Geotechnical and Geological Engineering (2006) 24: 1803–1819
DOI 10.1007/s10706-006-7172-1

Stability determination for layered soil slopes using the upper bound limit analysis

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(Received 12 May 2005; accepted 2 February 2006)

Abstract. By using the upper bound limit analysis, a method has been proposed in this article to compute stability numbers for layered soil slopes. The rupture surface was assumed to be a combination of different logarithmic spiral arcs with a common focus. The analysis ensures the kinematic admissibility of the collapse mechanism with respect to the rigid rotation of the bounded soil mass about the focus of the logarithmic spiral. For the sake of illustration, the computations, however, were exclusively performed only for a two layered soil slope. The effect of the pore water pressure and horizontal earthquake body forces was also incorporated in the computations. The computational procedure was validated by making a comparison of the obtained stability numbers with those reported in literature.

Key words. failure, limit analysis, numerical modelling, plasticity, slopes.

1. Introduction

Limit equilibrium techniques based upon the methods of slices are often used to assess the stability of embankment slopes (Fellenius, 1936; Bishop, 1955; Bishop and Morgenstern, 1960; Morgenstern and Price, 1965). The results obtained from these methods are reasonably accurate when compared with the advanced rigorous computational techniques based on finite and boundary element methods. The method of slices is extremely popular on account of its simplicity in dealing with layered soils, complex slope geometry, and the presence of pore water pressure as well as pseudo static earthquake body forces. However, this method has a limitation that it does not address the issue of kinematics and, therefore, the obtained solution may not always be correct. On the other hand, in the upper bound limit analysis, the kinematic admissibility of the chosen collapse mechanism has always to be guaranteed. A number of investigations have been performed in the recent past dealing with the stability of slopes using the upper bound limit analysis (Chen et al., 1969; Karal, 1977a, b; Chen and Liu, 1990; Michalowski, 1994, 1995, 2002a; Kumar, 2000, 2004). By making use of the upper bound theorem of limit analysis, Chen et al. (1969) has introduced a method for obtaining the critical heights for homogenous soil slopes. A rotational collapse mechanism bounded by an arc of the logarithmic spiral rupture surface was used in this analysis. Karal (1977a, b) later suggested a

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method for finding directly the factor of safety of the slope rather than obtaining either the stability number or the critical height. Michalowksi (1994) also used the upper bound theorem of limit analysis in order to obtain the stability numbers for soil slopes in the presence of pore water pressure. Using the upper bound limit analysis, Michalowski (1995) later presented a different approach which was based on a translational collapse mechanism comprising of a number of vertical slices. Michalowksi (2002a) computed the stability numbers for homogeneous slopes in the presence of pore water pressures as well as pseudo-static horizontal earthquake body forces. On the basis of methodology proposed by Karal (1977a, b), Kumar (2000) has framed a detailed step wise procedure for computing the factor of safety of a slope. Since the theorem of upper bound limit analysis is based on the applicability of an associated flow rule, all the available methods can not be simply used for a non-associated flow rule material. Kumar (2004) has recently used the concept of energy balance to compute stability numbers of soil slopes dealing with non-associated coaxial and non-coaxial flow rules. The available studies in literature based upon the concept of the upper bound limit analysis/energy balance have been reported only for homogeneous soil slopes; by using the upper bound limit analysis although a few studies are available (Michalowski, 2002b; Kumar, 2003) for computing the collapse loads for foundations and anchors placed on layered ground. In the present paper by using the upper bound theorem of limit analysis, a method has been proposed to determine the stability numbers for a layered soil slope; the stability numbers can be subsequently used to interpret the factor of safety of a given slope (Taylor, 1948). A rigid rotation of the collapse mechanism bounded by the arcs of different logarithmic spiral was chosen in the mechanism. All the chosen logarithmic spiral arcs were assumed to have a common focus as well as common radii at the interface of the two layers. For the purpose of illustration, calculations, however, were exclusively carried out only for a two layered soil slope. The effect of the pore water pressure and the pseudo-static horizontal earthquake body forces was also incorporated in the computations. The obtained values of the stability numbers were compared with those reported in literature. The computations in this paper have although been performed only for a two layered soil slope; however, the methodology can be implemented for multi-layered soil slopes with different values of cohesion (c), friction angle (ϕ) and unit weight (γ).

2. Definition of the Problem

Consider a two dimensional soil slope with three different horizontal layers as shown in Figure 1. Each layer is defined by means of its thickness H_i, cohesion c_i, angle of internal friction ϕ_i and unit weight γ_i; where $i = 1$, 2 and 3. The sloping surface is inclined at an angle β with horizontal. It is to determine the critical height (H_c) of the slope so that the sloping mass is on the verge of collapse (shear failure).

3. Collapse Mechanism

The rupture surface was assumed to be a combination of different (three as shown in Figure 1) logarithmic spiral arcs passing through the toe of the slope. The possibility that the rupture surface may pass below the toe for very small values of ϕ and β was not explored while doing the computations in the present study. In Figure 1, three different logarithmic spiral arcs were chosen with a common focus F. At the point of interface of two layers, the radius of the logarithmic spiral arc was kept unchanged. The expression for any logarithmic spiral arc was based on the friction angle of the layer through which the arc passes. As a result, $r_1 = r_0 \exp(\theta_0 \tan \phi_1)$, $r_2 = r_1 \exp(\theta_1 \tan \phi_2)$ and $r_3 = r_2 \exp(\theta_2 \tan \phi_3)$; where $r_0 =$ radius FB_0, $r_1 =$ radius FB_1, $r_2 =$ radius FB_2, $r_3 =$ radius FC, $\theta_0 =$ angle B_1FB_0, $\theta_1 =$ angle B_2FB_1 and $\theta_2 =$ angle CFB_2. For given height (H) and inclination (β) of the slope, the rupture surface, passing through the toe of the slope, can be completely defined by positioning the focus (F) of the logarithmic spiral with respect to the toe of the slope; the position of a focus can be defined by means of any two independent variables (for instance either coordinates x and y or the angles θ_a and θ_c as shown in Figure 1, where $\theta_a =$ angle GFB_0 and $\theta_c =$ angle GFC).

4. Analysis

For chosen values of H and β, a rupture surface can be drawn. The soil mass $A_0A_1A_2CB_2B_1B_0$ bounded by the logarithmic spiral arcs and the periphery of the slope was assumed to rotate about the focus of the logarithmic spiral as a single rigid unit. If the angular velocity of this block at collapse is taken equal to ω, the resultant

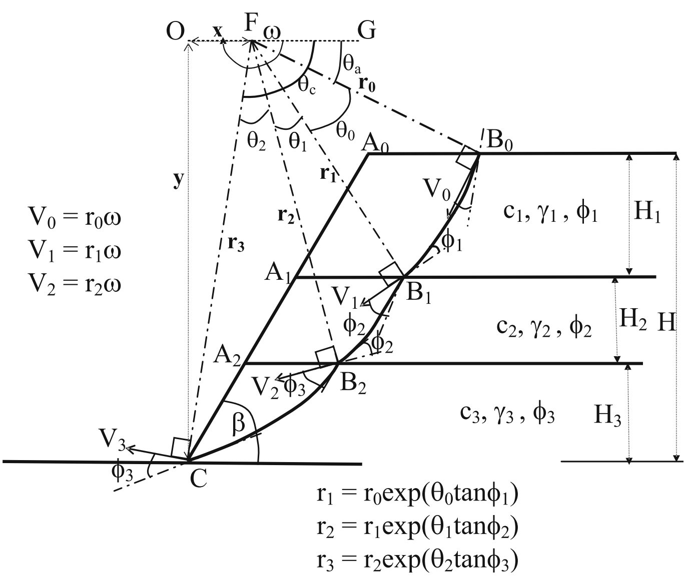


Figure 1. Collapse mechanism and velocity hodographs for a layered slope.

velocity (V_s) at any point (P) within the block at a radial distance s from F will become equal to s ω; the direction of V will become perpendicular to the radial line FP. By defining the velocities everywhere within the block $A_0A_1A_2CB_2B_1B_0$, the rate of the work done due to the body forces within an element of soil mass can be computed by making use of the following expression:

$$\mathrm{d}\dot{W} = V_v \gamma d(\mathrm{vol}) + V_h k_h \gamma d(\mathrm{vol}) \tag{1}$$

In the above expression, d(vol) is the volume of an infinitesimal element efgh (see Figure 2c); V_v and V_h are the vertical (downward positive) and horizontal (towards toe positive) velocities of the element, respectively; γ is the unit weight of the element ; and k_h is the magnitude of horizontal earthquake acceleration coefficient (positive towards toe). On the basis of Equation (1), the rate of total work done by the body forces within the block $A_0A_1A_2CB_2B_1B_0$ can be determined on the basis of numerical integration, that is,

$$\dot{W}_{\mathrm{Total}} = \int_{\mathrm{Volume}} V_v \gamma d(\mathrm{vol}) + V_h k_h \gamma d(\mathrm{vol}) \tag{2}$$

By following the assumption of an associative flow rule, subsequently, the rate of dissipation of internal energy can be computed. Since the soil block $A_0A_1A_2CB_2B_1B_0$ was assumed to be completely rigid, the dissipation of the internal energy will occur only due to the movement of the block along the periphery of the

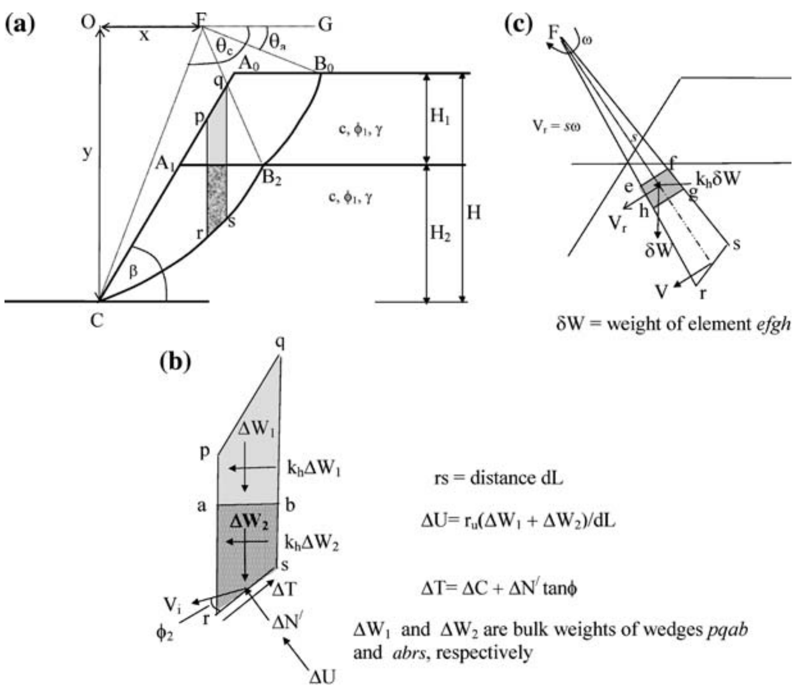


Figure 2. Collapse mechanism for a two layered slope.

rupture surface $CB_2B_1B_0$ (refer Figure 1). It should be noticed that due to logarithmic spiral shape of the rupture line, the direction of the resultant velocity at any point on the rupture line will incline at an angle ϕ with the corresponding tangent drawn at the rupture line; ϕ is the internal friction angle of the material appropriate at that point. The expression for computing the rate of dissipation of total internal energy along the periphery (L) of the rupture surface in the presence of pore water is provided below:

$$\dot{E}_{\text{Total}} = \int_{\text{L}} V(c\cos\phi - u\sin\phi)dl \tag{3}$$

In the above expression, V is the velocity jump at any point on the rupture surface, c and ϕ are the values of soil cohesion and angle of internal friction relevant at the same point, u is magnitude of pore water pressure and dl is the infinitesimal length of the element along the rupture surface. If value of the pore water coefficient (r_u) is defined at a point then the magnitude of the pore water pressure can be computed with the help of the following expression:

$$u = r_u \frac{\sum \Delta W}{dl} \tag{4}$$

$\Delta W = \Delta W_1 + \Delta W_2$ for Figure 2(b); ΔW_1 and ΔW_2 are the respective total weights (per unit length since the problem is two dimensional) of the elements abqp and abrs, respectively. Therefore, on the basis of Equations (3) and (4), for a chosen rupture mechanism, the rate of total work done by the body forces (including those by the surface traction if any) and the rate of dissipation of total internal energy can be determined. For a slope to be in critical state of failure, the condition ($\dot{E}_{\text{Total}} = \dot{W}_{\text{Total}}$) needs to be satisfied. For the chosen values of θ_a, θ_c (or x and y) and β, this condition can only be satisfied for a particular value of H. This value of H can be numerically determined by trial and error so that the value of ($\dot{E}_{\text{Total}} - \dot{W}_{\text{Total}}$) becomes almost equal to zero. After determining the magnitude of H for which the condition ($\dot{E}_{\text{Total}} = \dot{W}_{\text{Total}}$) is satisfied, the values of the parameters θ_a and θ_c can then be independently varied so as to determine the minimum value of H ($H = H_{cr}$) for a given slope inclination β. On this basis, using the upper bound limit analysis, the value of H_{cr} can, therefore, be computed for a given geometry of the slope.

5. Illustration of the Proposed Method

For the sake of illustrations, the computations were exclusively carried out in this study only for a two layered soil slope as shown in Figure 2(a). Both the layers were assumed to have same values of soil cohesion (c) and unit weight (γ) but with different values of ϕ. The upper and lower layers were assumed to have friction angle ϕ_1 and ϕ_2 , and thickness H_1 and H_2, respectively. The stratum below the toe of the slope was assumed to have same soil properties as that of the lower layer. It was aimed to determine the stability number (N_s) of the soil slope in the presence of pore water pressure and horizontal earthquake body forces; the stability number was defined in the same fashion as earlier introduced by Taylor (1948); where $N_s = \frac{\gamma H_c}{c}$, and H_c is the critical total height of the slope.

6. Solution Procedure

By equating the rate of total work done by all the body forces to the rate of dissipation of total internal energy, the value of H can be obtained. It should be noted that the value of H will become a function (F) of the following parameters:

$$H = \frac{c}{\gamma} F\left(\theta_c, \theta_a, r_u, k_h, \frac{H_1}{H}, \beta\right) \tag{5}$$

where, the angles θ_c and θ_a have been defined in Figure 2(a). For a given geometry of the slope, the function F can be minimized with respect to two independent variables θ_a and θ_c (or x and y) so as to obtain the critical height (H_c) of the slope.

7. Results

For a chosen two layered soil slope, the values of the stability numbers were obtained by varying β from 45° to 90°, ϕ_1 from 10° to 30°, ϕ_2 from 20° to 40°, r_u from 0 to 0.25, k_h from 0 to 0.1 and H_1/H from 0 to 1. All the results are presented in Figures 3 to 6. Following observations were made:

1. For $\phi_1 < \phi_2$, the values of the stability number were found to decrease continuously with increase in H_1/H. It should be noted that the effect of the relative thickness of the two layers is quite significant especially for small values of β (mild slopes). On the other hand, for very steep slopes, the effect of the relative thickness of the two layers is found to be only marginal.
2. As it was expected, the values of the stability numbers have been found to decrease continuously with (i) increase in k_h; (ii) increase in r_u; and (iii) increase in slope angle β .

8. Failure Patterns

By keeping the values of $H_1/H = 0.4$, $\phi_1 = 10°$, $\phi_2 = 30°$ and $\beta = 60°$, the failure patterns were plotted for four different cases, namely: (i) $k_h = 0$, $r_u = 0$; (ii) $k_h = 0$, $r_u = 0.25$; (iii) $k_h = 0.1$, $r_u = 0$; and (iv) $k_h = 0.1$, $r_u = 0.25$. These failure patterns are illustrated in Figures 7a–d. It can be noted from these figures that the size of the soil mass bounded by the periphery of the rupture surface and the sloping surface decreases continuously with increases in the values of r_u and k_h.

9. Comparisons

In order to validate the computational procedure, the obtained values of the stability numbers for homogeneous soil slopes were compared with the results reported by

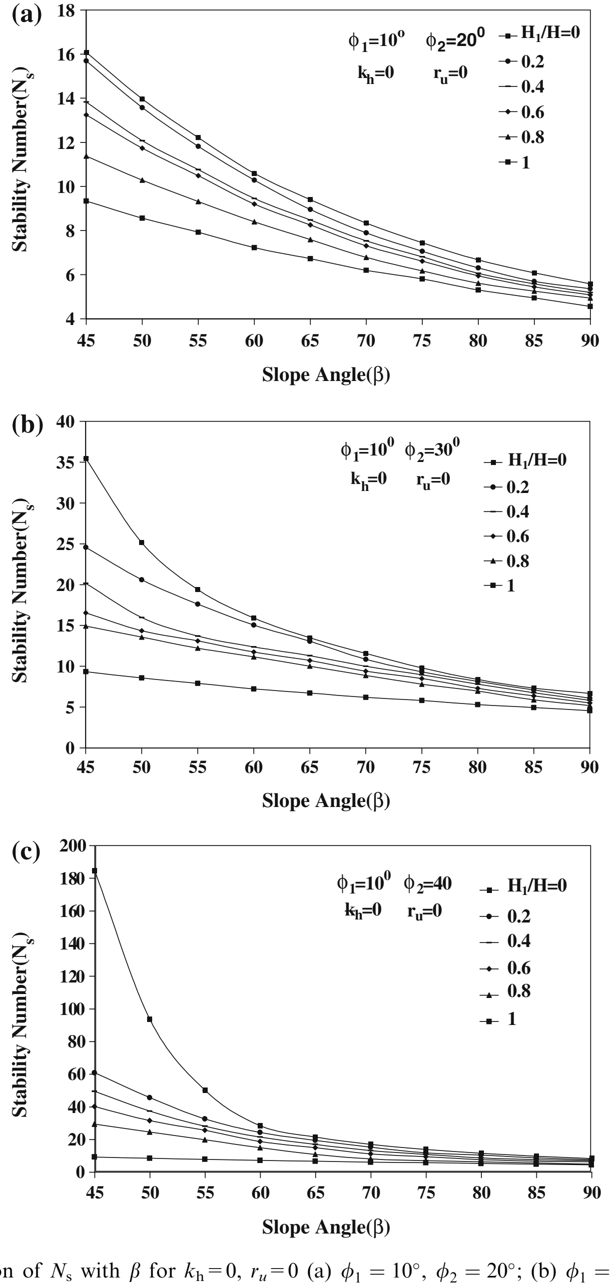


Figure 3. Variation of N_s with β for $k_h = 0$, $r_u = 0$ (a) $\phi_1 = 10°$, $\phi_2 = 20°$; (b) $\phi_1 = 10°, \phi_2 = 30°$; (c) $\phi_1 = 10°, \phi_2 = 40°$; (d) $\phi_1 = 20°, \phi_2 = 30°$; (e) $\phi_1 = 20°, \phi_2 = 40°$; and (f) $\phi_1 = 30°, \phi_2 = 40°$.

Chen et al. (1969), Michalowski (2002a) and Kumar (2004) on the basis of the upper bound limit analysis. Chen et al. (1969) obtained the results only with $r_u = k_h = 0$. On the other hand, Michalowski (2002a) has included the effect of r_u and k_h. On the

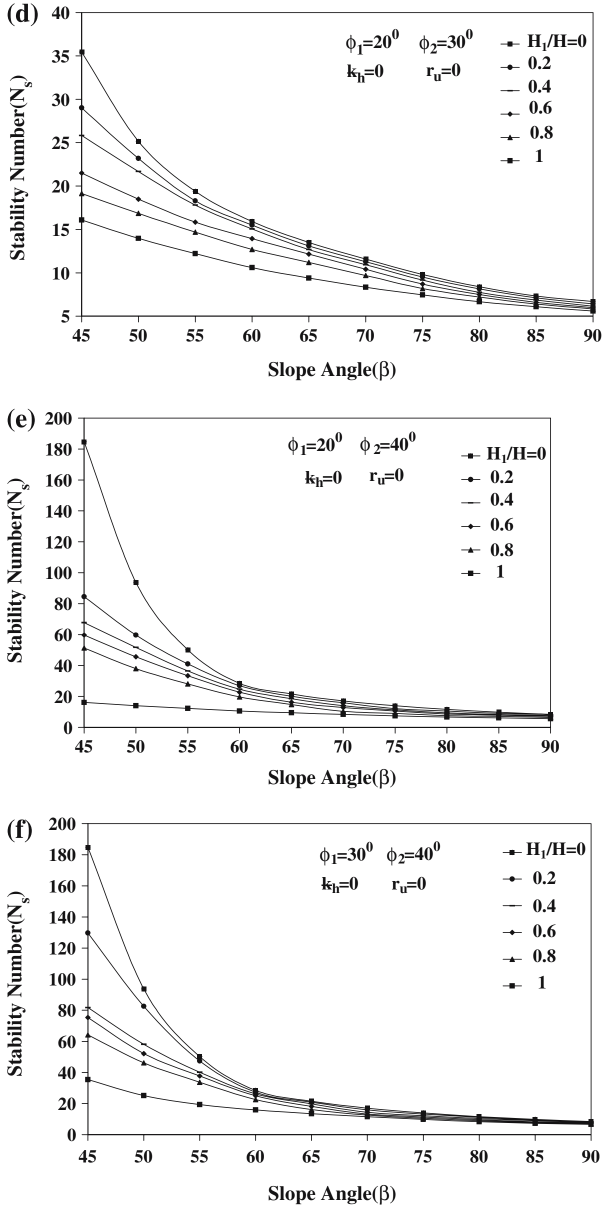


Figure 3. Continued.

contrary, the analysis of Kumar (2004) only incorporates the effect of r_u. The comparison of all these results has been presented in Tables 1–3. It can be noted that the present values of N_s are almost the same as those reported by Chen et al. (1969), Michalowski (2002a) and Kumar (2004); the results of Michalowski (2002a) have been provided only up to one decimal precision as these values were read from the graphs.

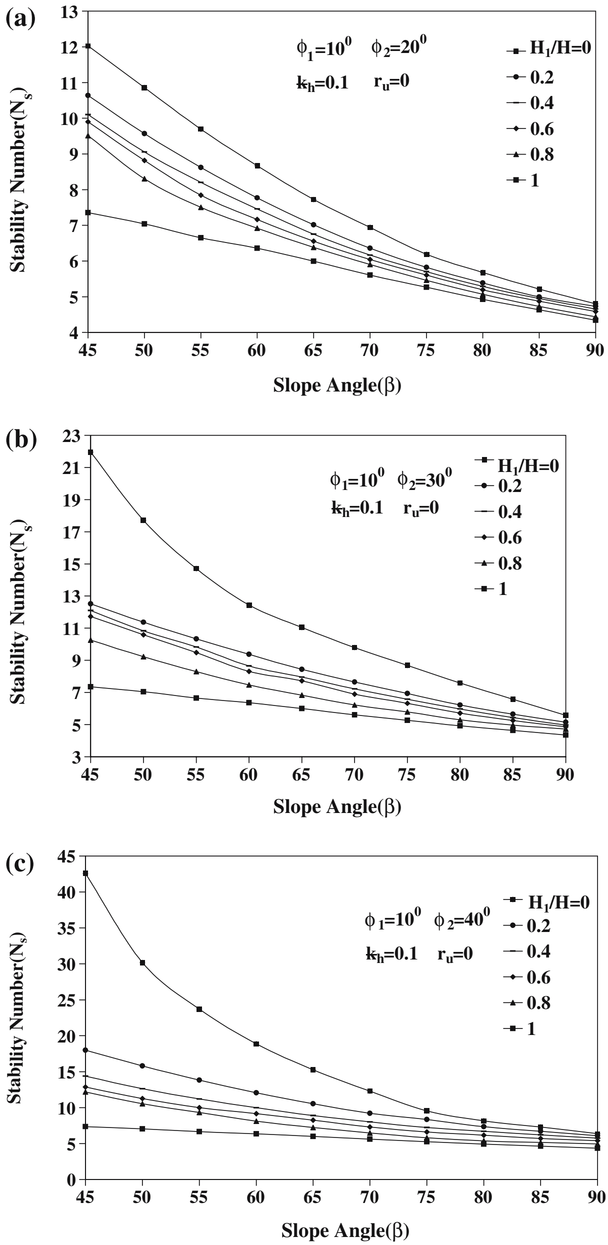


Figure 4. Variation of N_s with β for $k_h = 0.1$, $r_u = 0$ (a) $\phi_1 = 10°, \phi_2 = 20°$; (b) $\phi_1 = 10°, \phi_2 = 30°$; (c) $\phi_1 = 10°, \phi_2 = 40°$; (d) $\phi_1 = 20°, \phi_2 = 30°$; (e) $\phi_1 = 20°, \phi_2 = 40°$; and (f) $\phi_1 = 30°, \phi_2 = 40°$.

10. Remarks

It should be mentioned that the results have been obtained with an assumption that the soil mass obeys an associated flow rule. The dilatancy predicted on the basic of

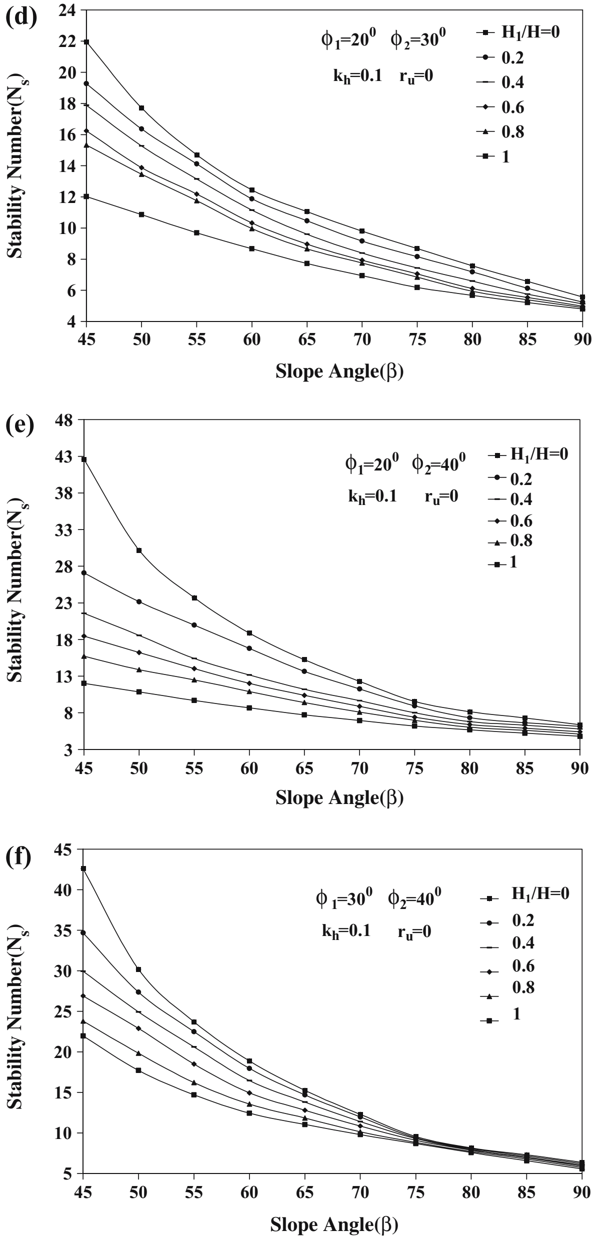


Figure 4. Continued.

this assumption is usually much greater than observed in most soils. It has already been shown by Drescher and Detournay (1993) that the assumption of an associated flow rule material overpredicts the magnitudes of collapse loads. As a result, the true stability number for slopes with non- associated flow rule material will be lower than

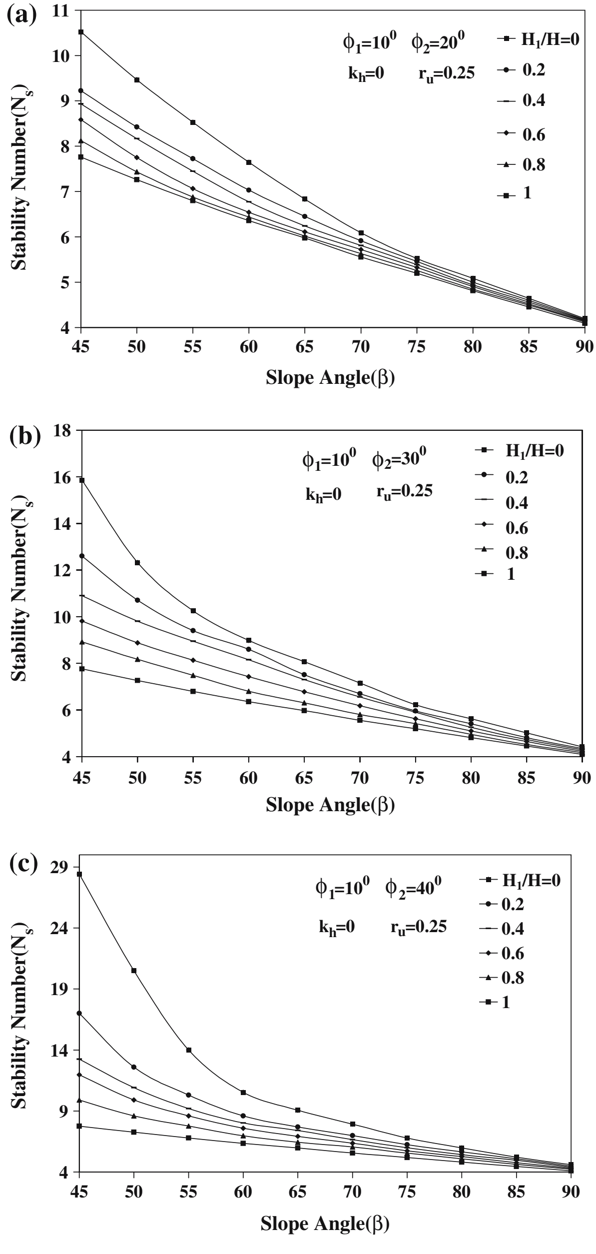
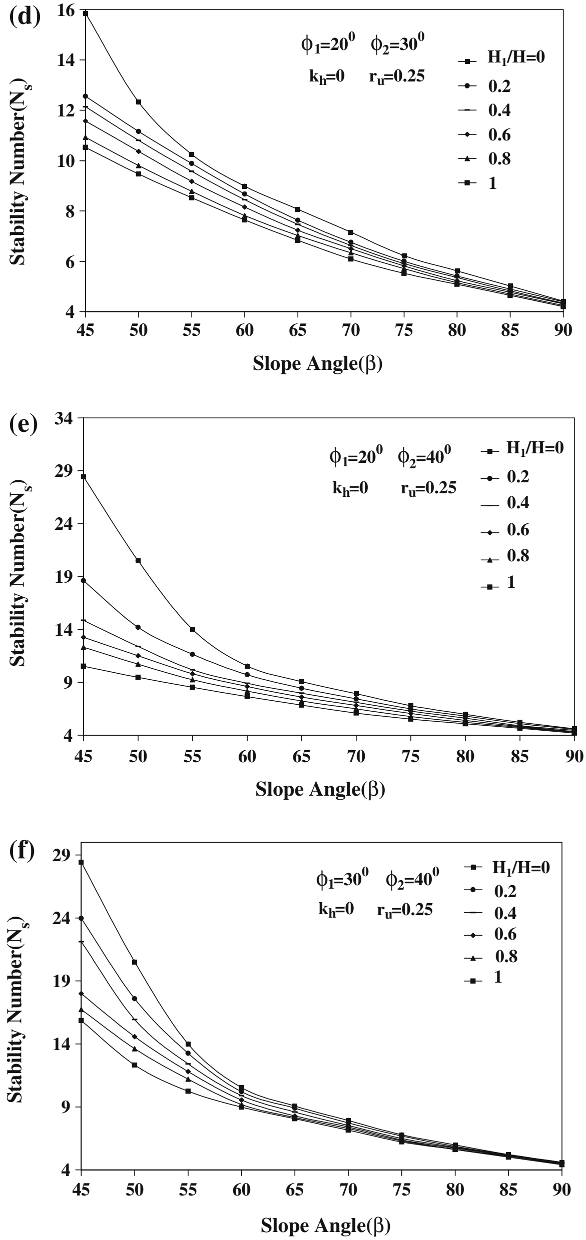


Figure 5. Variation of N_s with β for $k_h = 0$, $r_u = 0.25$ (a) $\phi_1 = 10°, \phi_2 = 20°$; (b) $\phi_1 = 10°, \phi_2 = 30°$; (c) $\phi_1 = 10°, \phi_2 = 40°$; (d) $\phi_1 = 20°, \phi_2 = 30°$; (e) $\phi_1 = 20°, \phi_2 = 40°$; and (f) $\phi_1 = 30°, \phi_2 = 40°$.

those determined on the basis of the present study. The effect of the non-associativity of the flow rule can be taken into account with the recent methodology suggested by Kumar (2004); however, an assumption is required in this approach to define the distribution of the normal stresses along the boundary of the rupture surface.



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11. Conclusions

Based upon the theorem of upper bound limit analysis, a method has been suggested to compute the stability numbers for layered soil slopes. As compare to the limit

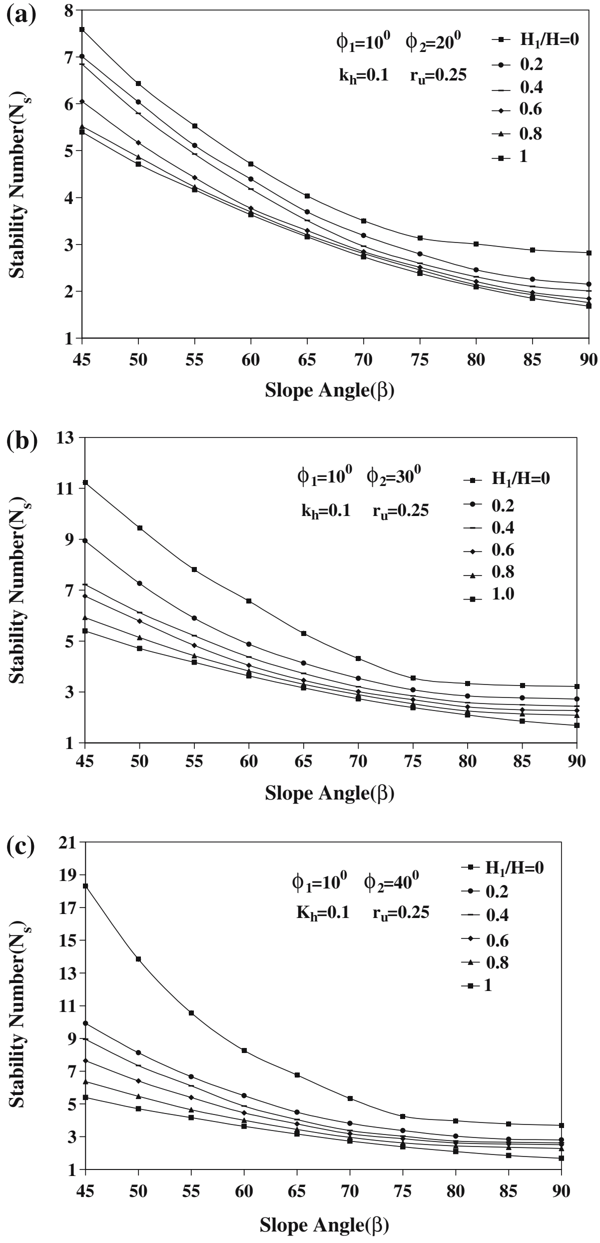

Figure 6. Variation of N_s with β for $k_h = 0.1$, $r_u = 0.25$ (a) $\phi_1 = 10°, \phi_2 = 20°$; (b) $\phi_1 = 10°, \phi_2 = 30°$; (c) $\phi_1 = 10°, \phi_2 = 40°$; (d) $\phi_1 = 20°, \phi_2 = 30°$; (e) $\phi_1 = 20°, \phi_2 = 40°$; and (f) $\phi_1 = 30°, \phi_2 = 40°$.

equilibrium method, the proposed technique has an advantage that it ensures the kinematic admissibility of the chosen collapse mechanism. The method was successfully applied for a two layered soil slope and a series of non-dimensional charts

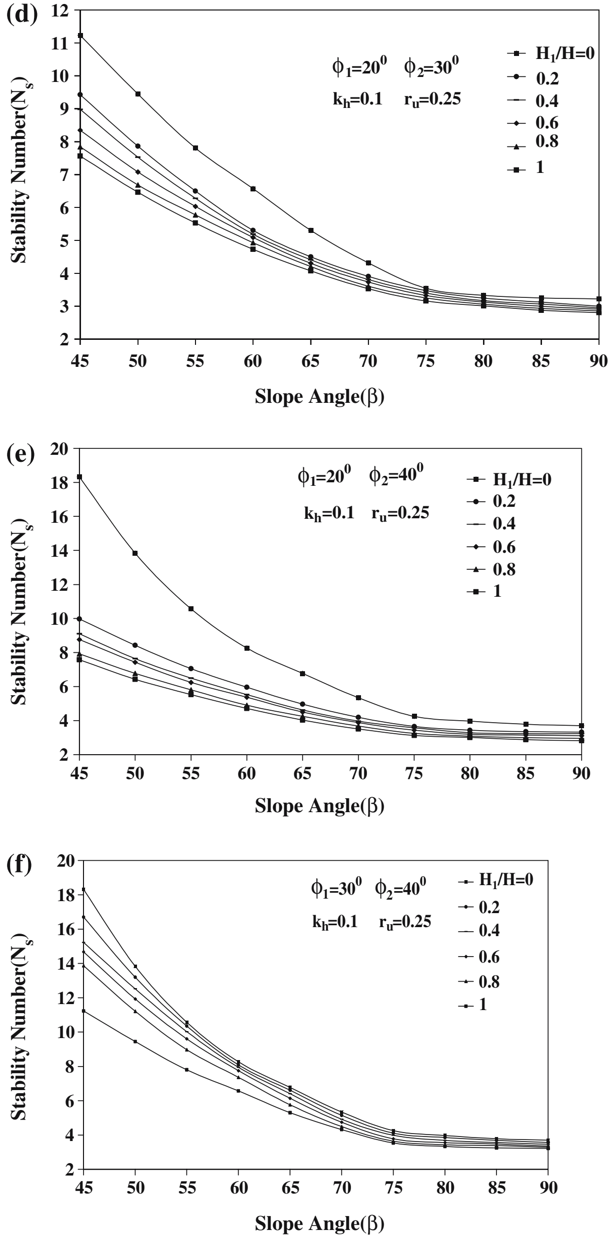

Figure 6. Continued.

providing the values of stability numbers were developed by incorporating the effect of pore water pressure as well as horizontal earthquake body forces. For given values of γ and c, a reduction in N_s with an increase in the relative thickness of the strata having smaller value of ϕ, especially for mild slopes, was clearly noted. The value of

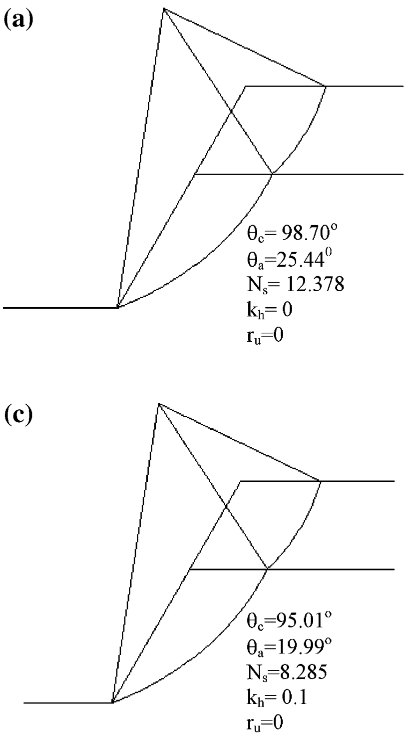

Figure 7. Failure patterns for $\beta = 60°$, $H_1/\ H_c = 0.4, \phi_1 = 10°$ and $\phi_2 = 30°$ for (a) $k_h = 0$, $r_u = 0$; (b) $k_h = 0$, $r_u = 0.25$; (c) $k_h = 0.1$, $r_u = 0$; and (d) $k_h = 0.1$, $r_u = 0.25$.

Table 1. Comparison of stability number (N_s) for $k_h = 0$ and $r_u = 0$

ϕ (Deg)	β (Deg)	Present study	Chen et al. (1969)	Kumar (2004)
10	90	4.565	4.59	4.583
	75	5.809	5.80	5.801
	60	7.229	7.26	7.259
	45	9.344	9.32	9.312
20	90	5.587	5.51	5.505
	75	7.451	7.48	7.477
	60	10.596	10.39	10.392
	45	16.080	16.18	16.181
30	90	6.679	6.69	6.689
	75	9.805	9.96	9.941
	60	15.915	16.11	16.052
	45	35.453	35.63	35.578
40	90	8.443	8.30	8.293
	75	13.965	14.00	13.977
	60	28.462	28.99	29.019
	45	184.778	185.60	187.574

Table 2. Comparison of stability number (N_s) for $k_h = 0$, $r_u = 0.25$

ϕ (Deg)	β (Deg)	Present study	Michalowski (2002a)	Kumar (2004)
10	90	4.093	4.0	4.134
	75	5.196	5.2	5.186
	60	6.357	6.3	6.389
	45	7.761	7.7	7.955
20	90	4.197	4.2	4.336
	75	5.521	5.5	5.771
	60	7.645	7.7	7.702
	45	10.524	10.6	10.948
30	90	4.415	4.4	4.394
	75	6.222	6.2	6.277
	60	8.983	9.0	9.262
	45	15.855	15.5	15.861
40	90	4.604	4.7	4.251
	75	6.788	6.8	6.660
	60	10.538	10.6	11.254
	45	28.434	28.9	27.663

Table 3. Comparison of stability number (N_s) for $k_h = 0.1$, $r_u = 0$

ϕ(Deg)	β(Deg)	Present study	Michalowski (2002a)
10	90	4.341	4.3
	75	5.265	5.2
	60	6.360	6.3
	45	7.357	7.3
20	90	4.803	4.8
	75	6.189	6.2
	60	8.670	8.7
	45	12.027	12.2
30	90	5.574	5.5
	75	8.687	8.6
	60	12.445	12.4
	45	21.961	21.9
40	90	6.366	6.3
	75	9.549	9.5
	60	18.878	18.8
	45	42.585	42.5

the stability number as well as the size of the soil mass bounded by the rupture surface decreases with an increase in the magnitude of r_u and k_h.

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