frac{qr_1}{2E_s} \left[2(1 - \nu) \left(\frac{\pi}{2} - \alpha \right) - \sin \alpha \cos \alpha \right] \quad (21)$$

where $\alpha = \arctan(r_1/H)$ and r_1 and H as before. Equation 21 is rewritten in the form of Eq. 17 as follows:

$$\rho = q(2r_1) \frac{1 - \nu^2}{E_s} I_s I_E \quad (22)$$

where,

$$I_s = \frac{1}{2} \left[\left(\frac{\pi}{2} - \alpha \right) - \frac{\sin \alpha \cos \alpha}{1 - \nu} \right] \quad (23)$$

E_{eq} is then obtained from Eq. 16 using the I_s factor of Eq. 23.

2.3 Derivation of the Proposed E_{eq} for Flexible Rectangular Footings Founded on Soil Medium with Linearly Increasing Elastic Modulus

Considering now that, a flexible rectangular footing rests on “Gibson’s soil” (Gibson 1967), that is, on a soil with elastic modulus linearly increasing with depth as follows:

$$E_{s,z} = E_{s,0} + k \cdot z \quad (24)$$

Equation 6 then becomes:

$$\varepsilon_{z,G} = \frac{q}{E_{s,0}} \int_{-B/2}^{B/2} \int_{-L/2}^{L/2} \frac{z(1 + \nu)(3z^2 - 2(x^2 + y^2 + z^2)\nu)}{\left(1 + \frac{k}{E_{s,0}}z\right) 2\pi(x^2 + y^2 + z^2)^{5/2}} dx dy \quad (25)$$

where, $E_{s,0}$ is the soil modulus immediately below the footing, k is the rate of increase of the modulus per unit depth and z is the depth measured from the foundation level.

On the other hand, the strain at depth z for the equivalent homogenous medium is:

$$\varepsilon_{z,eq} = \frac{q}{E_{eq}} \int_{-B/2}^{B/2} \int_{-L/2}^{L/2} \frac{z(1 + \nu_{eq})(3z^2 - 2(x^2 + y^2 + z^2)\nu_{eq})}{2\pi(x^2 + y^2 + z^2)^{5/2}} dx dy \quad (26)$$

Assuming, now, that the soil medium has thickness H , the equivalent elastic modulus, E_{eq} , derives from the numerical solution of the equation below:

$$\int_0^H \varepsilon_{z,eq} dz = \int_0^H \varepsilon_{z,G} dz \quad (27)$$

The results are given in chart form in Fig. 2 for various aspect ratio of footing L/B and $\nu_{eq} = \nu$ (assuming that all layers have the same Poisson’s ratio value) or $\nu_{eq} = 0$. H in Fig. 2 is the influence depth of footing as proposed by Terzaghi et al. (1996), i.e.:

$$z_l = 2B \left(1 + \log \frac{L}{B} \right) \quad (28)$$

Recalling Eq. 24, the depth where the elastic modulus of soil is equal to the equivalent one is:

$$z_{eq} = (E_{eq} - E_{s,0})/k \quad (29)$$

In this respect, z_{eq}/z_l versus k/E_0 charts for various L/B cases for $\nu_{eq} = \nu$ ($0.1 \leq \nu \leq 0.5$) are given in Fig. 3. It is noted that, Eqs. 25 and 26 refer to the center of footing.

3 Application Examples

3.1 Schmertmann’s (1970) Stratified Medium

The multilayer soil profile used by Schmertmann’s (1970; “Figure 6”) is also used here. This consists of eleven (horizontal) soil strata over bedrock; the thickness and modulus of each soil stratum is given in Table 1. A perfectly flexible, a smooth rigid and a rough rigid circular footing of 2.6 m diameter resting on the surface of the stratified medium were considered. Each one of these footings carries uniform load of 200 kPa. The Poisson’s ratio value for all strata is the same and equal to 0.4. The derived homogenous equivalent soil mediums were compared against the original stratified medium in a finite element analysis framework using Rocscience’s RS2. The criterion

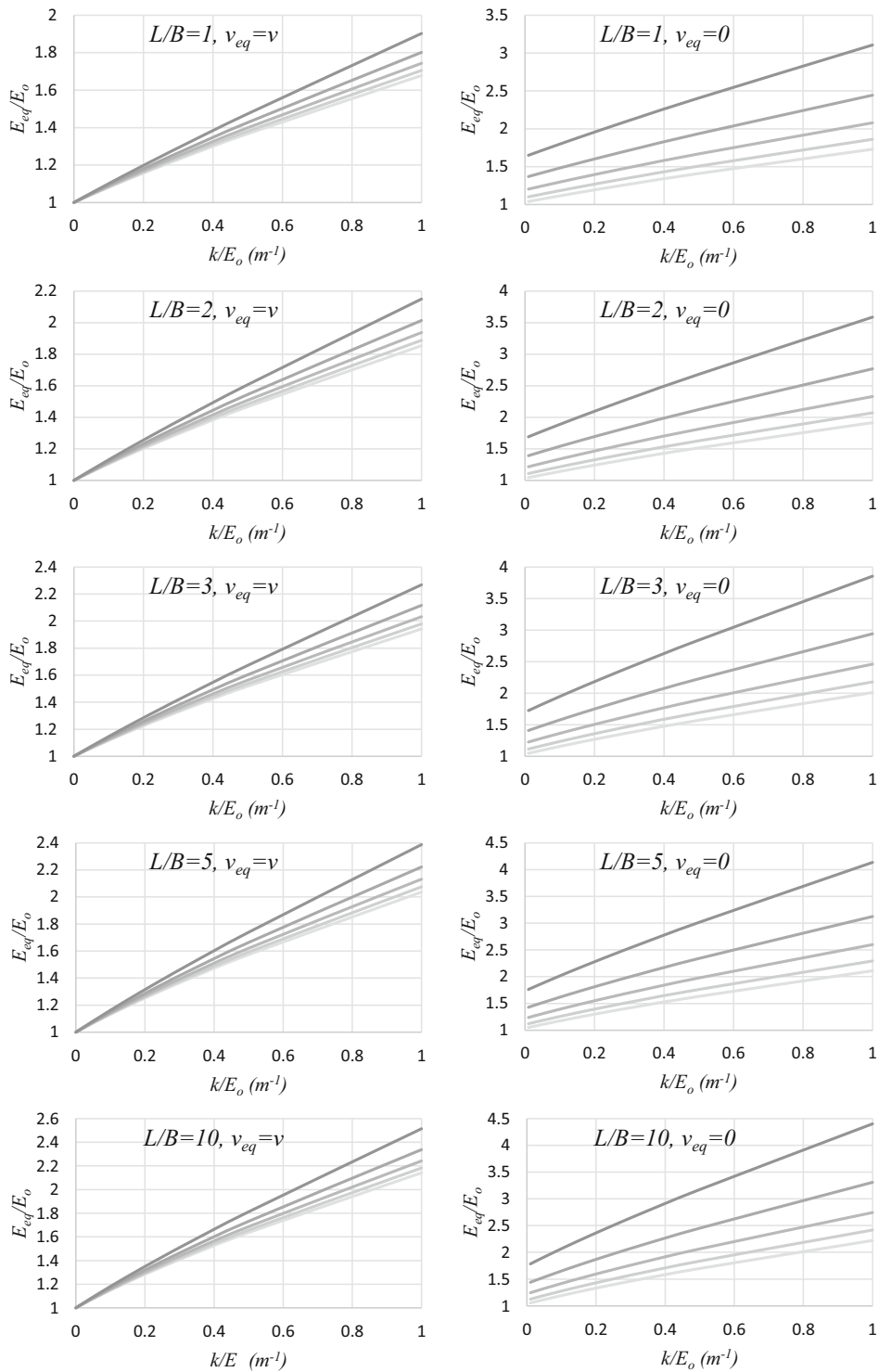


Fig. 2 E_{eq}/E_0 versus k/E_0 charts for various L/B cases for $v_{eq} = v$ (left) and $v_{eq} = 0$ (right). v ranges from 0.1 to 0.5 (from the lower to the upper curve) with 0.1 interval

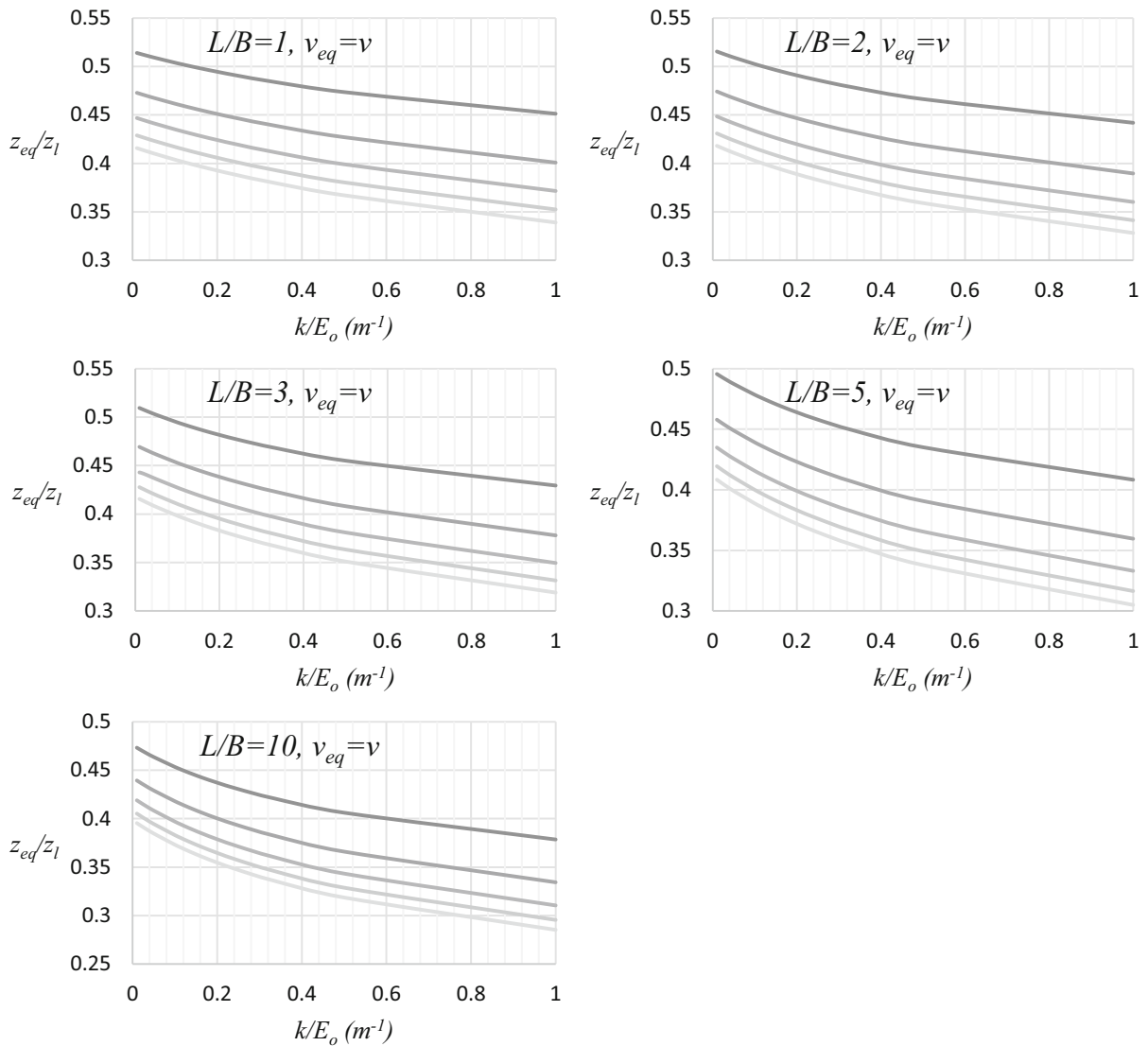


Fig. 3 z_{eq}/z_l versus k/E_0 charts for various L/B cases for $v_{eq} = v$. v ranges from 0.1 to 0.5 (from the lower to the higher curve) with 0.1 interval

used was that a homogenous soil medium is equivalent to a stratified one if under the same loading conditions, the two footings over them yield the same settlement. Prior to running the finite element models, the validity of the models was checked against analytical results (see Appendix).

Considering that $v_{eq} = 0.4$, Eq. 16 (with the use of the I_s factor of Eq. 19) gave $E_{eq} = 9132$ kPa; for $v_{eq} = 0$ it was found that $E_{eq} = 11,108$ kPa. The settlement values obtained are summarized in Table 2. As shown, the error due to the use of the equivalent elastic

constants is negligible. It is very interesting that, as shown in Fig. 4, not only the maximum settlement values do not differ but also the whole settlement profiles of the equivalent mediums are almost similar to the original one. Indeed, the same profile is taken using any (v_{eq}, E_{eq}) pair of values derived from Eq. 16; these pair of values are shown in Fig. 5. In the same figure, the relative difference (R_d) in E_{eq} is also given; the following formula for R_d was used:

$$R_d = 1 - \frac{E_{eq}(v_{eq} = v)}{E_{eq}(v_{eq} = 0)} \quad (30)$$

Table 1 E_s values and vertical extend of each layer (based on q_c given in Schmertmann (1970); also, according to Schmertmann et al. (1978), $E_s(\text{kPa}) = 2.5 q_c$ for the axisymmetric case)

| # of layer | $z_{up} - z_{lo}$ (m) | E_s (kPa) | # of layer | $z_{up} - z_{lo}$ (m) | E_s (kPa) |
|------------|-----------------------|-------------|------------|-----------------------|-------------|
| 1 | 0.0–1.0 | 6622 | 7 | 6.5–7.5 | 15,206 |
| 2 | 1.0–3.0 | 9074 | 8 | 7.5–8.5 | 24,525 |
| 3 | 3.0–3.5 | 17,658 | 9 | 8.5–10 | 10,301 |
| 4 | 3.5–4.5 | 7848 | 10 | 10–11 | 16,677 |
| 5 | 4.5–5.0 | 21,337 | 11 | 11–14 | 29,430 |
| 6 | 5.0–6.5 | 41,693 | 12 | 14–∞ | Bedrock |

Table 2 Comparison of settlements derived from Schmertmann’s (1970) original example medium and its equivalent both for $\nu_{eq} = 0.4$ and 0

| Case | Original, stratified soil medium | Equivalent medium | |
|---------------------------|----------------------------------|---|---|
| | | $\nu_{eq} = 0.4, E_{eq} = 9132 \text{ kPa}$ | $\nu_{eq} = 0, E_{eq} = 11,108 \text{ kPa}$ |
| Flexible footing (center) | 45.7 mm | 44.1 mm | 44.1 mm |
| Rigid smooth footing | 31.8 mm | 31.1 mm | 28.6 mm |
| Rigid rough footing | 31.8 mm | 31.1 mm | 28.6 mm |

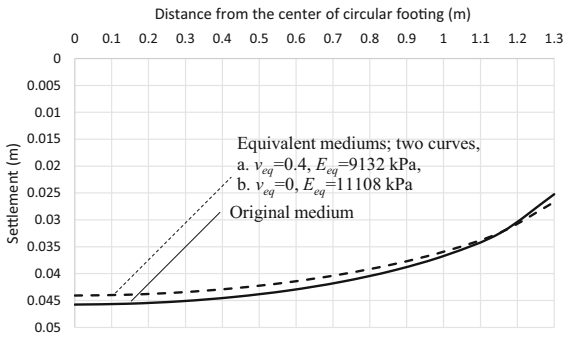


Fig. 4 Comparison of settlement profiles derived from Schmertmann’s (1970) original example medium and its equivalent both for $\nu_{eq} = 0.4$ ($E_{eq} = 9132 \text{ kPa}$) and 0 ($E_{eq} = 11,108 \text{ kPa}$)

It is a common practice, however, a study to be restricted to a specific influence zone depth (recall Eq. 28), due to the phenomenon of cementation of soils. Cementation refers to the additional bonding between particles through certain extraneous chemical substances present and hence exhibit higher resistance against deformation (Nagaraj et al. 1995); it contributes to the shear strength of a soil only where the shear strain is kept at low levels (0.001%; see Budhu

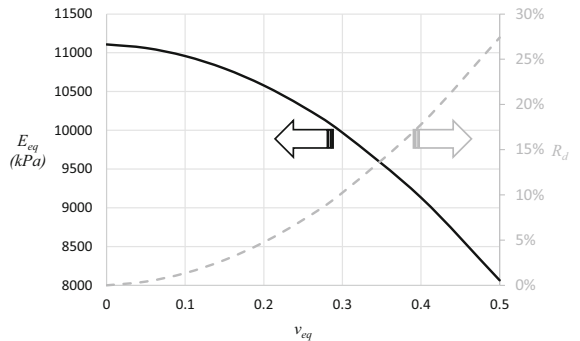


Fig. 5 Black, solid line: (ν_{eq}, E_{eq}) combinations giving the same settlement profile in the example considered (obtained from Eq. 16). Gray, dashed line: Relative difference of the E_{eq} value between the $\nu_{eq} = \nu$ and the $\nu_{eq} = 0$ case (Eq. 30)

2011). Considering an influence zone depth equal to two times the footing diameter (value suggested for circular and square footings; see Schmertmann et al. 1978 and Terzaghi et al. 1996), the equivalent modulus of elasticity, E_{eq} , for the example studied herein is 8336 kPa and 10,580 kPa for $\nu_{eq} = 0.4$ and 0 respectively; the variation of E_{eq} with the ‘(normalized) influence zone depth’ for the present example is also given (see Fig. 6) for both $\nu_{eq} = 0.4$ and 0.

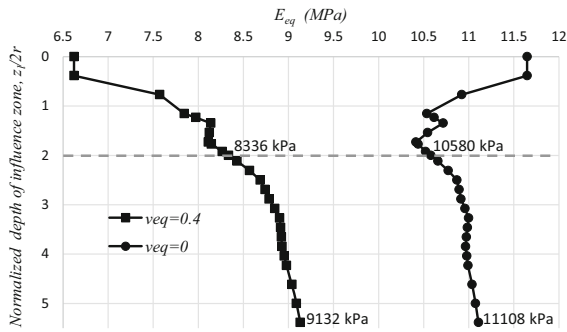


Fig. 6 Variation of E_{eq} with the normalized influence zone depth for the example considered for both the $\nu_{eq} = 0.4$ and 0 case

Some more examples are given in Table 3 for comparison purposes. From the table in question it is inferred that footing rigidity also affects the E_{eq} value; in this respect, E_{eq} is greater for rigid footings. The effect of footing area on E_{eq} is also shown; the difference in E_{eq} between footings of different footprint area is attributed to the fact that, as known, bigger footings have greater influence zone depth (bigger stress bulb) and thus, the E_{eq} value is affected by the deeper strata. Moreover, as shown in Table 3, the effect of the footing point (on plan-view) considered

on E_{eq} is significant for square (flexible) footings and negligible for (flexible) long footings. It is additionally noted that, the relative difference between any $\nu_{eq} = \nu$ case and the $\nu_{eq} = 0$ case can be taken by the simple empirical formula $R_d = \nu^2$; that is, the same R_d value is obtained for any $B \times L$ footing, footing width or location on footing’s plan-view (see Table 3). The E_{eq} value corresponding to $\nu_{eq} = 0$ and infinite influence zone depth can be taken by the following equation:

$$E_{eq}(\nu_{eq} = 0) = \frac{E_{eq}(\nu_{eq} = \nu)}{1 - \nu^2} \tag{31}$$

For comparison purposes “Bowles” (Egorov and Nichiporovich 1961; Bowles 1996) E_{eq} value is also given in Table 3:

$$E_{eq} = \frac{\sum H_i E_i}{\sum H_i} \tag{32}$$

where H_i and E_i are the thickness and modulus of the i -th layer.

“Bowles” method is probably the most widely used method among practitioners; it is noted however that, it does not consider the effect of the Poisson’s ratio of soils, the footing shape and size, the location on the plan-view of footing and more importantly, the

Table 3 The equivalent elastic modulus for various footing cases on Schmertmann’s (1970) stratified medium both for $\nu_{eq} = 0.4$ and 0 (the depth of influence zone was considered to be the whole thickness of Schmertmann’s (1970) medium)

| Method | Footing type | Location on footing’s plan-view | Footing dimensions (m) $2r$ or $B \times L$ | E_{eq} (kPa) | | R_d (%) |
|----------------|-------------------|---------------------------------|--|--------------------|--------------------|-----------|
| | | | | $\nu_{eq} = 0.4$ | $\nu_{eq} = 0$ | |
| Proposed | Flexible circular | Center | 2.6 | 9132 | 11,108 | 17.8 |
| | | Center | 5.2 | 10,188 | 13,239 | 23.0 |
| | Rigid circular | n/a | 2.6 | 9920 | 12,069 | 17.8 |
| | | n/a | 5.2 | 11,953 | 14,918 | 19.9 |
| | Flex. square | Center | 1.0×1.0 | 8109 | 9654 | 16.0 |
| | | Center | 2.6×2.6 | 10,167 | 12,104 | 16.0 |
| | | Center | 5.2×5.2 | 12,491 | 14,870 | 16.0 |
| | | Corner | 2.6×2.6 | 12,491 | 14,870 | 16.0 |
| | Flex. rectangular | Center | 5.2×5.2 | 15,449 | 18,391 | 16.0 |
| | | Center | 1.0×10 | 9213 | 10,967 | 16.0 |
| Center | | 2.6×26 | 10,464 | 12,457 | 16.0 | |
| Corner | | 2.6×26 | 10,883 | 12,956 | 16.0 | |
| Bowles’ (1996) | Any | Any | Any | 19629 ^a | 19629 ^a | 0 |

^aThe same value is used in all footing and ν cases; Considering Bowles’ (1996) $z_i = 5B = 13$ m suggestion, $E_{eq} = 18,875$ kPa

vertical distance of each soil layer from the foundation level. In every case, for the example cases considered herein, Bowles' E_{eq} value is very high and thus, on the non-conservative side (see Table 3). Finally, for circular flexible footings, the diameter seems to affect the relative difference in a neither significant nor negligible manner, whilst for rigid circular footings, the footing diameter plays less significant role.

3.2 Reducing the Elastic Modulus of a Homogenous Medium to a Value Suitable for a Winkler Spring Analysis

One of the important outcomes of the present paper is that, since the actual deformation pattern of soil mediums defers from the respective one assuming an array of Winkler springs, it is mandatory that, the elastic modulus of soil in a Winkler's spring analysis be modified as to correspond to a $\nu = 0$ medium, even when the soil is homogenous. The $\nu_{eq} = 0$ value corresponds to compression with no lateral expansion, that is, to the deformation pattern of springs. For example, if the soil below a circular foundation of diameter 2.6 m is homogenous with $\nu = 0.2$ and $E_s = 5000$ kPa, the correct E_s value for a Winkler type analysis, according to Eq. 16 (for $n = 1$), is 5332 kPa if $z_l/2r = 2$, and 5208 kPa if $z_l/2r = \infty$ (both values stand for $\nu = 0$); the 5208 kPa value can alternatively be obtained from Eq. 31, i.e. $5000/(1-0.2^2) = 5208$. If, now, the soil below the same foundation had $\nu = 0.4$ and $E_s = 5000$ kPa, the correct E_s value would be 6346 kPa for $z_l/2r = 2$ and 5952 kPa for $z_l/2r = \infty$ (again both values stand for $\nu = 0$).

4 Summary and Conclusions

The two elastic constants of soil are essential parameters for assessing the rigidity of shallow foundations (e.g. Gupta 2007), calculating the critical column spacing (e.g. ACI Committee 336, Ulrich et al. 1988) and calculating the modulus of subgrade reaction of soil, k_s (e.g. Vesic 1961). The latter is used as the spring constant in a Winkler type analysis. These springs must effectively represent the deformability of the ground in every case, that is, in homogenous soils, stratified mediums or homogenous soils with linearly increasing modulus; an effective representation

involves the use of the E_{eq} value corresponding to $\nu_{eq} = 0$ so that the equivalent soil to be consistent with the deformation pattern of Winkler's springs (compression with no lateral expansion). This paper presents a method for the determination of the equivalent elastic constants, E_{eq} and ν_{eq} , of the above-mentioned cases. The method in question is based on the theory of elasticity combining the principle of vertical and horizontal superposition. Flexible rectangular footings over stratified mediums or over homogenous mediums with linearly varying modulus, as well as rigid and flexible circular footings over stratified mediums were considered. The results are presented either as closed-form expressions or in chart form. A number of application examples are also given. As shown, the derived E_{eq} values not only return reliable settlement results (reliable as for the "equivalency"), but also settlement profiles that are similar to those corresponding to the original soil mediums.

Appendix: Numerical Modeling and Validation with Analytical Modeling

The numerical modeling carried out using Rocscience's RS2 (v9.028) was checked against analytical modelling. A flexible circular footing (axisymmetric model) having radius equal to one meter was considered (Fig. 7). The mesh used is shown in Fig. 7; this extends 15 m horizontally and 10 m vertically, while the geometry consists of 5651, 6-noded, triangular elements (77 nodes per square meter on average). The nodes of the external boundaries were fixed, while the nodes along the axis of rotation were free to slide in the vertical direction indicating a perfectly smooth boundary. The model of Fig. 7 refers to a homogenous soil with $E = 10,000$ kPa and $\nu = 0.499$. The surcharge is uniform and equal to 100 kPa, simulating a circular, flexible footing. Perfectly elastic conditions were considered. The maximum settlement obtained (at the center of the footing), was equal to 0.013458 m (shown in Fig. 7). The analytical modelling, on the other hand, gave 0.013507 m (use of Eq. 18 with $I_E = 1$).

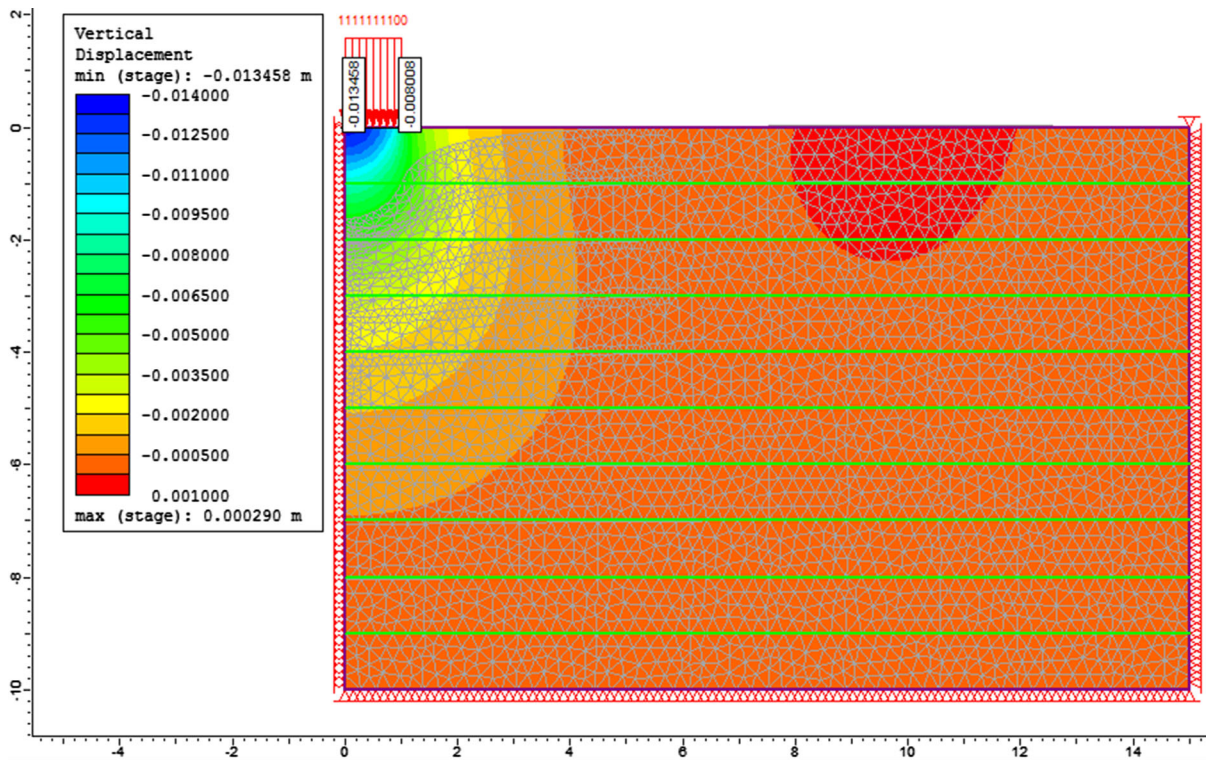


Fig. 7 Validation example: (deformed) finite element mesh and loading and boundary conditions

References

- Abu-Farsakh MY, Chen Q (2012) Evaluation of the base/sub-grade soil under repeated loading: phase II, in-box and ALF cyclic plate load tests. Louisiana Transportation Research Center
- Barden L (1962) Distribution of contact pressure under foundations. *Geotechnique* 12:181–198
- Barnes G (2016) *Soil mechanics: principles and practice*. Palgrave Macmillan, London
- Barros ST (1966) Deflection factor charts for two and three layer elastic system. *Highw Res Rec* 145:83–108
- Boussinesq J (1885) Application des potentiels à l'étude de l'équilibre et du mouvement des solides élastiques: principalement au calcul des déformations et des pressions que produisent, dans ces solides, des efforts quelconques exercés sur une petite partie de leur surface. Gauthier-Villars, Paris
- Bowles LE (1996) *Foundation analysis and design*. McGraw-Hill, New York
- Brahma P, Mukherjee S (2010) A realistic way to obtain equivalent Young's modulus of layered soil. In: Indian geotechnical conference, Bombay, India, pp 305–308
- Budhu M (2011) *Soil mechanics and foundations*, 3rd edn. Firewall Media, Hoboken
- Das BM (2007) *Fundamentals of geotechnical engineering*, 3rd edn., Thomson-Engineering.
- Egorov KE, Nichiporovich AA (1961) Research on the deflection of foundations. In: Proceedings of the 5th international conference on soil mechanics and foundation engineering, pp 861–866
- Fox L (1948) The mean elastic settlement of a uniformly loaded area at a depth below the ground surface. In: 2nd international conference on soil mechanics and foundation engineering, p 129
- Fraser RA, Wardle LJ (1985) Numerical analysis of rectangular rafts on layered foundations. *Golden Jubil Int Soc Soil Mech Found Eng Commem*, vol 223
- Gerrard CM (1982) Equivalent elastic moduli of a rock mass consisting of orthorhombic layers. *Int J Rock Mech Min Sci Geomech Abstr* 19:9–14
- Gibson RE (1967) Some results concerning displacements and stresses in a non-homogeneous elastic half-space. *Géotechnique* 17:58–67. <https://doi.org/10.1680/geot.1967.17.1.58>
- Gorbunov-Possadov MI, Malikova TA (1973) Calculation of structures on elastic foundation. (In Russian). Stroizdat, Moscow
- Gupta SC (2007) *Raft foundations: design and analysis with a practical approach*. New Age International
- HariBharghav M, Madhav MR, Padmavathi V (2017) Estimation of Deformation Moduli of Reinforced Foundation Beds from Load Tests. In: Indian Geotech Conf 2017 GeoNest

- Hirai H (2008) Settlements and stresses of multi-layered grounds and improved grounds by equivalent elastic method. *Int J Numer Anal Methods Geomech* 32:523–557
- Hirai H, Kamei T (2003) A method to calculate settlement, stress and allowable stress of multi-layered ground. *J StructConstrEng* 573:81–88
- Hirai H, Kamei T (2004) A method to calculate settlement, stress, failure and allowable stress of multi-layered ground by equivalent thickness theory. *J Struct Constr Eng* 69:79–86
- Kézdi A, Rétháti L (1988) 4 Soil Mechanics of Earthworks, Foundations and Highway Engineering, Handbook of Soil Mechanics, Volume 3. Elsevier
- Nagaraj T, Srinivasa B (1994) Analysis and prediction of soil behaviour. Taylor & Francis, Abingdon-on-Thames
- Nagaraj T, Srinivasa B, Vatsala A (1995) Analysis and prediction of soil behaviour. New Age Publications, India
- Odemark N (1949) Investigations as to the Elastic Properties of Soils and Design of Pavements According to the Theory of Elasticity
- Pantelidis L (2019) The equivalent modulus of elasticity of layered soil mediums for designing shallow foundations with the Winkler spring hypothesis: a critical review. *EngStruct* 201:109452. <https://doi.org/10.1016/j.engstruct.2019.109452>
- Sadrekarami J, Akbarzad M (2009) Comparative study of methods of determination of coefficient of subgrade reaction
- Salamon MDG (1968) Elastic moduli of a stratified rock mass. *Int J Rock Mech Min Sci Geomech Abstr* 5:519–527
- Schmertmann JH (1970) Static cone to compute static settlement over sand. *J Soil Mech Found Div* 96:1011–1043
- Schmertmann JH, Hartman JP, Brown PR (1978) Improved strain influence factor diagrams. *J GeotechGeoenvironEng* 104(8):1131–1135
- Sridharan A, Gandhi N, Suresh S (1990) Stiffness coefficients of layered soil systems. *J GeotechEng* 116:604–624
- Steinbrenner SW (1934) Tafeln zur setzungsberechnung. Die StraBe 1
- Terzaghi K, Peck RB, Mesri G (1996) Soil mechanics in engineering practice. Wiley, Hoboken
- Ueshita K, Meyerhof GG (1967) Deflection of multilayer soil systems. *J Soil Mech Found Div* 93(5):257–282
- Ulrich EJ, Shukla SN, Baker CN Jr, Ball SC, Bowles JE, Colaco JP, Davisson XI et al (1988) Suggested analysis and design procedures for combined footings and mats. *J Am Concr Inst* 86:304–324
- Vesic AB (1961) Beams on elastic subgrade and the Winkler's hypothesis. In: Proceedings of the fifth international conference on soil mechanics and foundation engineering
- Wardle LJ, Gerrard CM (1972) The “equivalent” anisotropic properties of layered rock and soil masses. *Rock Mech Rock Eng* 4:155–175

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