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Reliability Analysis of Earth Slopes Considering Spatial Variability

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Abstract The paper presents a computational procedure for reliability analysis of earth slopes considering spatial variability of soils under the framework of the Limit Equilibrium Method. In the reliability analysis of earth slopes, the effect of spatial variability of soil properties is generally included indirectly by assuming that the probabilistic critical slip surface is the same as that determined without considering spatial variability. In contrast to this indirect approach, in the direct approach, the effect of spatial variability is included in the process of determination of the probabilistic critical surface itself. While the indirect approach requires much less computational effort, the direct approach is definitely more rigorous. In this context this paper attempts to investigate, with the help of numerical examples, how far away are the results obtained from the indirect approach from that obtained from the direct approach. In both the approaches, it is required to use a model of discretization of random fields into finite random variables. A few such models are available in the literature for one-dimensional (1D) as well as two-dimensional (2D) spatial variability. The developed computational scheme is based on the First Order Reliability Method (FORM) coupled with the Spencer Method of Slices valid for limit equilibrium analysis of general slip surfaces. The study includes bringing out the computational advantages and disadvantages of the three commonly used discretization models. The sensitivity of the reliability index to the magnitudes of the scales of fluctuation has also been studied.

Keywords Slope stability · Slip surface · Random variable · Spatial variability · Reliability analysis

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1 Introduction

In geotechnical engineering, slope stability analysis is perhaps the area which is most dominated by uncertainty (El-Ramly et al. 2002). As in other areas of geotechnical engineering, the uncertainties associated with earth slopes can be broadly categorized as the aleatory uncertainty and the epistemic uncertainty. The sources of aleatory uncertainty are the inherent spatial variability of soil caused by variations in mineral composition, environmental conditions during deposition, stress history and variation in moisture content. The sources of epistemic uncertainty, on the other hand, are the limited availability of information due to limited site investigation effort as well as imperfect information due to measurement errors, test imperfection, soil sample disturbance etc. (Bourdeau and Amundaray 2005; Ang and Tang 2007). It is now more than four decades ago that Vanmarcke (1977a, b) published his pioneering work on the modeling of spatial variability of engineering soil properties. However, in the vast majority of probabilistic slope stability analyses (reliability analyses) reported in the literature, only the epistemic uncertainty (statistical uncertainty and measurement errors) was considered; in other words, the aleatory uncertainty due to spatial variability was not taken into account.

In most of the early works reliability analysis on earth slopes were conducted on the deterministic critical slip surface (the surface of minimum factor of safety) and the reliability index associated with this surface was taken as the reliability index for the slope (Tobutt and Richards 1979; Chowdhury et al. 1987; Chowdhury and Xu 1992). However, the deterministic critical slip surface may not be the same as the probabilistic critical slip surface (surface of minimum reliability index or the maximum probability of failure), especially for non-homogeneous slopes. Subsequently, Hassan and Wolff (1999) proposed a procedure for the determination of the probabilistic critical slip surface and the associated minimum reliability index. The method has been developed based on their observation that the critical probabilistic surface was found generally to coincide with that obtained by setting one dominant parameter (random variable) to a low value. However, as pointed out by Crum (2001) and others, even though the proposed method provides a practical and efficient tool to locate the critical probabilistic surface, it does not have a mathematical basis and, therefore, cannot be proven to yield a lower bound for the reliability index. Further, Li and Cheung (2001) have cautioned that the critical probabilistic surface does not always coincide with that obtained by setting one dominant parameter to a low value. More recently, Zhang et al. (2013) have found that for slopes with complex geometry, the Hassan and Wolff method is less accurate in locating the most critical slip surface. Bhattacharya et al. (2003) proposed a procedure which is similar to the procedure for the determination of the deterministic critical slip surface. Subsequently quite a few research work were reported based on the Hassan and Wolff approach (El-Ramly et al. 2002, 2003a; Zhang et al. 2013) and the Bhattacharya et al. approach (Khajehzadeh et al. 2010; Liang and Xue-song 2012; Metya and Bhattacharya 2014). However, all these studies considered only one kind of uncertainty, namely, the epistemic uncertainty.

Studies on reliability analysis of earth slopes considering spatial variability were conducted under the framework of both Limit Equilibrium Method (LEM) and the Finite Element Method (FEM). Those based on the LEM include Li and Lumb (1987), El-Ramly et al. (2002), Low (2003), Babu and Mukesh (2004), Cho (2007), Hong and Roh (2008), Wang et al. (2011), Ji et al. (2012) and Li et al. (2013). El-Ramly et al. (2002) modeled the spatial variability of each input variable along the slip surface by a 1D stationary random field describing an elaborate spatial variability discretization model. A few others (e.g., Hong and Roh 2008; Wang et al. 2011; Li et al. 2013) also modeled the spatial variability of soil properties by a 1D random field; but they considered spatial variation along the vertical direction. It was, however, argued that if only the vertical autocorrelation distance is considered, it might result in some of the variables having no effect on the critical slip surface (Li et al. 2013). Low (2003), Cho (2007) and Ji et al. (2012) adopted the slicewise discretization of the 2D random field. Cho (2007), however, proposed a local averaging method combined with numerical integration to discretize random fields of soil properties in two-dimensional space; while Low (2003); Ji et al. (2012) used the midpoint discretization of random field known as the method of autocorrelated slices. Ji et al. (2012), however, proposed another method known as the method of interpolated autocorrelations. The authors, however, have concluded that the method of autocorrelated slices is more accurate and it should be used as the benchmark for the development of the method of interpolated autocorrelations. However, with the exception of Ji et al. (2012), most of these studies were made to determine the probability of failure (or reliability index) of a predetermined slip surface.

On the other hand, some researchers studied the influence of spatial variation on the slope reliability based on the Random Finite Element Method (RFEM) (Griffiths and Fenton 2004; Griffiths et al. 2009; Hicks and Spencer 2010) as well as the Stochastic Finite Element Method (SFEM) (Faraha et al. 2011; Jiang et al. 2014). In spite of their key advantage in assuming no failure mechanism, both the RFEM and the SFEM suffer from excessive computational efforts since the strength reduction method calculates the factor of safety by progressively reducing or increasing the shear strength of the material in order to bring the slope to a state of limiting equilibrium (Cho 2010). Moreover, as pointed out by Ji et al. (2012), the number of spatially correlated random variables assigned to elements is commonly very large in the advanced RFEM so that only Monte Carlo simulation can be employed.

Studies contained in this paper, concerning reliability analysis of earth slopes considering spatial variability of soil properties, are based on the LEM. As stated before, all the above mentioned LEM based studies considered spatial variability but the reliability analyses were conducted on predetermined slip surfaces, specifically, the probabilistic critical slip surfaces determined without considering spatial variability. It may be stated that these studies are based on an indirect approach for taking the effect of spatial variability into account. Only one or two researchers (e.g., Ji et al. 2012) have addressed the problem of direct determination of the probabilistic critical slip surface considering spatial variability during the process of determination itself. It may be stated, therefore, that such a study is based on a direct approach for taking the effect of spatial variability into account. While the direct approach is a more logical of the two approaches, the indirect approach is computationally simpler. It is therefore necessary to investigate to what extent the results obtained based on the two approaches differ, as well as which approach leads to a more conservative estimate of the safety of a slope. Further, whichever approach is used, the results would vary depending on the spatial variability discretization model used as well as the magnitudes of the scale of fluctuation in the horizontal and the vertical direction in a 2D spatial variability situation.

To this end, the purpose of the paper is to draw a comparison between the results of reliability analysis of earth slope obtained from both the indirect and the direct approaches for considering spatial variability of soil properties. Three different spatial variability discretization models, namely, the 1D model proposed by El-Ramly et al. (2002); the 2D model proposed by Cho (2007) and the 2D model of autocorrelated slices proposed by Ji et al. (2012) have been selected for the study. The magnitudes of the scale of fluctuation in the horizontal and the vertical direction have been varied from within the guidelines of their ranges as available from the literature.

2 Adopted Methodologies

2.1 Evaluation of Factor of Safety

Out of the numerous LEMs of slices currently available for slope stability analysis, the Spencer method valid for general slip surfaces (Spencer 1973) is regarded as one of the rigorous methods as it does not make any a priori assumption regarding the shape of the slip surface and satisfies both the force and the moment equilibrium conditions (Duncan and Wright 1980). In this study, therefore, this method is chosen for calculation of the factor of safety (FS) and hence for evaluation of the performance function for the reliability analysis. The method of solution for FS of a given slip surface is cast as a mathematical programming problem and solved using the well-known Sequential Quadratic Programming (SQP) (Rao 2009) technique in the MATLAB environment. The adoption of the Sequential Quadratic Programming (SQP) technique is based on Hong and Roh (2008) who reported that ‘an extensive comparative study of nonlinear programming codes presented by Schittkowski (1980) ranked the performance of the SQP method to be the highest’.

2.2 Deterministic Critical Slip Surface

The problem of determination of the critical slip surface and the associated minimum factor of safety (FS_{min}) is, as usual, cast as a mathematical programming problem, and, once again, the Sequential Quadratic Programming (SQP) technique in the MATLAB environment is employed to solve this problem.

2.3 Probabilistic Analysis

The First Order Reliability Method (FORM), being the most versatile among the FOSM methods of reliability analyses (Haldar and Mahadevan 2000), has been adopted in this study. In this method, the reliability index β is defined as the minimum distance from the origin to the failure surface in the standard normal space, using a linearization of the performance function around the design point as originally proposed by Hasofer and Lind (1974). The limit state function for the slope stability is usually defined as $g(\mathbf{X}) = FS - 1.0$, $\mathbf{X}$ being a vector of basic state (or design) variables of the system consisting of the uncertain geotechnical parameters (i.e. geometry and soil properties). The determination of the reliability index β is, thus, a problem of optimization, and as indicated by Wang et al. (2011), the successful application of FORM relies on the selection of a robust optimization algorithm for multi-dimensional minimization, the Sequential Quadratic Programming (SQP) in the MATLAB environment is employed again to solve this problem. The solution yields the design point on the failure surface and the corresponding reliability index β. Then the failure probability can be expressed as $p_F = \Phi(-\beta)$, where Φ (.) denotes the standard normal cumulative distribution function.

2.4 Search Algorithm for the Probabilistic Critical Slip Surface

In deterministic slope stability analysis, it is conventional to use an optimization based algorithm to search for the deterministic critical slip surface (surface with the minimum factor of safety) based on some suitable slope stability model. In most algorithms, the problem of locating the deterministic critical slip surface associated with the minimum factor of safety, FS_{min}, is formulated as an optimization problem as follows [Eq. (1)]:

$$FS_{min} = \min_{X} FS(\mathbf{P}, \mathbf{X}) \tag{1}$$

where, $\mathbf{P}$ = set of input geotechnical parameters: c_1, ϕ_1, c_2, ϕ_2, etc. $\mathbf{X}$ = set of co-ordinates defining the shape and location of the slip surface: x_1, y_1, x_2, y_2, etc. FS = factor of safety for a given set of geotechnical parameters $\mathbf{P}$ and a given geometry of the slip surface defined by the location parameters $\mathbf{X}$.

Bhattacharya et al. (2003) proposed a computational procedure for locating the probabilistic critical slip surface (surface of the minimum reliability index, β_{min}) for earth slopes, which is conceptually no different from that of the deterministic critical slip surface. The problem of locating the probabilistic critical slip surface associated with the minimum reliability index, β_{min}, had been formulated in exactly the same way as for the deterministic critical slip surface, viz.,

$$\beta_{min} = \min_{X} \beta(\mathbf{P}, \mathbf{X}) \tag{2}$$

where β = reliability index for a given set of geotechnical parameters (including the statistical properties) and a given geometry of the slip surface defined by its location parameters.

It is, thus, evident that the proposed computational procedure for the determination of the probabilistic critical slip surface involves a 3-tier analysis: (i) Evaluation of performance function requires the evaluation of Spencer's factor of safety involves the first tier of analysis; (ii) Evaluation of the reliability index, β based on FORM involves the second tier of analysis, and (iii) Search for the probabilistic critical slip surface and the associated minimum reliability index β_{min} involves the third tier of analysis. As mentioned before, for the first two tiers, the optimization problem has been solved using the Sequential Quadratic Programming (SQP) in the MATLAB environment. The third tier of analysis has also been solved using the SQP technique when the slip surfaces are assumed to be of circular shape. However, for the slip surface of general shape, an efficient random search technique (Greco 1996) has been employed. The reason for such choice is based on experience. For slip surfaces of general shape, the random search technique of Greco (1996) has been found to yield a lower minimum compared to the SQP technique.

3 Modeling of Spatial Variability

It is well known that even within nominally homogeneous soil layers, engineering properties of soils may exhibit considerable variation from one point to another (Vanmarcke 1977a; Lacasse and Nadim 1996; Elkateb et al. 2003) and this phenomenon is known as the spatial variability of soil. The statistical parameters such as the mean and variance are one-point statistical parameters and cannot capture the features of the spatial variability of the soil properties. The most common way to deal with spatial variation of soil properties is the random field theory (Vanmarcke 1983). For a field of interest, the soil parameters at a particular location are random variables due to the spatial variation, but are correlated with those at adjacent locations. The set of random variables at all locations in the field is referred to as a random field (Ji et al. 2012). It may not be out of place to mention that the cross correlation coefficients, commonly called correlation coefficients, are used to describe the interrelationship between two different random parameters. On the other hand, the spatial correlation coefficients involved in the discretization models of spatial variability are used to describe the interrelationship between the values of the same random parameter at different spatial locations.

The point to point variation of a random field is very difficult to obtain in practice and is often of no practical significance. Local averages over a spatial local domain (e.g., the average soil properties over the area of the slip surface) are of much greater value to geotechnical engineers. The variance of the strength, spatially averaged over some domain, is less than the variance at the point. As the size of the domain over which the soil property is being averaged increases, the variance decreases (El-Ramly et al. 2002). Vanmarcke (1977a) introduced a dimensionless variance function, $\Gamma^2(L)$, as the ratio between point variance to the variance of the spatial average of any parameter over a domain of length L, in order to measure the reduction in the point variance under local averaging.

As already indicated, the in situ soil property values at neighboring point are correlated to each other and an autocorrelation function is needed to describe this (Babu and Mukesh 2004). Quite a few autocorrelation functions are available in the literature (Vanmarcke 1977a). In general, any of these autocorrelation functions (based on statistical data) could be used and the development of such a model should be based on statistical data. This is not an easy task since acquisition of large quantity of statistical data needs much effort. This particular aspect, however, is beyond the scope of this study. Moreover, results of slope reliability analyses are generally found to be rather less sensitive to the form of autocorrelation function (Li and Lumb 1987). An autocorrelation function decays over the separation distance between spatial quantities and the distance up to which this correlation exists is termed as the autocorrelation distance, r_o (Babu and Mukesh 2004). For most commonly used autocorrelation functions, Vanmarcke (1977a), showed that the variance function $\Gamma^2(L)$ can be approximated by

$$\begin{aligned} \Gamma^2(L) &= 1 && \text{for } L \le \delta \\ \Gamma^2(L) &= \delta/L && \text{for } L \ge \delta \end{aligned} \tag{3}$$

where δ is the scale of fluctuation. In concept, the scale of fluctuation has the same meaning as the autocorrelation distance but differs in numeric value (El-Ramly et al. 2002). For the common exponential and Gaussian autocorrelation functions, the scale of fluctuation is equal to 2 and π times the autocorrelation distance, r_o, respectively (Vanmarcke 1977a). A large autocorrelation distance implies a very uniform material, and a small autocorrelation distance implies a material whose properties change over short distances (Christian et al. 1994). DeGroot (1996) and Lacasse and Nadim (1996) illustrated the estimation of r_o. But these techniques require not only substantial amount of data, but also data at very close spacing. According to El-Ramly et al. (2003b), the autocorrelation distance varies from 10 to 40 m in the horizontal direction, while in the vertical direction it ranges from 1 to 3 m. In the absence of adequate data to estimate a site or formation-specific autocorrelation distance, El-Ramly et al. (2003b) suggested that estimates of autocorrelation distance could be inferred from within these ranges. As per Mostyn and Li (1993), the ranges of distances are 2–30 m and 0.1–5 m respectively. Recently Salgado and Kim (2014) regrouped the values of scale of fluctuation already reported in the literature. Babu and Mukesh (2004) and Cho (2007) have shown that assumption of isotropic random field is always conservative. Therefore, apart from choosing or assuming autocorrelation distances in two directions and furthermore noting the significant difference in the values of the vertical and horizontal autocorrelation distances, El-Ramly et al. (2003a) suggested to identify the direction in which the variability of properties has a dominant impact on the analysis. For example, the earth pressure on a retaining structure is controlled by the variability of the coefficient of earth pressure with depth. In such cases, spatial variability in the vertical direction is likely to be more important to the analysis. In contrast, the stability of slopes in pre-sheared or jointed formations with shallow dip angles is controlled by the variability of strength parameters along near-horizontal rupture surfaces. Autocorrelation distances in horizontal directions are thus more relevant.

Quite a few models of discretization of random fields into finite random variables are available in the literature for one-dimensional (1D) as well as two-dimensional (2D) spatial variability. Three such models have been made use of in this study. For the sake of ready reference, brief outlines of these models are presented in the following.

3.1 Discretization Model I (El-Ramly et al. 2002)

In this model, proposed by El-Ramly et al. (2002), the spatial variability of each input variable along the slip surface is approximated by a 1D stationary random field. The point to point variability of each input variable along the slip surface is resembled by the variability of its local averages over segments of the slip surface. For the arbitrary positions of intermediate slip surfaces within the search domain, entire length of the slip surface is at first divided into strips within the various layers present in the soil profile. The portion of each strip length within the subject layer is divided into segments of length L not exceeding the scale of fluctuation δ. The local average, m(L), of each input variable over the length, L, of any of these segments, is considered a segment variable. The CDF of this segment variable m(L) is the same as the CDF of the original input variable, with no variance reduction. This is because, as per Vanmarcke (1977a), the variance function $\Gamma^2(L)$, [Eq. (3)], equals unity for $L \leq \delta$.

In studying the uni-dimensional situation depicted in Fig. 1, local averages $X(\Delta z)$ and $X'(\Delta z')$, of the variable x over the intervals Δz and $\Delta z'$, are spatially correlated. The correlation coefficient, $\rho\ (X_{\Delta z}, X'_{\Delta z'})$, between $X(\Delta z)$ and $X'(\Delta z')$ is given by Eq. (4) (Vanmarcke 1983; El-Ramly et al. 2002). It is a function of the lengths of the two intervals Δz and $\Delta z'$, the separation, Z_o, between them (Fig. 1), and the variance function of the variable x being averaged.

$$\rho(X_{\Delta z}, X'_{\Delta z'}) = \frac{Z_o^2\Gamma(Z_o) - Z_1^2\Gamma(Z_1) + Z_2^2\Gamma(Z_2) - Z_3^2\Gamma(Z_3)}{2\Delta z\Delta z'[\Gamma(\Delta z)\Gamma(\Delta z')]^{0.5}} \tag{4}$$

where, Z_1 is the distance from the beginning of the first interval to the beginning of the second interval, Z_2 is the distance from the beginning of the first interval to the end of the second interval, and Z_3 is the distance from the end of the first interval to the end of the second interval. Choosing the length of segments equal to δ eliminates the correlation coefficients between most of the segment variables and greatly simplifies this process. Figure 2 is a schematic illustration of this model based on the solution of an example problem solved by El-Ramly et al. (2002), where the discretization of random soil properties e.g., the shear strength of the Layer 4 (S) and the unit weight of the Layer 1 (γ) are clearly described.

3.2 Discretization Model II (Cho 2007)

Cho (2007) proposed a local averaging method combined with numerical integration to discretize both isotropic and anisotropic random fields of soil properties in two-dimensional space. Two most commonly used auto-correlation functions were taken to describe isotropic [Eq. (5)] and anisotropic [Eq. (6)] random field respectively.

$$\rho(z) = \exp\left\{-\frac{2}{\delta_X}|z|\right\} \tag{5}$$

$$\rho(x,y) = \exp\left\{-\left(\frac{2}{\delta_{X(x)}}|x| + \frac{2}{\delta_{X(y)}}|y|\right)\right\} \tag{6}$$

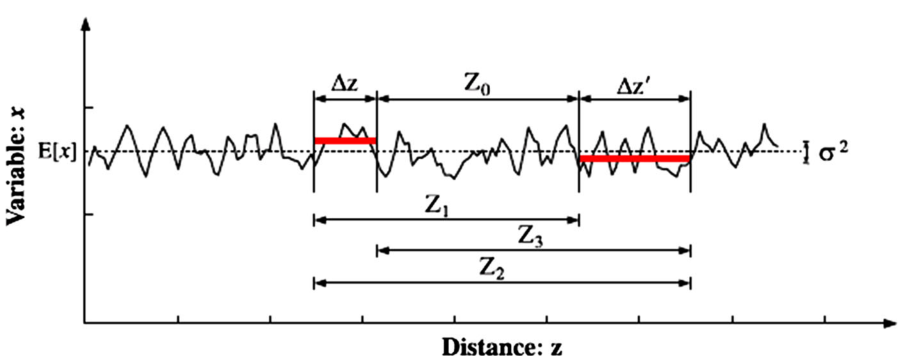

Fig. 1 A realization of a 1D random field of a variable x with a mean $E[x]$, variance σ^2, and cumulative probability distribution function $F(x)$, showing local averages over intervals Δz and $\Delta z'$ (After El-Ramly et al. 2002)

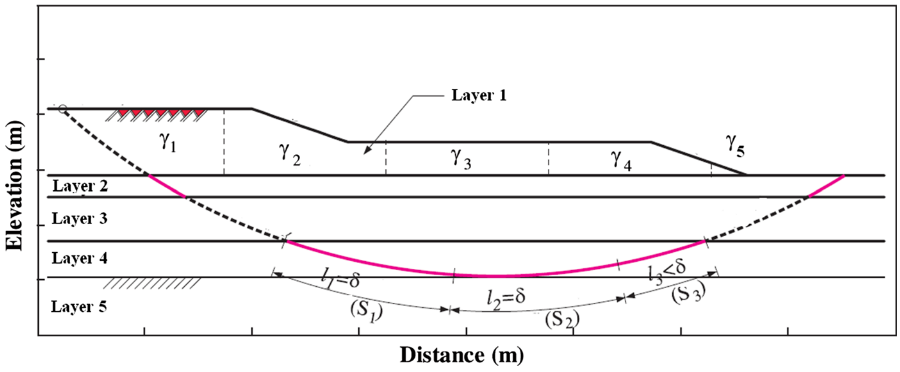

Fig. 2 Modeling the spatial variability of the input parameters along the failure surface (After El-Ramly et al. 2002)

where, $\delta_{X(x)}$ is the horizontal scale of fluctuation, and $\delta_{X(y)}$ is the vertical scale of fluctuation. Using this model of discretization, the length of the slip surface is first divided into several segments by grouping slices (Fig. 3).

The field within each segment is described in terms of the spatial average of the field over the segment base. The variance of the average strength parameters are reduced by multiplying variance function for each segment with the point variance of the random field. The variance function for the isotropic [Eq. (7)] and anisotropic [Eq. (8)] random field are as follows:

$$\gamma(L) = \frac{\delta_X^2}{2L^2}\left[\frac{2L}{\delta_X} - 1 + \exp\left\{-\frac{2L}{\delta_X}\right\}\right] \tag{7}$$

$$\gamma(L) = \frac{\left[2L\left(\frac{\cos\alpha}{\delta_{X(x)}} + \frac{\sin\alpha}{\delta_{X(y)}}\right) - 1 + \exp\left\{-2L\left(\frac{\cos\alpha}{\delta_{X(x)}} + \frac{\sin\alpha}{\delta_{X(y)}}\right)\right\}\right]}{2L^2\left(\frac{\cos\alpha}{\delta_{X(x)}} + \frac{\sin\alpha}{\delta_{X(y)}}\right)^2} \tag{8}$$

The correlation coefficients between these segments are estimated using Eq. (9).

$$\rho(L_i, L_j) = \frac{1}{L_i L_j}\int_0^{L_i}\int_0^{L_j}\rho(z)djdi \tag{9}$$

where z is the distance between the two arbitrarily situated points, one on segment i and another on segment j.

3.3 Discretization Model III (Ji et al. 2012)

Ji et al. (2012) proposed two models namely the method of autocorrelated slices and the method of interpolated autocorrelations for the probabilistic slope analysis involving 2-D spatial variation. In the method of autocorrelated slices, the strength parameter at the midpoint along the base of each slice is taken as a random variable and spatial variation between these random variables is modeled based on the random fields theory.

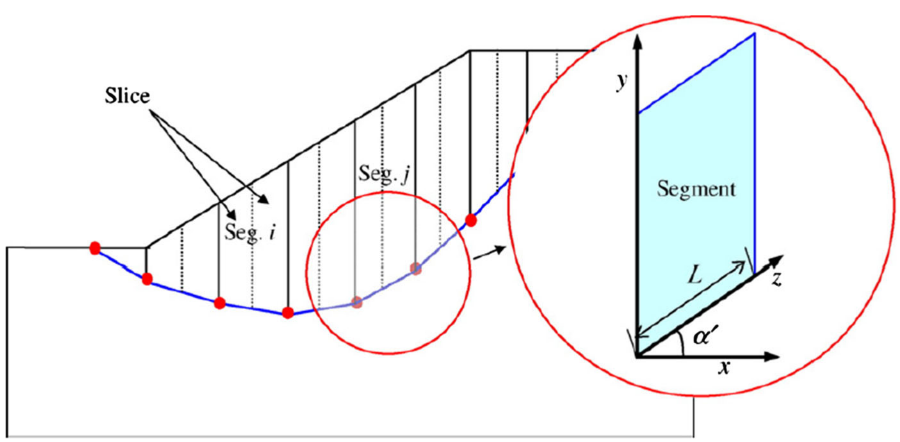

Fig. 3 Discretization of the random fields over the slip surface (After Cho 2007)

The correlation between spatial quantities is described by negative exponential autocorrelation function (as suggested by Li and Lumb 1987) as given in Eq. (10).

$$\rho_{ij} = \rho(x, y) = \exp\left(-\frac{|x_i - x_j|}{\delta_x} - \frac{|y_i - y_j|}{\delta_y}\right) \tag{10}$$

where (x_i, y_i) denotes the position of a random variable (Fig. 4); δ_x and δ_y are the horizontal and vertical autocorrelation distances respectively.

4 Developed Computer Programs

In order to carry out the computations involved in solving the numerical examples included in the paper, the following computer programs have been developed in the MATLAB environment:

1. Computer Program I: Computer Program to search for the deterministic critical slip surface.
2. Computer Program II: Computer Program based on FORM to search for the probabilistic critical slip surface without considering spatial variability.
3. Computer Program III: Computer Program based on MCS to calculate the probability of failure for a given slip surface considering spatial variability based on the discretization model II (Cho 2007).
4. Computer Program IV: Computer Program to search for the probabilistic critical slip surface incorporating spatial variability based on the discretization model I (El-Ramly et al. 2002).
5. Computer Program V: Computer Program to search for the probabilistic critical slip surface incorporating spatial variability based on the discretization model II (Cho 2007).
6. Computer Program VI: Computer Program to search for the probabilistic critical slip surface incorporating spatial variability based on the discretization model III (Ji et al. 2012).

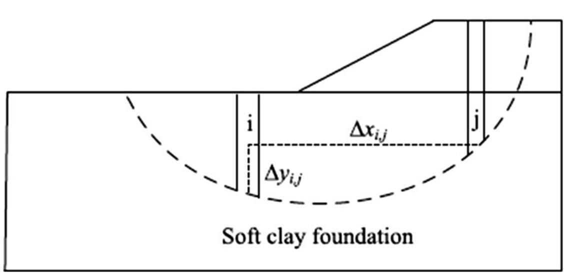

Fig. 4 Discretization of the 2-D random fields over the slip surface according to slices (After Ji et al. 2012)

5 Illustrative Examples

For the purpose of numerical demonstration of the results of the investigations proposed in this paper two example problems (Example 1 and Example 2) on layered soil slopes have been selected from the literature (Cho 2007; Ji et al. 2012) and are described in the following subsections.

5.1 Example 1: (Cho 2007)

5.1.1 Description

Example 1 (Fig. 5) is of a layered slope in clay bounded by a hard stratum below and the layer boundaries are parallel to the ground surface. The strength parameters, namely, the cohesion c, the angle of shearing resistance ϕ and the unit weight γ for each of the two layers are treated as random variables and their statistical properties are as in Table 1. No water table or external water is considered. This example was previously analysed by Cho (2007).

5.1.2 Results of Deterministic Analysis

Before searching for the deterministic critical slip surface using the developed computer program I, in order to validate the subroutine for the evaluation of the factor of safety, the deterministic critical slip surface reported by Cho (2007) has been scaled down from his paper and re-evaluated using the mean values of the input parameters in Table 1. Following Cho (2007) the total number of slices is taken as 12. The factor of safety FS is obtained as 1.592 which is identical with that reported by Cho (2007). This observation has served to validate the subroutine for the evaluation of factor of safety of a given slip surface in the computer program I.

Next, using the computer program I and assuming the soil properties to be deterministic with values equal to their mean values in Table 1, the deterministic critical slip surface has been determined using Spencer method (Spencer 1973) coupled with the Sequential Quadratic Programming (SQP) available in

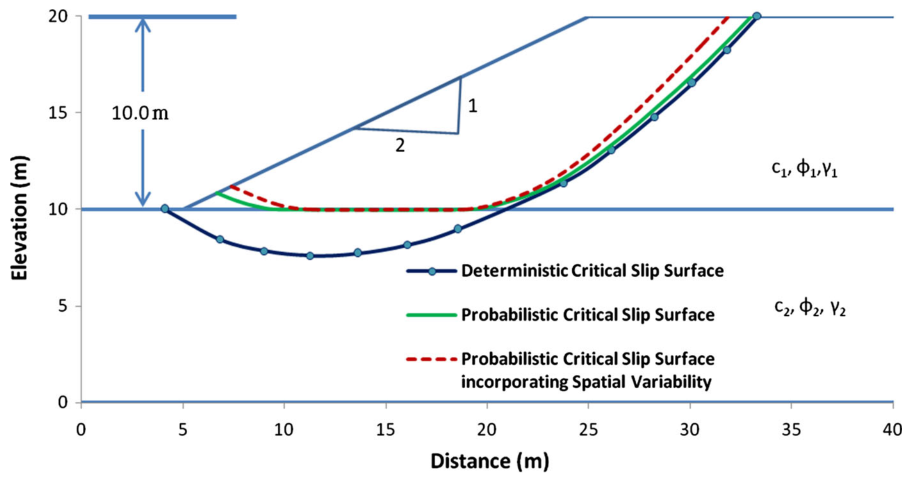

Fig. 5 Slope section and critical slip surfaces in Example 1

Table 1 Statistical properties of soil parameters for Example 1

Parameter (1)	Mean (2)	Coefficient of variation (COV)	
		Case 1 (3)	Case 2 (4)
c_1	38.31 kN/m^2	0.20	0.40
ϕ_1	0.00	–	–
γ_1	18.00 kN/m^3	0.05	0.05
c_2	23.94 kN/m^2	0.2	0.2
ϕ_2	12°	0.1	0.1
γ_2	18.00 kN/m^3	0.05	0.05

the MATLAB environment and is as shown in Fig. 5. The associated minimum factor of safety is obtained as $FS_{min} = 1.582$ which is slightly lower than 1.592 reported by Cho (2007).

5.1.3 Results of Probabilistic Analysis

The results of probabilistic analyses are presented under the following sub-headings:

I. Determination of probabilistic critical slip surface without considering spatial variability
II. Re-analysis of the above surface considering spatial variability
III. Determination of probabilistic critical slip surface considering spatial variability.

5.1.3.1 Determination of Probabilistic Critical Slip Surface Without Considering Spatial Variability Before searching for the probabilistic critical slip surface using the developed computer program II, in order to validate the subroutine for the evaluation of the reliability index β, the probabilistic critical slip surface reported by Cho (2007) has been scaled down from his paper and re-evaluated. Following Cho (2007) all the geomechanical parameters in Table 1 (except ϕ_1) are treated as random variables having lognormal distribution. The total number of slices is again taken as 12. The reliability index β is obtained as 2.602 for COV case 1 and 1.227 for COV case 2, which are very close to 2.604 and 1.227 reported by Cho (2007). This observation has served to validate the subroutine for the evaluation of reliability index of a given slip surface.

Next, using the computer program II, the probabilistic critical slip surfaces have been determined for both the COV cases 1 and 2 and shown in Fig. 5. The two surfaces are almost coincident. The associated minimum reliability indexes (β_{min}) are obtained as 2.449 and 1.146 for COV case 1 and COV case 2 respectively, which are significantly lower than 2.604 and 1.227 reported by Cho (2007). From Fig. 5 it can be observed that the two types of critical slip surfaces are markedly different in shape and location. While the deterministic critical slip surface extends well into the lower c-ϕ layer, the probabilistic critical slip surfaces are confined within the upper clay layer. Table 2 presents a summary of the results for the deterministic analysis and the initial probabilistic analysis without considering spatial variability for Example 1.

Table 2 Summary of Results for Example 1 without considering spatial variability

Studies	FS_{min}	β_{min}	
		COV Case 1	COV Case 2
Cho (2007)	1.592	2.604	1.227
Present study	1.582 (1.592)	2.449 (2.602)	1.146 (1.227)

Figures in the parentheses indicate those obtained by re-evaluating the critical slip surfaces reported by Cho (2007)

5.1.3.2 Re-Analysis of the Above Probabilistic Slip Surface Considering Spatial Variability The determination of the probabilistic critical slip surface(s) together with the associated minimum reliability index, as presented in the preceding section, was done without considering spatial variability of the soil properties. Now, to investigate the effect of consideration of spatial variability, this surface is re-analysed using the spatial variability discretization models I, II and III. But before doing this, it is desirable to validate the computer programs developed for the discretization models I, II and III.

Validation of Sub-Programs Developed for the Discretization Models I, II and III For the Sake of Convenience, the Validation of Discretization Model II (Cho 2007) is taken up first. Now, noting that Cho (2007) carried out his analysis using the Monte-Carlo Simulation method (MCS), in order to reproduce his results, a computer program (computer program III), has been developed based on the discretization model II coupled with the MCS exclusively for the purpose of this validation. The probabilistic critical slip surface reported by Cho (2007) (for COV case 2) has been scaled down from his paper and re-analysed using computer program III. Table 3 presents the results obtained by using the Program III, which agree well with those available from Cho's paper. The small disagreement at some places could be attributed to the error in scaling down of numerical values from graphical presentation of results in Cho's paper. Table 3 thus serves to validate the sub-program for the discretization model II.The computer program developed for analysis of a given slip surface using discretization model III (Ji et al. 2012) has been validated with respect to Ji et al.'s results and this validation is detailed at a later section (Example 2). The computer program developed for analysis of a given slip surface using discretization Model I (El-Ramly et al. 2002) has also been validated with respect to El-Ramly et al's results. However, details of this validation is not presented here for the sake of brevity and to save space.

Re-Analysis of the Probabilistic Critical Slip Surface Using the validated computer programs the re-analysis of the probabilistic critical slip surface determined for COV case 2 is now taken up to study the effect of spatial variability for a given slip surface. The reason for the choice of this particular surface is based on the fact that the COV case 2 has higher uncertainty level. During the re-analysis, the following studies have been conducted: Study 1: Comparison of reliability index with and without consideration of spatial variability, Study 2: Variation in reliability index with variation in the scale of fluctuation. As mentioned before, in each study all the three discretization models have been used.

Study 1: Comparison of Reliability Index With and Without Consideration of Spatial Variability For this study, the values of scale of fluctuation in the horizontal and the vertical direction (δ_x and δ_y) are taken as 20 m (assumed range 10–50 m) and 2 m (assumed range 1–5 m) respectively. Results are

Table 3 Validation of sub-program for discretization model II

Value of horizontal scale of fluctuation (δ_x)	Probability of failure (p_F)							
	Cho (2007)				Present study			
	Isotropic random field	Anisotropic random field			Isotropic random field	Anisotropic random field		
		$\delta_x/\delta_y = 1.0$	$\delta_x/\delta_y = 0.6$	$\delta_x/\delta_y = 0.2$		$\delta_x/\delta_y = 1.0$	$\delta_x/\delta_y = 0.6$	$\delta_x/\delta_y = 0.2$
$\delta_x = 10$	7.0×10^{-3}	3.2×10^{-3}	1.8×10^{-3}	1.2×10^{-4}	6.0×10^{-3}	3.0×10^{-3}	1.6×10^{-3}	1.8×10^{-4}
$\delta_x = 15$	1.7×10^{-2}	1.0×10^{-2}	7.5×10^{-3}	1.4×10^{-3}	1.7×10^{-2}	1.0×10^{-2}	7.1×10^{-3}	1.3×10^{-3}

Table 4 Comparison between values of reliability index with and without consideration of spatial variability for Example 1

Reliability index			Difference (%)
Without considering spatial variability		1.146	
Considering spatial variability	Using discretization model I[a] ($\delta = 20$ m)	1.402	22.34
	Using discretization model II ($\delta_x = 20$ m; $\delta_y = 2$ m)	2.711	136.56
	Using discretization model III ($\delta_x = 20$ m; $\delta_y = 2$ m)	1.992	73.82

[a] Not directly comparable with the values from the other two models

presented in Table 4. From Table 4 it is observed that for a given slip surface consideration of spatial variability increases the reliability index significantly. Further, for the particular combination of scale of fluctuations selected for this study, β value obtained from discretization model III is more conservative than that from model II. β value obtained from discretization model I is not directly comparable with the values from the other two models as it considers 1D spatial variability along the slip surface. However the β value obtained using $\delta = \delta_x$ (=20 m) is furnished in the table merely for the sake of completeness and having a rough idea about its magnitude.

Study 2: Variation in Reliability Index With Variation in the Scale of Fluctuation This is a parametric study in which the scale of fluctuation in the horizontal direction has been varied within a range of 10–50 m and that in the vertical direction from 1 to 5 m. However, a combination of exceptionally high value of 1000 m in both the horizontal and the vertical direction has also been considered and it is of academic interest only. The reliability indexes are calculated using all the three discretization models. The results are presented graphically in Fig. 6.

Results presented in Fig. 6 also indicate that the observations from Study 1 (Table 4) for an arbitrarily chosen combination of the horizontal and the vertical scale of fluctuation are found to be valid for other combinations also. Further, decrease in the value of scale of fluctuation in any direction results in an increase in the value of reliability index, and vice versa. As expected, assumptions of very high values of scale of fluctuations ($\delta_x = 1000$; $\delta_y = 1000$), results in a value of reliability index which is the same as that obtained without considering spatial variability. A close examination of Fig. 6 (together with the detailed program output) also reveals that the reliability index is more sensitive to the scale of fluctuation in the vertical direction (δ_y) as compared to that in the horizontal direction (δ_x). For example, for Model III, keeping δ_y constant at 2 m, if δ_x is increased by 100 % (from 20 to 40 m), the β value decreases by 5 %. But, keeping δ_x constant at 20 m, if δ_y is increased by 100 % (from 2 to 4 m), the β value decreases by 13 %.In case of the discretization model I (El-Ramly et al. 2002), however, the scale of fluctuation (δ) is considered along the slip surface. Therefore, any value of δ in excess of the length of the slip surface (30.025 m), there is virtually no effect of spatial variability as the soil properties behave like a single random variable. Therefore, as seen from Fig. 6a, the values of reliability index considering spatial variability with δ greater than approximately 30.0 m merges with that obtained without considering spatial variability.

5.1.3.3 Search for Probabilistic Critical Slip Surface Considering Spatial Variability Using the developed computer programs IV, V and VI, the probabilistic critical slip surfaces corresponding to the discretization models I, II and III have been determined for the COV case 2.

For the particular case of $\delta_x = 20$ m and and $\delta_y = 2$ m, the three critical slip surfaces corresponding to the discretization models I, II and III are plotted in Fig. 5 which shows that these surfaces are very close to one another but substantially different from the probabilistic critical slip surface determined without considering spatial variability. Table 5 presents the values of β_{min} associated with these surfaces. It is interesting to note that these values are markedly different though the surfaces are close to one another.

A comparison between Tables 5 and 4 indicates that the observations made from Table 4 for β values are also valid for β_{min} values in Table 5. Further, values of β_{min} in Table 5 are lower than the

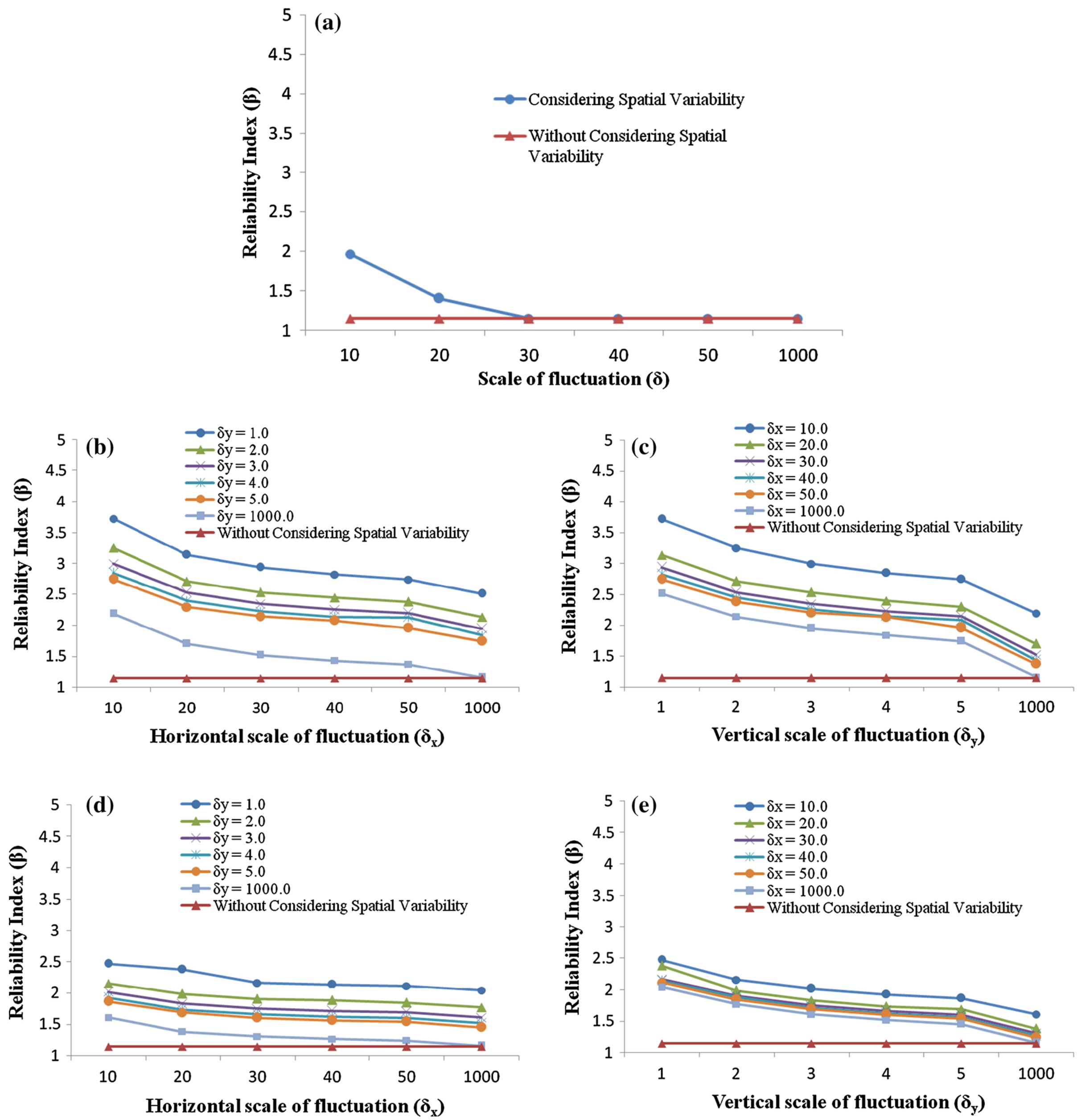

Fig. 6 Results of parametric studies for re-analysed surface (Example 1). **a** β versus δ for model I, **b** β versus δ_x for model II, **c** β versus δ_y for model II, **d** β versus δ_x for model III, **e** β versus δ_y for model III

corresponding values of β in Table 4 but only by a small degree. The largest difference is nearly 3 % corresponding to the discretization model III.

Using programs IV, V and VI, for the sake of a parametric study, a series of other probabilistic critical slip surfaces and the associated minimum reliability indexes have been obtained (for the COV case 2) by varying the values of δ_x within a range of 10–50 m and δ_y within a range of 1–5 m. The determined critical slip surfaces are found to lie within a narrow band around the probabilistic slip surface shown in Fig. 3 for the particular values of $\delta_x = 20$ m and $\delta_y = 2$ m. These surfaces are, however, not shown in Fig. 3 for the sake of clarity. Figure 7 presents the results of this

Table 5 Comparison between values of minimum reliability index with and without consideration of spatial variability for example 1

Minimum reliability index			Difference (%)
Without considering spatial variability		1.146	
Considering spatial variability	Using discretization model I[a] ($\delta = 20$ m)	1.370	19.46
	Using discretization model II ($\delta_x = 20$ m; $\delta_y = 2$ m)	2.659	131.82
	Using discretization model III ($\delta_x = 20$ m; $\delta_y = 2$ m)	1.938	68.992

[a] Not directly comparable with the values from the other two models, as noted in Table 4

parametric study. As pointed out before, a combination of exceptionally high value of 1000 m in both the horizontal and the vertical direction has been considered and it is of academic interest only.

Results presented in Fig. 7 also indicate that the observations from Fig. 6 for an arbitrarily chosen combination of the horizontal and the vertical scale of fluctuation ($\delta_x = 20$ m; $\delta_y = 2$ m) are found to be valid for other combinations also. Further, decrease in the value of scale of fluctuation in any direction results in an increase in the value of minimum reliability index, and vice versa. As expected, assumptions of very high values of scale of fluctuations ($\delta_x = 1000$; $\delta_y = 1000$), results in a value of minimum reliability index which is the same as that obtained without considering spatial variability. A close examination of Fig. 7 also reveals that the minimum reliability index is more sensitive to the scale of fluctuation in the vertical direction (δ_y) as compared to that in the horizontal direction (δ_x). For example, for Model III, keeping δ_y constant at 2 m, if δ_x is increased by 100 % (from 20 to 40 m), the β_{min} value decreases by 3 %. But, keeping δ_x constant at 20 m, if δ_y is increased by 100 % (from 2 to 4 m), the β_{min} value decreases by 11 %. This observation, again, is in agreement with those reported earlier (Ji et al. 2012).

5.2 Example 2: (Ji et al. 2012)

5.2.1 Description

Figure 8 shows an embankment underlain by soft clay foundation, taken from Ji et al. (2012). The undrained shear strength of the soft clay c_2 is assumed to be normally distributed random variable with a mean value of 25 kN/m^2 and a coefficient of variation equal to 0.25. All the other strength parameters of the problem are assumed to have deterministic values as given in Fig. 8.

It may be pointed out that previously Ji et al. (2012) analysed this problem assuming slip surfaces to be of circular shape.

5.2.2 Results of Deterministic Analysis

Taking the value of cohesion of the foundation material as 25 kPa (equal to its mean value), the deterministic critical slip surface having minimum factor of safety of 1.47 has been located (Fig. 8) using the computer program-I. The number of slices is taken as 24 for the stability analysis of this slope (as used by Ji et al. 2012). In order to compare the results with that of Ji et al. (2012), slip surfaces were assumed as circular and the Spencer method (Spencer 1967) has been used. A minimum factor of safety of 1.462 has been reported by Ji et al. (2012) using Spencer method which is close to that obtained in the present analysis.

5.2.3 Results of Probabilistic Analysis

As was done in case of Example 1, the results of probabilistic analyses are presented under the following sub-headings:

I. Determination of probabilistic critical slip surface without considering spatial variability
II. Re-analysis of the above surface considering spatial variability
III. Determination of probabilistic critical slip surface considering spatial variability

5.2.3.1 Determination of Probabilistic Critical Slip Surface Without Considering Spatial Variability Using the computer program-II, the probabilistic critical slip surface has been determined using the FORM method. Figure 8 shows the probabilistic critical slip surface alongside the deterministic critical slip surface. The two critical

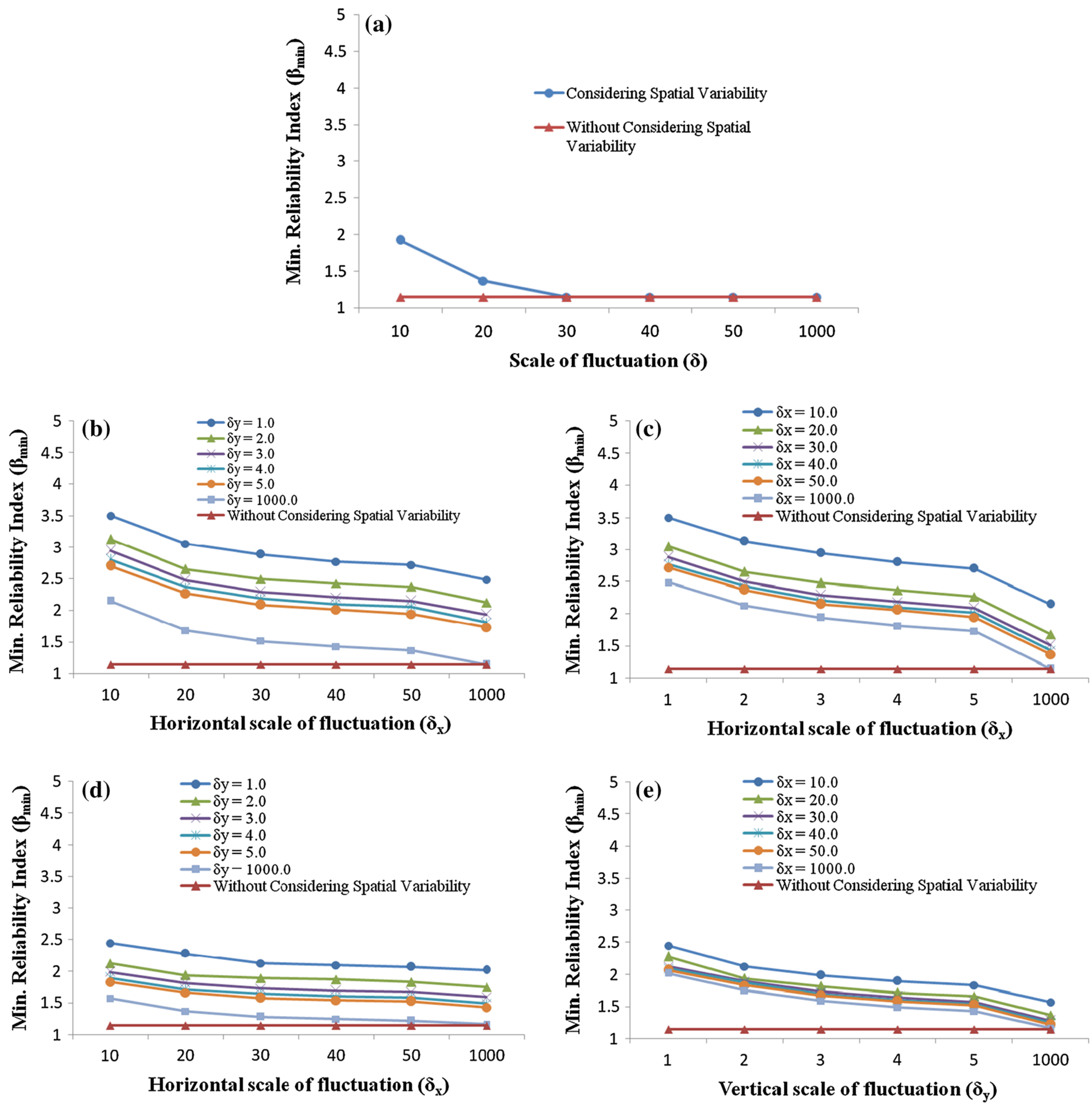

Fig. 7 Results of parametric studies for critical slip surface (Example 1). **a** β_{min} versus δ for model I, **b** β_{min} versus δ_x for model II, **c** β_{min} versus δ_y for model II, **d** β_{min} versus δ_x for model III, **e** β_{min} versus δ_y for model III,

slip surfaces are very similar in shape and very close to each other. This resemblance could be attributed to the fact that in this analysis only a single random variable has been considered. The associated minimum reliability index β_{min} equals 1.36 which is close to the value of 1.32 reported by Ji et al. (2012).

5.2.3.2 Re-Analysis of the Above Slip Surface Considering Spatial Variability As has been done in case of Example-1, to investigate the effect of consideration of spatial variability, the probabilistic critical slip surface determined as above is re-analysed using the discretization models I, II and III. Model II has been validated earlier in connection with Example 1. Now for the validation of model III (used as a subprogram in computer program VI), the critical slip surface reported by Ji et al. (2012) has been scaled down and re-analysed. Table 6 presents the results

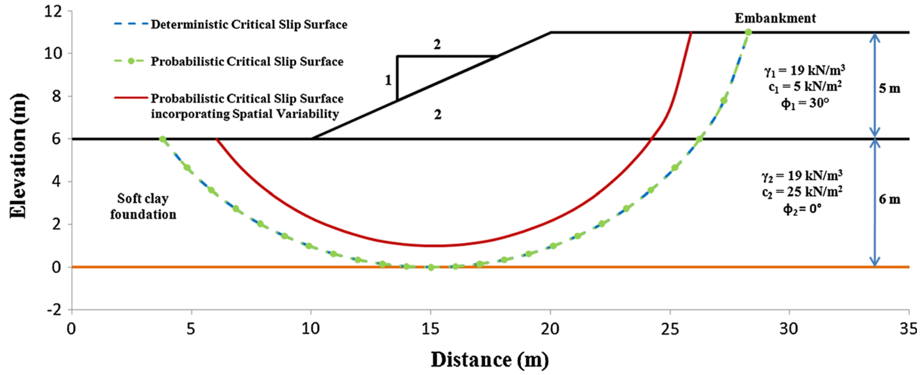

Fig. 8 Slope section and critical slip surfaces in Example 2

obtained by using the Program VI, which agree well with those available from Ji et al.'s paper. The small disagreement could be attributed to the differences in the LEM of slices used for computation of the factor of safety of a slip surface. More specifically, both the authors' analysis and the Ji et al., analysis have used the Spencer method. However, there is a difference in the consideration of the interslice force function: while in the authors' analysis, interslice forces are assumed to be parallel, Ji et al. considered interslice forces with varying inclinations. Table 6 thus serves to validate the sub-program for discretization Model III in Program VI.

Using the validated computer programs the re-analysis of the probabilistic critical slip surface is now taken up to study the effect of spatial variability for a given slip surface. As in case of Example-1, the following studies have been conducted: Study 1: Comparison of reliability index with and without consideration of spatial variability, Study 2: Variation in reliability index with variation in the scale of fluctuation. In each study all the three discretization models have been used.

Table 6 Validation of sub-program for discretization model III

Scale of fluctuation (m)		Reliability index	
δ_x	δ_y	Ji et al. (2012)	Present study
20	2	2.256	2.264
1000	2	1.997	1.986
20	1000	1.711	1.698
1000	1000	1.487	1.412

Study 1: Comparison of Reliability Index With and Without Consideration of Spatial Variability For this particular study, the values of scale of fluctuation in the horizontal and the vertical direction (δ_x and δ_y) are again taken as 20 and 2 m respectively. Results are presented in Table 7. From Table 7 it is observed that for a given slip surface consideration of spatial variability increases the reliability index significantly. Further, for the particular combination of scale of fluctuations selected for this study, β value obtained from discretization model III is more conservative than that from model II. As discussed before, β value obtained from discretization model I is not directly comparable with the values from the other two models as it considers 1D spatial variability along the slip surface. However the β value obtained using $\delta = \delta_x$ (=20 m) is furnished in the table merely for the sake of completeness and having a rough idea about its magnitude.

Study 2: Variation in Reliability Index With Variation in the Scale of Fluctuation As has been done in case of Example-1, this is a parametric study in which the scale of fluctuation in the horizontal direction has been varied within a range of 10–50 m and that in the vertical direction from 1 m to 5 m. However, a combination of exceptionally high value of 1000 m in both the horizontal and the vertical direction has also been considered and it is of academic interest only. The reliability indexes are calculated using all the three discretization models. The results are presented in Fig. 9.

Results presented in Fig. 9 also indicate that the observations from Study 1 (Table 7) for an arbitrarily chosen combination of the horizontal and the vertical

Table 7 Comparison between values of reliability index with and without consideration of spatial variability for Example 2

Reliability index			Difference (%)
Without considering spatial variability		1.36	
Considering spatial variability	Using discretization model I[a] ($\delta = 20$)	1.575	15.81
	Using discretization model II ($\delta_x = 20$; $\delta_y = 2$)	3.109	128.60
	Using discretization model III ($\delta_x = 20$; $\delta_y = 2$)	2.320	70.61

[a] Not directly comparable with the values from the other two models, as noted in Table 4

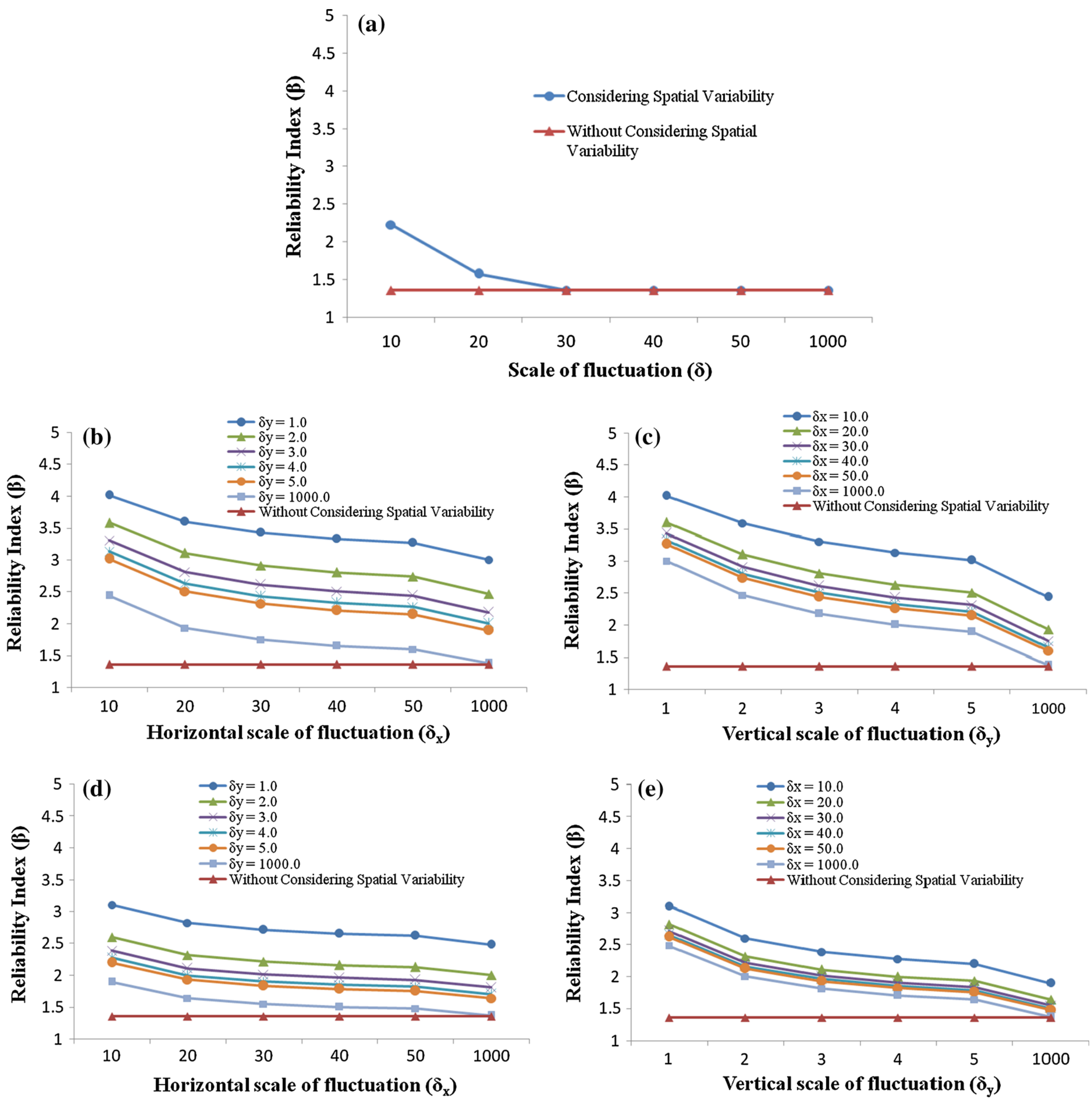

Fig. 9 Results of parametric studies for re-analysed surface (Example 2). **a** β versus δ for model I, **b** β versus δ_x for model II, **c** β versus δ_y for model II, **d** β versus δ_x for model III, **e** β versus δ_y for model III,

scale of fluctuation are found to be valid for other combinations also. Further, decrease in the value of scale of fluctuation in any direction results in an increase in the value of reliability index, and vice versa. As expected, assumptions of very high values of scale of fluctuations ($\delta_x = 1000$; $\delta_y = 1000$), results in a value of reliability index which is the same as that obtained without considering spatial variability. A close examination of Fig. 6 also reveals that the reliability index is more sensitive to the scale of fluctuation in the vertical direction (δ_y) as compared to that in the horizontal direction (δ_x). For example, for Model III, keeping δ_y constant at 2 m, if δ_x is increased by 100 % (from 20 to 40 m), the β value decreases by 7 %. But, keeping δ_x constant at 20 m, if δ_y is increased by 100 % (from 2 to 4 m), the β value decreases by 14 %. This observation is in agreement with those reported earlier (Ji et al. 2012).In case of the discretization model I (El-Ramly et al. 2002), however, the scale of fluctuation (δ) is considered along the slip surface. Therefore, any value of δ in excess of the length of the slip surface within the soft clay layer (26.5 m out of total length 31.92 m), there is virtually no effect of spatial variability as the soil properties behave like a single random variable. Therefore, the values of reliability index considering spatial variability with δ greater than approximately 30.0 m merges with that obtained without considering spatial variability.

5.2.3.3 Search for Probabilistic Critical Slip Surface Considering Spatial Variability Using the developed computer programs IV, V and VI, the probabilistic critical slip surfaces corresponding to the discretization models I, II and III have been determined. For the particular case of $\delta_x = 20$ m and $\delta_y = 2$ m, the three critical slip surfaces corresponding to the discretization models I, II and III are plotted in Fig. 8 which shows that these surfaces are very close to one another but substantially different from the probabilistic critical slip surface determined without considering spatial variability. Table 8 presents the values of β_{min} associated with these surfaces. It is interesting to note that these values are markedly different though the surfaces are close to one another.

A comparison between Tables 8 and 7 indicates that the observations made from Table 7 for β values are also valid for β_{min} values in Table 8. Further, values of β_{min} in Table 8 are lower than the corresponding values of β in Table 7 but only by a small degree. The largest difference is nearly 5 % corresponding to the discretization model I.

Using programs IV, V and VI, for the sake of a parametric study, a series of other probabilistic critical slip surfaces and the associated minimum reliability indexes have been obtained by varying the values of δ_x within a range of 10–50 m and δ_y within a range of 1–5 m. The determined critical slip surfaces are found to lie within a narrow band around the probabilistic slip surface shown in Fig. 8 for the particular values of $\delta_x = 20$ m and $\delta_y = 2$ m. These surfaces are, however, not shown in Fig. 8 for the sake of clarity of the figure. Figure 10 presents the results of this parametric study. As pointed out before, a combination of exceptionally high value of 1000 m in both the horizontal and the vertical direction has been considered and it is of academic interest only.

Results presented in Fig. 10 also indicate that the observations from Table 8 for an arbitrarily chosen combination of the horizontal and the vertical scale of fluctuation are found to be valid for other combinations also. Further, decrease in the value of scale of fluctuation in any direction results in an increase in the value of minimum reliability index, and vice versa. As expected, assumptions of very high values of scale of fluctuations ($\delta_x = 1000$; $\delta_y = 1000$), results in a

Table 8 Comparison between values of minimum reliability index with and without consideration of spatial variability for example 2

Minimum reliability index			Difference (%)
Without considering spatial variability		1.36	
Considering spatial variability	Using discretization model I[a] ($\delta = 20$)	1.502	10.43
	Using discretization model II ($\delta_x = 20$; $\delta_y = 2$)	3.021	122.15
	Using discretization model III ($\delta_x = 20$; $\delta_y = 2$)	2.264	66.46

[a] Not directly comparable with the values from the other two models, as noted in Table 4

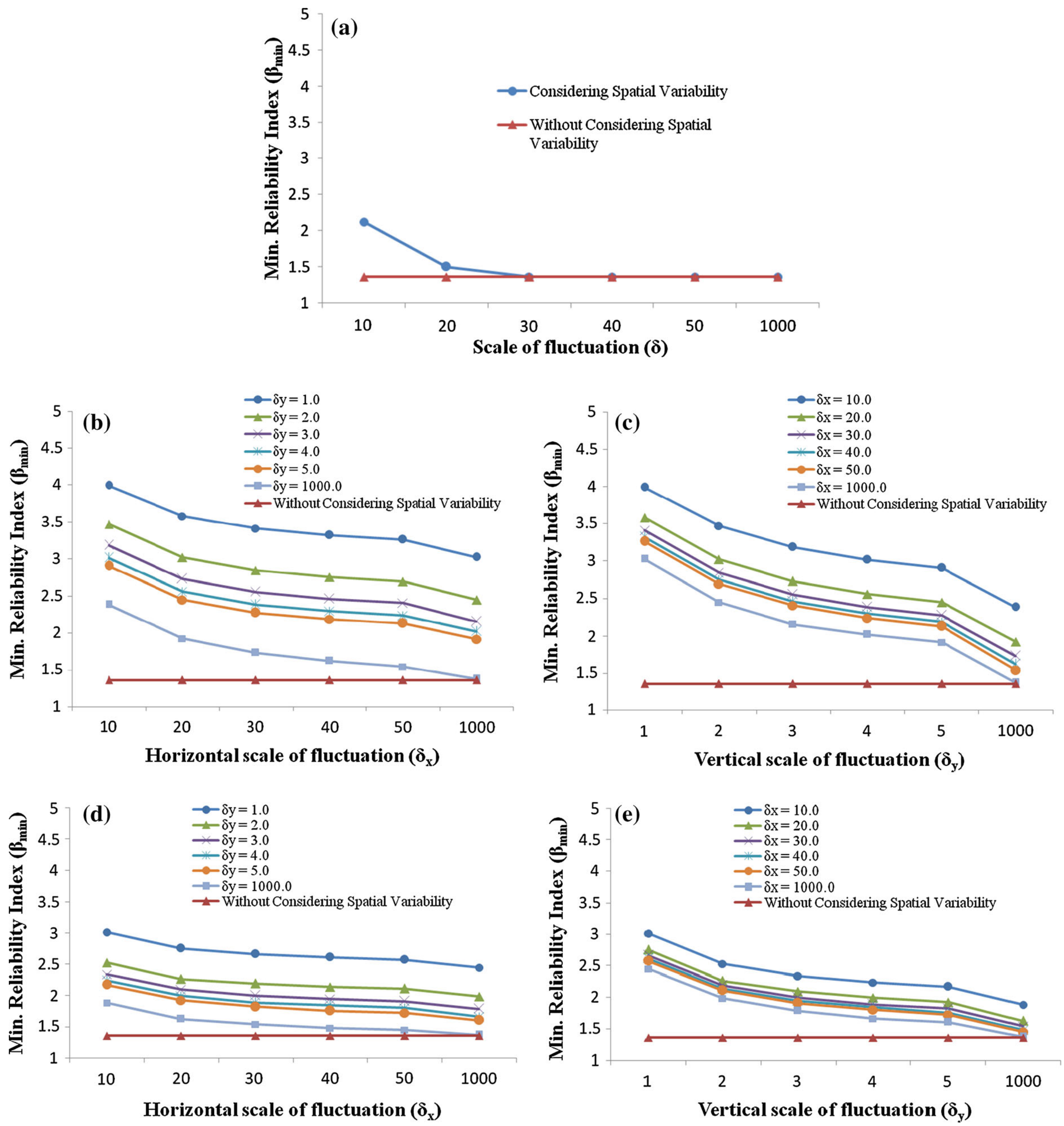

Fig. 10 Results of Parametric studies for critical slip surface (Example 2). **a** β_{min} versus δ for model I, **b** β_{min} versus δ_x for model II, **c** β_{min} versus δ_y for model II, **d** β_{min} versus δ_x for model III, **e** β_{min} versus δ_y for model III

value of minimum reliability index which is the same as that obtained without considering spatial variability. A close examination of Fig. 9 also reveals that the minimum reliability index is more sensitive to the scale of fluctuation in the vertical direction (δ_y) as compared to that in the horizontal direction (δ_x). For example, for Model III, keeping δ_y constant at 2 m, if δ_x is increased by 100 % (from 20 to 40 m), the β_{min} value decreases by 5 %. But, keeping δ_x constant at 20 m, if δ_y is increased by 100 % (from 2 to 4 m), the β_{min} value decreases by 12 %. This observation is in agreement with those reported earlier (Ji et al. 2012).

6 Discretization Models: Computational Advantages and Disadvantages

Based on the experience gained during this study in respect of using the three discretization models, the following computational advantages and disadvantages can be enumerated.

(1) An advantage with the slicewise discretization model proposed by Cho (2007) (Model II) as well as by Ji et al. (2012) (Model III) is that the discretization is the same as used for the conventional limit equilibrium analysis using method of slices. On the other hand, as per the El-Ramly's Model (Model I), based on the scale of fluctuation and segmental length within each layer, the conventional slicewise discretization has to be modified.

(2) In both the models proposed by Cho (2007) and by Ji et al. (2012), the size of the correlation matrix and the number of random variables essentially depend on the total number of slices. Before the search process of the reliability analysis, fixing the number of random variables as well as the size of the correlation matrix may not be the ideal choice. Therefore, it is required to carry out a sensitivity analysis to study the influence of the number of slices on the reliability index, β. [For example, in the case of example problem 2, Ji et al. (2012) studied the influence of the number of slices and found out the optimum number of slices as 24 and for the case of example problem 1, though there is no explicit mention in the paper, it is believed that the 12 number of slices used in the analysis was arrived at based on such a sensitivity analysis.] On the other hand, as per the model proposed by El-Ramly et al. (2002), the number of random variables as well as the size of the correlation matrix need not be fixed initially; it can be varied depending on the locations of the trial intermediate slip surfaces, and hence, no sensitivity analysis is necessary. Ji et al. (2012), however, proposed another model known as the method of interpolated autocorrelations which is free from this shortcoming.

(3) In the model II (Cho 2007) or model III (Ji et al. 2012), in cases of layered slopes, by assuming slice to slice correlation, inter-layer correlation of soil properties are implicitly considered which may not really exist. This does not arise in model I proposed by El-Ramly et al. (2002).

(4) Using the modeling proposed by El-Ramly et al. (2002), unlike Cho's model and Ji et al.'s model, the information on the horizontal and the vertical scales of fluctuation cannot be taken into account directly.

(5) In a layered slope, if the values of the scale of the fluctuation are different from layer to layer, Cho (2007)'s model and Ji et al. (2012)'s model cannot be used. However, the model proposed by El-Ramly et al. (2002) can automatically handle this situation.

7 Conclusions

Based on the studies undertaken in this paper, the following concluding remarks can be made:

(1) In slope reliability analysis, with the exception of Ji et al. (2012), published works on this topic have adopted an indirect approach to take spatial variability of soil properties into account. In this approach, initially the probabilistic critical slip surface is determined without considering spatial variability and then the reliability index associated with this predetermined slip surface is modified to consider spatial variability. In contrast, the direct approach (e.g., as adopted by Ji et al. 2012), is to directly search out the probabilistic critical slip surface and the associated minimum reliability index (β_{min}) by minimizing the reliability index computed considering spatial variability.

(2) The studies undertaken in this paper have revealed that the two approaches might yield reliability results which are significantly different. Specifically, the probabilistic critical slip surfaces obtained from the direct approach (surface searched considering spatial variability) are found to be widely different from those from indirect approach (surface searched without considering spatial variability). Further, values of the minimum reliability indices (β_{min}) associated with the probabilistic critical slip surfaces obtained from the direct approach are found to be lower than those from the

indirect approach. In other words, adoption of indirect approach might lead to an overestimation in the β_{min} value (underestimation in the probability of failure, p_F of a slope). For the two example problems this overestimation in the β_{min} value is found to be rather small (<5 %). It is also noteworthy that while using the direct approach, the different discretization models yield critical slip surfaces which are rather close to one another. Both the above observations could be attributed to the small number of random variables as well as to the simple geometry of the slopes and layer boundaries in the example problems, and that no pore water pressures are considered. However, in the case of complex slope situations such as zoned dams and levees with complex layering and pore pressure conditions, the underestimation in the probability of failure are likely to be substantial. Such types of research are in progress and expected to be reported in the near future.

(3) Irrespective of whether the direct or the indirect approach is adopted, it is known that consideration of spatial variability results in substantial increase in the reliability index (or a decrease in the probability of failure). However, the amount of increase depends not only on the magnitude of the scale of fluctuation in the horizontal direction (δ_x) and in the vertical direction (δ_y) in a general 2D spatial variability situation, but also on the spatial variability discretization model used in the analysis.

(4) Parametric studies conducted in this paper for the two layered slope examples reveal that:

  (i) Between the two 2D discretization models, Ji et al.'s Model (Model III) is hugely more conservative than Cho's model (Model II). El Ramly et al.'s model (Model I) being a 1D model cannot be directly compared with the other two models; however, for this model if the value of the scale of fluctuation is taken equal to the value of δ_x, it appears to be the most conservative of the three models.

  (ii) Effect of variation of the vertical scale of fluctuation δ_y on the reliability results is much more than that of the horizontal scale of fluctuation δ_x. This observation is in agreement with those reported earlier (Ji et al. 2012).

(5) For the two slope example problems considered here, observations from studies on different spatial variability discretization models as also the magnitudes of the scale of fluctuation in the horizontal and the vertical directions based on the indirect approach of analysis are found to be similar to that based on the direct approach of analysis.

(6) There are computational advantages and disadvantages associated with each model; but in applying the Cho's model (Model II) or the Ji et al.'s model (Model III) to case of layered slopes, by assuming slice to slice correlation, inter-layer correlation of soil properties are implicitly considered which may not really exist. This deficiency is not there in the El-Ramly et al.'s model (Model I). Further, in a layered slope, if the values of the scale of the fluctuation are different from layer to layer, Cho (2007)'s model and Ji et al. (2012)'s model cannot be used. However, the model proposed by El-Ramly et al. (2002) can easily handle this situation. In view of the above, it appears that it will be computationally handy if it is possible to extend the El-Ramly et al.'s model from a 1D to a 2D spatial variability model.

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