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Reliability based assessment of shallow foundations using mathcad

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Abstract. A reliability based method was used to design and analyse shallow foundations using first-order Taylor series approximation. The computer program Mathcad was used to facilitate all mathematical and computional efforts. This method is an effective tool to assist the foundation designers and analyists to investigate how reliable their designs or analyses are in relation to the ultimate bearing capacity of the foundations. The approach presented in this paper provides a reliable alternative for design and analysis of shallow foundations, rather than the conventional design methods, which employs the assumptions of a specified saftey factor. Several examples were presented for design and analysis of strip footings embedded in sandy soil, and rectangular and square footings analysis embedded in clayey soils. The program input and output of each example are also presented and discussed.

Key words. bearing capacity, Mathcad, reliability, shallow foundation, Taylor's series.

1. Introduction

Reliability based design (RBD) has been an important subject for the last two decades. The reliability based methods to design shallow foundations are becoming accepted as powerful tools to assist designers investigate how reliable their designs are. The lack of use of these methods is due to the pre-requirements of knowledge in the fields of statistics and probability, which most designers are not familiar with. Preparing ready design charts or computer software for this purpose increases the widespread use of such methods. The simplicity of Mathcad's input/output interface allaws the designer to easily check the probability of failure or the reliability of his/her design.

The application of reliability analysis to bearing capacity problems started in the early 1970s. Simple examples were presented in several publications (e.g. Ang and Tang, 1975; Harr, 1977). The variability of soil properties was studied extensively even earlier. Lumb (1974) published one of the most comprehensive reviews in 1974. This paper presented a collection of variability of some basic soil properties that affect the bearing capacity.

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Because of the variability and randomness of soil properties, it is expected that the ultimate bearing capacity (q_{ult}) of a footing will be a random variable. The probabilistic methods that may be used to determine the distribution of q_{ult} and consequently the reliability of the chosen factor of safety falls into four categories:

1. Point estimation methods (PEM, Rosenblueth method).
2. First- and second-order approximation methods usually making use of Taylor's series expansion.
3. Monte Carlo methods.
4. The exact methods. (e.g., Cherubini, 1990; Easa, 1992).

The first and the second categories are simple and may yield reasonably accurate results in practice. The Monte Carlo method is best used when studying the effect of variability of different properties on the ultimate bearing capacity, If a general solution is required by this category the computer time will be relatively higher. By "exact methods" reference is made to methods consisting of mathematical determination of the distribution function of q_{ult} depending on the inverse distribution functions of the independent variables such as the angle of internal friction (ϕ), the cohesion (c), and the unit weight (γ) of the soil.

Although the latter method (exact methods) is more rigorous due to the complicated relation of the bearing capacity with many parameters, this makes obtaining the derivatives of the different parameters with respect to the ultimate bearing capacity extremely difficult. As these derivatives are necessary for the determination of the distribution function of the ultimate bearing capacity from the inverse distributions. Approximate modified bearing capacity equations (see e.g., Krizek, 1965) have been adopted, but still the problem is difficult to solve analytically.

In this paper, Taylor's series is used to expand the distribution function of the ultimate bearing capacity. The method is supported by several authors, (Kapur and Lamberson, 1977; Harr, 1977; Basheer and Najjar, 1998; Duncan, 2000).

2. The ultimate bearing capacity equation

Many modifications have been published since the original bearing capacity equation presented by Terzaghi (1943), In this paper, the following equation and factors will be adopted.

For a strip footing, the ultimate bearing capacity may be taken as

$$q_{ult} = cN_c + \bar{q}N_q + \frac{1}{2}\gamma BN_\gamma \tag{1}$$

where B is the width of the footing, c the cohesion intercept of the soil,$\bar{q}$ is the soil effective overburden pressure at the foundation base level $= \gamma D_f$ with D_f denoting depth of the footing N_c, N_q and N_γ are the bearing capacity factors and are functions of ϕ as follows, see Vesic (1973):

$$N_q = \tan^2(45^\circ + \phi) \cdot e^{\pi \tan \phi}$$
$$N_c = (N_q - 1) \cot \phi \tag{2}$$
$$N_\gamma = 2(N_q + 1) \cdot \tan \phi$$

It may be noticed that no problems may occur when calculating these factors for any values of ϕ except in the case where $\phi = 0$, $N_q = 1$ and $N_\gamma = 0$ but N_c yields no definite number. Usually N_c for $\phi = 0$ is taken as the limit, 5.14. In this paper and for computer input a value of $\phi = 0.001^\circ$ (1.74533×10^{-3}) will be used instead of zero, this will give $N_c = 5.14182$, $N_q = 1.00009$, and $N_\gamma = 0.00007$. To distinguish the case of $\phi = 0$, an if statement will be used in the program.

For other shapes of the foundation, each term in Equation (1) is multiplied by a shape factor

$$q_{\text{ult}} = cN_cS_c + \bar{q}N_qS_q + 0.5\gamma BN_\gamma S_\gamma \tag{3}$$

Many definitions were suggested for these shape factors. Table 1 lists the values suggested by Hanson (Arora, 1987) with L being the long dimension, while B is the short dimension of the footing. These shape factors are convenient and may be adopted in practice. The effects of depth, inclination of load, base tilting, ground slope, and water table depth are ignored and may be taken into consideration in later work.

3. Variability of soil properties

In data analysis and density function distribution, two essential parameters are required, the mean μ and the variance V. The standard deviation, σ, being the square root of the variance, is usually relied upon when measuring dispersion of the data about the mean. Comparing the dispersion of different groups requires a dimensionless variable, the coefficient of variation, $CV = \sigma/\mu$.

Related to bearing capacity analysis of shallow foundation, the soil properties needed for calculation are the angle of internal friction, the cohesion, and the unit weight. Most researchers agree that these properties are, in fact, random variables and are controlled by normal distribution. Much of effort has been exerted to assess the coefficient of variation of these and other soil properties. Lumb (1974) stated that

Table 1. Shape factors adopted

Shape of Footing	S_c	S_q	S_γ
Strip	1.0	1.0	1.0
Rectangular	$1+0.2\ B/L$	$1+0.2\ B/L$	$1-0.4\ B/L$
Square	1.3	1.2	0.8
Circular	1.3	1.2	0.6

most coefficients of variation of soil properties commonly range between 10 and 25% and values out of this range should be avoided or used cautiously. This range is relatively strict when dealing with values obtained from field tests, such as the standard penetration test, vane shear test, etc. Tests based on total stresses may also yield higher coefficients of variation. Table 2 presents recommended ranges for the coefficient of variation for soil parameters taken from different researchers.

Duncan (2000) suggested that the "3σ rule", σ = (highest value obtained − lowest value obtained)/6, may be used successfully to estimate the coefficient of variation. In

Table 2. Variability of soil properties

Property	CV (%)	Notes	Reference
ϕ (Sands)	5–15	Recommended 10%	Lee et al. (1983)
ϕ (Clays)	12–56		Lee et a1. (1983)
tan ϕ (Sands)	5–15		Lumb (1974)
ϕ'	4		Christian et al. (1994)
ϕ' estimated from PI	15–20	Clay	Phoon and Kulhway (1996)
ϕ' (Direct shear)	7–20	Sand, clay	Phoon and Kulhway (1996)
ϕ' (Triaxial)	10–15	Sands	Phoon and Kulhway (1996)
c_u (Sandy soils)	25–30	Recommended 30%	Lee et al. 1983
c_u (Clays)	20–50		Lee et al. (1983)
c_u (Clays)	20–50		Lumb (1974)
s_u	13–40		Duncan (2000)
s_u/σ'_{vo}	5–15		Duncan (2000)
s_u (Vane)	10–20		Duncan (2000)
s_u (Vane)	20–32		Christian et al. (1994)
s_u (Unconfined)	20–55	Clay	Phoon and Kulhway (1996)
s_u (UU test)	10–35	Clay	Phoon and Kulhway (1996)
s_u (CU test)	20–45	Clay	Phoon and Kulhway (1996)
s_u (Field vane)	15–50	Clay	Phoon and Kulhway (1996)
s_u (Estimated from N)	40–60	Clay	Phoon and Kulhway (1996)
s_u (Estimated from PI)	30–55	Clay	Phoon and Kulhway (1996)
γ_d Modified Proctor	1–7	Recommended, 5%	Lee et al. (1983)
γ Density	1–10	Recommended 3%	Lee et al. (1983)
γ	5–10		Lumb (1974)
γ	3–7		Duncan (2000)
γ_b	0–10		Duncan (2000)
γ	2		Christian et al. (1994)
Liquid limit (LL)	2–48	Recommended 10%	Lee et al. (1983)
Plasticity index (PI)	7–79	Recommended 30% for clays and 70% for sands and gravels	Lee et al. (1983)
N	15–45		Duncan (2000)
q_u Unconfined strength	6–100	Recommended 40%	Lee et al. (1983)
q_c (Elastic)	5–15		Duncan (2000)
q_c (Mechanical)	15–37		Duncan (2000)

fact, the uncertainty in estimating the coefficient of variation itself makes the use of 3σ rule plausible. Judgement and experience are valuable in such estimations.

4. Reliability and factor of safety

The applied contact pressure under a footing may be calculated as the gross load (Q) (including the weight of the footing and the soil above) divided by the footing area (A), If this pressure is denoted by g_{app}, then the factor of safety against failure will be

$$F_{\mathrm{s}} = q_{\mathrm{ult}}/q_{\mathrm{app}} \tag{4}$$

Sometimes the F_{s} is required, corresponding to the net pressure neglecting the weight of the footing and the refill soil and is thus calculated as

$$\begin{aligned} F_{\mathrm{s}} &= \frac{q_{\mathrm{ult}} - \gamma D_{\mathrm{f}}}{q_{\mathrm{app}} - \gamma D_{\mathrm{f}}} \\ &= (q_{\mathrm{ult}} - \gamma D_{\mathrm{f}})A/Q_{\mathrm{net}} \end{aligned} \tag{5}$$

where Q_{net} is the net load applied from the superstructure. In this paper Equation (4) will be adopted for the factor of safety calculation. It is assumed that the probability distribution of q_{ult} is a Gaussian normal one, which is not very accurate but the level of approximation is accepted. Other researchers, however, have used other distributions (see e.g., Basma, 1994; Basheer and Najjar, 1998).

Division of q_{ult} distribution by the value of F_{s} will also yield a normal bell type distribution for q_{app}. Figure 1 shows both distributions. The shaded area is the region indicative of the probability of failure, which may be defined in three ways as shown in the following equation

$$P_{\mathrm{f}} = P(q_{\mathrm{ult}} < q_{\mathrm{app}}) = P(q_{\mathrm{ult}} - q_{\mathrm{app}} < 0) = P(q_{\mathrm{ult}}/q_{\mathrm{app}} < 1) \tag{6}$$

while the reliability is simply

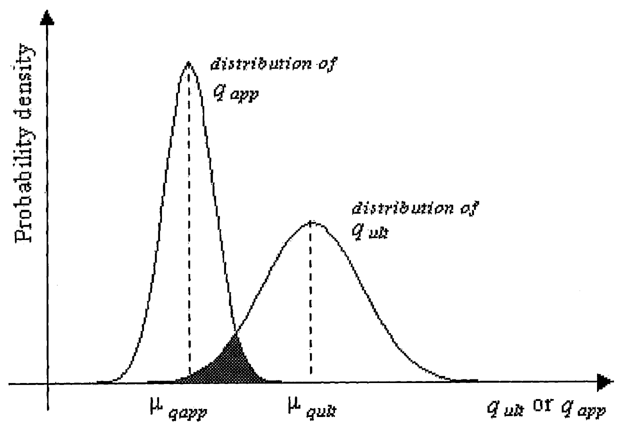

Figure 1. Probability of failure indicated by the shaded area, the intersection of the distributions of ultimate bearing capacity and applied load.

$$R = 1 - P_f$$

The first definition, i.e., $P_f = P(q_{ult} < q_{app})$, is depicted in Figure 2(a) where q_{ult} is distributed and required in the area up to $q_{ult} = q_{app}$ to measure the probability of failure. If the density function of $(q_{ult} - q_{app})$ is plotted, it will give a normal distribution shape with a mean of $(\mu_{q_{ult}} - \mu_{q_{app}})$ and a standard deviation of $(\sigma^2_{q_{ult}} - \sigma^2_{q_{app}})^{0.5}$ Figure 2(b) shows this distribution with the shaded area being the probability of failure, A similar distribution may be adopted for the factor of safety F_s being q_{ult}/q_{app}, as shown in Figure 2(c).

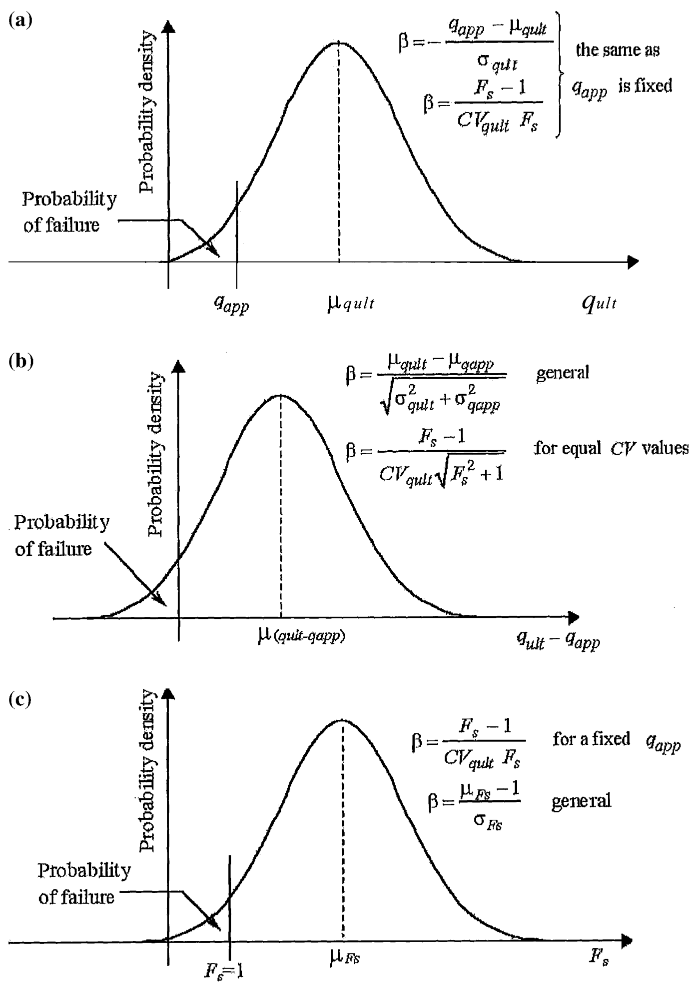

Figure 2. Definitions of the probability of failure with the corresponding reliability indeces and factors of safety.

Considering Figure 2(a), the convenient way to assess the probability of failure is to fix the value of q_{app} and seek the probability of $q_{ult} < q_{app}$. The random variable z of the standard normal distribution will be

$$z = \frac{q_{ult} - \mu_{q_{ult}}}{\sigma_{q_{ult}}} \tag{8}$$

and defining z_0 as z at which $q_{ult} = q_{app}$, i.e.,

$$z_0 = \frac{q_{app} - \mu_{q_{ult}}}{\sigma_{q_{ult}}} \tag{9}$$

the probability of failure will be $P_f = \Phi(z_0)$ where Φ is the standard normal distribution cumulative function. These definitions are adopted in the computer programs of this paper. For the second definition of the probability of failure, that is $P_f = P(q_{ult} - q_{app} < 0)$, it is possible to apply a distribution for q_{app} as well (Figure 1). If a distribution of $(q_{ult} - q_{app})$ is available (Figure 2(b)) the standard normal variate z will be

$$z = \frac{(q_{ult} - q_{app}) - (\mu_{q_{ult}} - \mu_{q_{app}})}{\sqrt{\sigma^2_{q_{ult}} + \sigma^2_{q_{app}}}} \tag{10}$$

and if we are seeking the probability of failure then the value of z corresponding to zero, the difference of q_{ult} and q_{app} will be

$$z_0 = -\frac{\mu_{q_{ult}} - \mu_{q_{app}}}{\sqrt{\sigma^2_{q_{ult}} + \sigma^2_{q_{app}}}} \tag{11}$$

and the probability of failure will be $P_f = \Phi(z_0)$ accordingly. If the standard deviation is substituted as $CV \cdot \mu$, the probability of failure may be written as

$$P_f = \Phi\left(-\frac{\mu_{q_{ult}} - \mu_{q_{app}}}{\sqrt{CV^2_{q_{ult}}\mu^2_{q_{ult}} - CV^2_{q_{app}}\mu^2_{q_{app}}}}\right) \tag{12}$$

If the coefficients of variation are assumed equal this may be directly related to the factor of safety substituting $\mu_{q_{ult}} = F_s \cdot \mu_{q_{app}}$ and $\sigma_{q_{ult}} = F_s \cdot \sigma_{q_{app}}$. So, the variable z_0 at zero difference will be

$$z_0 = -\frac{F_s - 1}{CV_{q_{ult}}\sqrt{F_s^2 + 1}} \tag{13}$$

The factor of safety refers to the one called "central factor of safety", which is calculated as the ratio of means of the ultimate and applied distributions. This may be used for rapid determination of an approximate value of the reliability using:

$$R = \Phi(-z_0) = \Phi\left(\frac{F_s - 1}{CV_{q_{ult}}\sqrt{F_s^2 + 1}}\right) \tag{14}$$

Notice that when F_s increases, the random variable within the parentheses approaches 1.0 $/CV_{q_{ult}}$, i.e., $\mu_{q_{ult}}/\sigma_{q_{ult}}$.

It is obvious that as F_s is increased, probability of failure is decreased and consequently the reliability is increased. For a convenient measure of risk, a reliability index is adopted as

$$\beta = -z_0 = \frac{\mu_{q_{ult}} - \mu_{q_{app}}}{\sqrt{\sigma^2_{q_{ult}} + \sigma^2_{q_{app}}}} \tag{15}$$

The value of β indicates the probability of failure or the reliability through their definitions. When the coefficients of variation for both ultimate capacity and applied pressure are taken as equal, β becomes:

$$\beta = \frac{F_s - 1}{CV_{q_{ult}}\sqrt{F_s^2 + 1}} \tag{16}$$

Phoon and Kulhawy (1996) stated that β for most geotechnical and structural components lies between 1 and 4. Probability of failure is expected to be 50% when β becomes 0, corresponding to a factor of safety that equals 1. For the last definition of probability of failure in Equation (6), i.e., $P(q_{ult}/q_{app} < 1)$, with reference to Figure 2(c), distribution for the factor of safety itself should be available and then it will become easy to calculate the probability of failure, $P_f = P\ (F_s < 1)$. The random variate z will be:

$$z = \frac{F_s - \mu_{F_s}}{\sigma_{F_s}} \tag{17}$$

with z_0 obtained at $F_s = 1$. If q_{app} is fixed, that is $CV_{q_{app}} = 0$, the variable z will be the same as the one adopted for the first definition in Equation (9) and CV_{F_s} will be equal to $CV_{q_{ult}}$. Substituting F_s as the expected mean and $CV_{q_{ult}}$ for CV_{F_s} a value for a reliability index in this case may be written as:

$$\beta = -z_0 = \frac{F_s - 1}{CV_{q_{ult}} F_s} \tag{18}$$

Table 3 lists the values of reliability and factor of safety for different values of β and coefficients of variation chosen for q_{ult}. It may be noticed that as the coefficient of variation increases, the factor of safety corresponding to a certain reliability decreases. The flatter the probability distribution for the q_{ult}, the lower the risk. It is also important to distinguish that when the reliability required is increased, high coefficients of variation yield zero or negative factors of safety. This could be partially overcome by applying other types of probability distributions.

5. Mean and variance of the ultimate bearing capacity

From the above descriptions and definitions one may estimate the reliability of the foundation safety for a certain factor of safety, otherwise the distribution of q_{app} and

Table 3. Central factors of safety for different values of reliability and coefficient of variation of ultimate capacity

		Central factors of safety									
β	R	$CV_{q_{ult}} = 0.05$	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
0	0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
		1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.25	0.598706	1.012	1.025	1.039	1.052	1.066	1.081	1.095	1.111	1.126	1.142
		1.017	1.036	1.054	1.073	1.092	1.112	1.132	1.152	1.173	1.194
0.50	0.691462	1.025	1.052	1.081	1.111	1.142	1.176	1.212	1.250	1.290	1.333
		1.036	1.073	1.112	1.152	1.194	1.238	1.284	1.333	1.384	1.437
0.75	0.773373	1.039	1.081	1.126	1.176	1.230	1.290	1.355	1.428	1.509	1.600
		1.054	1.112	1.173	1.238	1.308	1.384	1.465	1.554	1.651	1.758
1.00	0.841345	1.052	1.111	1.176	1.250	1.333	1.428	1.538	1.666	1.818	2.000
		1.073	1.152	1.238	1.333	1.437	1.554	1.686	1.836	2.010	2.215
1.25	0.894350	1.066	1.142	1.230	1.333	1.454	1.600	1.777	2.000	2.285	2.666
		1.092	1.194	1.308	1.437	1.585	1.758	1.964	2.215	2.530	2.942
1.50	0.933193	1.081	1.176	1.290	1.428	1.600	1.818	2.105	2.500	3.076	4.000
		1.112	1.238	1.384	1.554	1.758	2.010	2.332	2.763	3.377	4.341
1.75	0.959941	1.095	1.212	1.355	1.538	1.777	2.105	2.580	3.333	4.705	8.000
		1.132	1.284	1.465	1.686	1.964	2.332	2.849	3.647	5.068	8.414
2.00	0.977250	1.111	1.250	1.428	1.666	2.000	2.500	3.333	5.000	10.00	
		1.152	1.333	1.554	1.836	2.215	2.763	3.647	5.369	10.43	
2.25	0.987776	1.126	1.290	1.509	1.818	2.285	3.076	4.705	10.00		
		1.173	1.384	1.651	2.010	2.530	3.377	5.068	10.43		
2.50	0.993790	1.142	1.333	1.600	2.000	2.666	4.000	8.000			
		1.194	1.437	1.758	2.215	2.942	4.341	8.414			
2.75	0.997020	1.159	1.379	1.702	2.222	3.200	5.714	26.66			
		1.216	1.494	1.877	2.461	3.507	6.098	27.13			
3.00	0.998650	1.176	1.428	1.818	2.500	4.000	10.00				
		1.238	1.554	2.010	2.763	4.341	10.43				
3.25	0.999423	1.194	1.481	1.951	2.857	5.333	40.00				
		1.261	1.618	2.160	3.145	5.709	40.48				
3.50	0.999767	1.212	1.538	2.105	3.333	8.000					
		1.284	1.686	2.332	3.647	8.414					
3.75	0.999912	1.230	1.600	2.285	4.000	16.00					
		1.308	1.758	2.530	4.341	16.45					
4.00	0.999968	1.250	1.666	2.500	5.000						
		1.333	1.836	2.763	5.369						
4.25	0.999989	1.269	1.739	2.758	6.666						
		1.358	1.920	3.040	7.065						

Note: Cells that are not shaded are devoted for values of factor of safety calculated from Equation (18) while the shaded areas with light gray are for those calculated through Equation (16). The cells shaded with dark gray indicate unavailability of positive safety factors.

q_{ult} should be known. The main step is to determine the expected mean and variance of q_{ult} so that one may calculate β and consequently the probability of failure.

As it is mentioned in Introduction, the method adopted to determine $\mu_{q_{\mathrm{ult}}}$, $\sigma_{q_{\mathrm{ult}}}$ is the Taylor's series expansion. Using the first three terms of Taylor's series it can be shown that (see e.g., Kapur and Lamberson, 1977)

$$E(y) \approx f(\mu) + \frac{1}{2} f''(\mu) \cdot V(x) \tag{19}$$

where $E(y)$ is the expected value of $y = f(x)$. The variable x has a mean value μ, and f'' denotes the second derivative of the function f. For the determination of the variance of the function y, only two terms of Taylor's series are sufficient and $V(y)$ will be:

$$V(y) \approx [f'(\mu)]^2 \cdot V(x) \tag{20}$$

When a function of several variables is considered, $y = f(x_1, x_2, \ldots, x_n)$, the expansion of $E(y)$ and $V(y)$ by Taylor's series yields the following:

$$E(y) = f(\mu_1, \mu_2, \ldots, \mu_n) + \frac{1}{2} \sum_{i=1}^{n} \left[\frac{\partial^2 f}{\partial x_i^2} \bigg|_{x_i = \mu_i} \cdot V(x_i) \right] \tag{21}$$

$$V(y) = \sum_{i=1}^{n} \left\{ \left[\frac{\partial f}{\partial x_i} \bigg|_{x_i = \mu_i} \right]^2 \cdot V(x_i) \right\} \tag{22}$$

Next, these equations are applied to determine the expected mean and the variance of the ultimate bearing capacity, Returning to the form of q_{ult} in Equation (3), the independent variables are c,ϕ and γ.

The first- and second-order derivatives of q_{ult} based on Equation (3) may be written as:

$$\begin{aligned}
&\frac{\partial q_{\mathrm{ult}}}{\partial c} = S_c N_c \\
&\frac{\partial q_{\mathrm{ult}}}{\partial \phi} = c S_c \frac{\partial N_c}{\partial \phi} + \gamma D_{\mathrm{f}} S_q \frac{\partial N_q}{\partial \phi} + 0.5 \gamma B S_\gamma \frac{\partial N_\gamma}{\partial \phi} \\
&\frac{\partial q_{\mathrm{ult}}}{\partial \gamma} = D_{\mathrm{f}} S_q N_q + 0.5 B S_\gamma N_\gamma \\
&\frac{\partial^2 q_{\mathrm{ult}}}{\partial c^2} = 0 \\
&\frac{\partial^2 q_{\mathrm{ult}}}{\partial \phi^2} = c S_c \frac{\partial^2 N_c}{\partial \phi^2} + \gamma D_{\mathrm{f}} S_q \frac{\partial^2 N_q}{\partial \phi^2} + 0.5 \gamma B S_\gamma \frac{\partial^2 N_\gamma}{\partial \phi^2} \\
&\frac{\partial^2 q_{\mathrm{ult}}}{\partial \gamma^2} = 0
\end{aligned} \tag{23}$$

Hence, the expected mean and variance of q_{ult} may be found through the following equations:

$$E(q_{\text{ult}}) = q_{\text{ult}}(\mathbf{m}) + 0.5\mathbf{v}^{\mathrm{T}} \left.\frac{\partial^2 q_{\text{ult}}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{m}} \tag{24}$$

$$V(q_{\text{ult}}) = \mathbf{v}^{\mathrm{T}} \left.\frac{\partial^2 q_{\text{ult}}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{m}} \tag{25}$$

where

$$\mathbf{x} = (c \quad \phi \quad \gamma)^{\mathrm{T}}$$

$$\mathbf{m} = (\mu_c \quad \mu_\phi \quad \mu_\gamma)^{\mathrm{T}}$$

$$\mathbf{v} = (V_c \quad V_\phi \quad V_\gamma)^{\mathrm{T}}$$

$$q_{\text{ult}}(\mathbf{m}) = q_{\text{ult}}(\mu_c, \mu_\phi, \mu_\gamma)$$

$$\frac{\partial^2 q_{\text{ult}}}{\partial \mathbf{x}} = \left(\left(\frac{\partial q_{\text{ult}}}{\partial c}\right)^2 \quad \left(\frac{\partial q_{\text{ult}}}{\partial \phi}\right)^2 \quad \left(\frac{\partial q_{\text{ult}}}{\partial \gamma}\right)^2 \right)^{\mathrm{T}}$$

The variance, being the square of the standard deviation, is substituted as $V = (\mu \cdot CV)^2$. This is more convenient for practical and programming reasons as the usual statistical parameter given for a soil property is the coefficient of variation CV.

6. Procedure for analysis and design

Three cases may be considered according to the input and output parameters, one case for analysis and two cases for design. These cases are listed in Table 4 and the procedure is explained in the following.

In case 1, as dimensions are unknown, a trial dimension B for the width (or the diameter for the circular footings) is required. A convenient start may be 0.3 m. Using this trial width, the expected mean and variance of q_{ult} are calculated. Once the normal distribution parameters are determined, the value of q_{ult} corresponding to a given probability of failure, $P_{\mathrm{f}} = P(q_{\text{ult}} < q_{\text{app}} = Q/A) = 1 - R$, may be found using the inverse of the normal distribution cumulative function.

Table 4. Cases of the three programs with their input–output explained

Case no.	Case type	Input data	Output data
1	Design	Soil properties applied load reliability	Dimensions factor of safety
2	Design	Soil properties applied load factor of safety	Dimensions reliability
3	Analysis	Soil properties applied load dimensions	Factor of safety reliability

This value of q_{ult} is again used to evaluate B from $A = Q/q_{ult}$ and in turn, B is used to recalculate quit, and so on, until the difference in two successive resulting B values are very close. A limit for the number of iterations may be used as well. After this stage, the factor of safety may be calculated as (q_{ult} from Equation (3))/q_{app}. The central factor of safety may be calculated also as $\mu_{q_{ult}}/q_{app}$. It may be noticed that the central factor of safety is always higher than that obtained by the conventionally calculated bearing capacity. The reason is that the derivatives in Equation (23) are usually added to the value of $E(q_{ult})$ in Equation (24).

In design case 2, no iteration loop is required as the factor of safety is given. The dimensions are found directly from Equation (4) by substituting $q_{app} \cdot F_s$ for q_{ult} with $q_{app} = Q/A$. Consequently, A or B may be determined.

No problems are encountered when dealing with circular, square or strip footings calculations, as B may be used explicitly to define the area and the dimensions of the footing. In rectangular footings, the longer dimension L is also effective in calculations. This can be overcom by adding the ratio B/L as an input parameter, or else the dimension L should be specified as part of the input data. Thus, B will be the only unknown.

It is assumed that the footing is re-dimensioned for practical reasons, hence, a final check may be done through the analysis program prepared for case 3. In case 3, as all dimensions and properties are given, the factor of safety is calculated as in Equation (4). The probability of failure is found directly from the normal distribution cumulative curve using q_{app} as the value of the random variable (see Figure 2(a)). If the central factor of safety is required, the values obtained from Table 3 may be adopted. It may also be calculated dividing the mean q_{ult} by q_{app}. The program of case 3 may be used to study the effects of the different parameters such as the coefficients of variation for different input data.

7. Computer programs

The most important advantage of applying Mathcad to the programming of the reliability problem is to leave the tedious work of the first and the second-order differentiation of the function of the ultimate bearing capacity to the computer and limit the paper work to the conceptual level. This allows one to extend the application to the general bearing capacity equation in which many terms and variables are dependent on ϕ and it also allows for the consideration of other variabilities such as those for loading and dimensions.

Another advantage of Mathcad is that it has built-in statistical functions to calculate the probability corresponding to a certain value of the random variable, whether the density or the cumulative density is required, In addition the inverse function and the value of the variable corresponding to a certain probability may also be found. For the case of normal distribution, the functions "p_{norm}" and

"q_{norm}" are the cumulative probability function and the inverse cumulative probability function, respectively.

Three programs were prepared for each type of footings (strip, square, circular and rectangular) corresponding to cases 1, 2, and 3. For tracing how Mathcad statements and commands are used, a sample of the coding is presented in Appendix for each case. These three coding lists correspond to the design and analysis of strip footings, cases 1, 2 and 3, respectively. It will be easy to guess which statements should be modified for other types of footings. Statements and notation from Mathcad are simple and easily understood.

8. Application and discussion

Many examples have been solved using the stated computer programs. In this paper, five examples are presented. In each example, the main requirement is stated first with all corresponding input data. A discussion of the results follows, Through the discussion, due to some changes in the problem data and consequently the output, differences are evaluated and justified.

In the design examples, case 1, a tolerance value to stop the iteration is chosen as 1 mm. A limit of 100 iterations is also specified. In all problems, the ultimate bearing capacity and the mean of the ultimate capacity are computed according to Equation (3) and Equation (24), respectively, with their corresponding factors of safety.

8.1. EXAMPLE 1

Strip footing, sandy soil, design case 1

Given data and properties		Results and required output	
D_f (m)	1.0	$CV_{q_{ult}}$	0.434
Q (kN/m)	500	R	0.95
c (kN/m^2)	0	P_f	0.05
ϕ (°)	35	CF_s	3.499
γ (kN/m^3)	18.9	F_s	3.076
CV_c	0	$\mu_{q_{ult}}$ (kN/m^2)	1374
CV_ϕ	0.10	q_{ult} (kN/m^2)	1208
CV_γ	0.05	β	1.645
R	0.95	B (m)	1.274
		Iterations	9

In this example, a strip footing is placed at 1 m depth in a sandy soil. This example is given and has been solved by others (Basheer and Najjar, 1998). This is a design problem of case 1 and a loop of operations is required to obtain the proper width. The results of Basheer and Najjar (1998) were, $B = 1.1$ m, $CF_s = 2.8$, $\mu_{q_{ult}} = 1277$ kN/m^2, hence $q_{ult} = 1128$ kN/m^2. Basheer and Najjar (1998) applied a

"Beta distribution" and that is why the results are different. It can be noticed that although the reliability values are the same, the resulted width and factors of safety are different. The normal distribution yields a flatter density function and the probability of failure corresponding to a given applied pressure will be greater than that of a Beta distribution. To make a fair comparison, the given reliability value was changed several times until a width of 1.1 m was obtained. The corresponding reliability was 0.9326, i.e., a probability of failure of 0.0674. At this stage, the factors of safety became very close.

Although Basheer and Najjar (1998), used only iterations, the time required to use their design charts might be 100 times greater.

8.2. EXAMPLE 2

Strip footing, sand, design case 2

Given data and properties		Results and required output	
D_f (m)	0.7	$CV_{q_{ult}}$	0.428
Q (kN/m)	180	R	0.949836 ~0.95
c (kN/m^2)	0	P_f	0.05
ϕ (°)	30	CF_s	3.372
γ (kN/m^3)	18	F_s	3.0
CV_c	0	$\mu_{q_{ult}}$ (kN/m^2)	523.4
CV_ϕ	0.12	q_{ult} (kN/m^2)	465.7
CV_γ	0.10	β	1.643
F_s	3.0	B (m)	1.16

Basma (1994) applied an asymptotic extreme type II maxima distribution and utilized the first-order Taylor's series expansion for the mean and the variance of q_{ult}. Using a criterion for risk reduction, higher factors of safety were proposed and the corresponding probabilities of failure were computed and presented. The procedure consists of choosing a value for the probability of failure and a risk reduction factor is then determined using a relation between the probability of failure chosen and the coefficient of variation of the ultimate bearing capacity. Against this risk reduction factor the calculated factor of safety is enlarged and the dimensions are re-calculated accordingly.

The results of the program above may not be compared to those obtained by the method of risk reduction factor as the probability of failure is an input parameter in the latter method.

To make a comparison, one of the findings in Basma's results is detected here. In the example solved by the risk reduction factor method, an enlarged factor of safety of 4.92 was obtained for a given probability of failure of 0.01 (i.e., a reliability value of 0.99). The MATHCAD program was re-executed with this safety factor and the corresponding reliability was 0.970297, β = 1.885.

Applying the normal distribution to the ultimate bearing capacity with coefficients of variation greater than 0.35 (in this example 0.428), reliability values cannot reach 0.98 (see also Table 3), hence, a value R in the range 0.95–0.977 should be accepted while corresponding values obtained using other types of distribution are usually higher.

If one tries to obtain a high value for R, say 0.99, negative values for the width or the factor of safety are obtained and the number of iterations reaches its maximum limit.

8.3. EXAMPLE 3

Strip footing, sand, analysis case 3

Given data and properties		Results and required output	
D_f (m)	0.5	$CV_{q_{ult}}$	0.587
B (m)	1.0	R	0.872
Q (kN/m)	180	P_f	0.128
c (kN/m^2)	0	CF_s	3.005
ϕ (°)	30	F_s	3.223
γ (kN/m^3)	19.61	$\mu_{q_{ult}}$ (kN/m^2)	540.825
CV_c	0	q_{ult} (kN/m^2)	400.132
CV_ϕ	0.20	β	1.136
CV_γ	0		

This example is chosen to show how the probability of failure increases due to high $CV_{q_{ult}}$. The effect of the 20%, CV_ϕ is very clear. The example was originally solved by Cherubini (1990), where an "exact" distribution function was obtained but the bearing capacity equation used was the Krizek (1965) approximate equation. The values computed for the ultimate capacity using this equation were about 25% higher than those obtained using Equation (3). This made the comparison of safety factors insignificant. Only a comparison of the trend is possible and several trends may be noticed. When the CV_ϕ value is increased to 25% the probability of failure is increased to 0.134 and when it is decreased to 15% the corresponding probability of failure becomes 0.105. This is similar to the trend in Table 3 as $CV_{q_{ult}}$ value is directly related to CV_ϕ.

Meanwhile, a decrease in the angle of friction to 20° caused a decrease in P_f to 0.078 keeping the safety factor unchanged. This is due to the effects of the derivatives of the bearing capacity factors on the value of $CV_{q_{ult}}$ which decreased to 0.422. It is preferred to acquire higher factors of safety for higher ϕ values.

8.4. EXAMPLE 4

Rectangular footing, clayey sand, analysis case 3

Given data and properties		Results and required output	
D_f (m)	0	$CV_{q_{ult}}$	0.455
B (m)	0.5	R	0.899514
L (m)	2.0	P_f	0.100486
Q (kN)	353.3	CF_s	2.391
c (kN/m^2)	6.4	F_s	2.075
ϕ (°)	38.5	$\mu_{q_{ult}}$ (kN/m^2)	844.68
γ (kN/m^3)	15.69	q_{ult} (kN/m^2)	732.932
CV_c	0.30	β	1.279
CV_ϕ	0.10		
CV_γ	0.03		

This example is originally a test cited in Bowles (1988). The failure pressure was found to be 1060 kN/m^2 but the bearing capacity equation, Equation (3), gave an ultimate bearing capacity of 732.9 kN/m^2 which is relatively low compared to the field value (similar result is cited by Bowles, 1988). The coefficients of variation for soil properties were taken as recommended in Table 2.

Again, the value of $CV_{q_{ult}}$ is relatively high yielding a value for reliability, about 90%, relatively lower than what could be obtained by other skewed type distributions. Decreasing the applied load to half the value above, 176 kN/m, will increase the safety factor to about 4, and increase the reliability to about 0.96.

Decreasing CV_c, to 0.15 increases the reliability to 0.909475. while decreasing CV_ϕ to half its value, i.e., 0.05, causes a significant increase in R which becomes 0.966298. Reliability is more sensitive to the value of ϕ and its coefficient of variation rather than other properties.

8.5. EXAMPLE 5

Square footing, c–ϕ soil, analysis case 3

Given data and properties		Results and required output	
D_f (m)	0.5	$CV_{q_{ult}}$	0.29
B (m)	0.71	R	0.970327
L (m)	0.71	P_f	0.03
Q (kN)	132.3	CF_s	2.208
c (kN/m^2)	14.7	F_s	2.147
ϕ (°)	25	$\mu_{q_{ult}}$ (kN/m^2)	579.584
γ (kN/m^3)	14.7	q_{ult} (kN/m^2)	563.448
CV_c	0.30	β	1.886
CV_ϕ	0.10		
CV_γ	0.03		

This example is also a test cited in Bowles (1988). The bearing capacity equation adopted gave a very good estimation close to the experimental value, 540 kN/m^2. The applied load is taken at half of this value and Q is given as 132.3 kN. This should yield a factor of safety close to 2. Here, it can be noticed how the reliability is relatively high although the factor of safety is not too large. The reliability increases to 0.992 if the safety factor becomes 3.

9. Conclusions

Design and analysis of shallow foundations with respect to bearing capacity may be more reliable when the probability of failure or the reliability of safety factor are considered, as the designer may have a better assessment of the involved risk.

The use of Taylor's series method for the estimation of the mean and variance of the ultimate bearing capacity becomes easier when differentiation is made by Mathcad. The accuracy can also be increased taking more terms from Taylor's series without much paperwork.

The use of Mathcad may substitute the need for design charts or deep knowledge of statistics. Simple applicability of Mathcad programs may expand the use of the reliability based design methods.

It is found that the reliability of the footing system is more sensitive to the value of the angle of friction and its coefficient of variation rather than other soil properties such as the unit weight or the cohesion. Reliability increases as any of the coefficients of variation of soil strength properties decreases. Adopting higher safety factors for cases with higher values of the angle of internal friction is recommended as small variations in the angle of friction may cause higher variation in the ultimate bearing capacity and the risk increases.

The use of normal distribution was found to be safer than other distributions, that is, lower reliability values are obtained when applying normal distribution at the same factors of safety. Also, the use of the reliability index to determine the factor of safety may be considered an effective guide to assess the uncertainty of the ultimate bearing capacity through the calculated coefficient of variation as shown in Table 3.

Appendix

$D := 1$	notes
$Q := 500$	this program solves for case 1, design of
$c := 0$	strip footings
$\phi := 35$	Q = actual design load, kN/m
$\gamma := 18.9$	D = depth of footing

$CV_c := 0$
$CV_\phi := 0.1$
$CV_\gamma := 0.05$
$M := 100$
$\text{tol} := 0.001$
$R := 0.95$

$$\phi := (\phi + 0.001) \cdot \frac{\pi}{180}$$

$$N_{\text{q}}(\phi) := \left(\tan\left(\frac{\phi}{2} + \frac{\pi}{4}\right)\right)^2 \cdot \exp(\pi \cdot \tan(\phi))$$

$$N_\gamma(\phi) := 2 \cdot (N_q(\phi) + 1) \cdot \tan(\phi)$$

$$N_c(\phi) := (N_q(\phi) - 1) \cdot \cot(\phi)$$

$$S_c := 1$$

$$S_q := 1$$

$$S_\gamma := 1$$

$$V_c := (CV_{c \cdot c})^2$$

$$V_\phi := (CV_{\phi \cdot \phi})^2$$

$$V_\gamma := (CV_{\gamma \cdot \gamma})^2$$

$$q_{\text{ult}}(B, D, c, \phi, \gamma) := c \cdot N_c(\phi) \cdot S_c + \gamma \cdot N_q(\phi) \cdot S_q + 0.5B \cdot \gamma \cdot N_\gamma(\phi) \cdot S_\gamma$$

$$\sigma_{q_{\text{ult}}}(B, D, c, \phi, \gamma) := \left[V_c \cdot \left(\frac{\text{d}}{\text{d}c} q_{\text{ult}}(B, D, c, \phi, \gamma)\right)^2 + V_\phi \cdot \left(\frac{\text{d}}{\text{d}\phi} q_{\text{ult}}(B, D, c, \phi, \gamma)\right)^2 + V_\gamma \cdot \left(\frac{\text{d}}{\text{d}\gamma} q_{\text{ult}}(B, D, c, \phi, \gamma)\right)^2\right]^{0.5}$$

$$\mu_{q_{\text{ult}}}(B, D, c, \phi, \gamma) := q_{\text{ult}}(B, D, c, \phi, \gamma) + 0.5\left(V_c \frac{\text{d}^2}{\text{dc}^2} q_{\text{ult}}(B, D, c, \phi, \gamma) + V_\phi \cdot \frac{\text{d}^2}{\text{d}\phi^2} q_{\text{ult}}(B, D, c, \phi, \gamma) + V_\gamma \cdot \frac{\text{d}^2}{\text{d}\gamma^2} q_{\text{ult}}(B, D, c, \phi, \gamma)\right)$$

$$CV_{q_{ult}}(B,D,c,\phi,\gamma) := \frac{\sigma_{q_{ult}}(B,D,c,\phi,\gamma)}{\mu_{q_{ult}}(B,D,c,\phi,\gamma)}$$

$B1 := 0 \qquad B2 := 0.3$

$$\begin{aligned}
BB(B1,B2) := \; & \mathrm{I} \leftarrow 0 \\
& \text{while } [(|B1 - B2| \geq \text{tol}) \vee (B2 < 0)] \\
& \quad B1 \leftarrow B2 \\
& \quad I \leftarrow I + 1 \\
& \quad \text{break if } I = M \\
& \quad q_{\text{app}} \leftarrow q_{\text{norm}}(1 - \mathrm{R}, \mu_{q_{ult}}(B1,D,c,\phi,\gamma), \sigma_{q_{ult}}(B1,D,c,\phi,\gamma)) \\
& \quad B2 \leftarrow \frac{Q}{q_{\text{app}}} \\
& \begin{pmatrix} I \\ B2 \end{pmatrix}
\end{aligned}$$

$k := BB(B1,B2) \quad I := k_0 \quad B := k_1$

$$q_{\text{app}} := \frac{Q}{B}$$

$$CF_{\text{s}} := \frac{\mu_{q_{ult}}(B,D,c,\phi,\gamma)}{q_{\text{app}}}$$

$$F_s := \frac{q_{\text{ult}}(B,D,c,\phi,\gamma)}{q_{\text{app}}}$$

$$\beta := \frac{(CFs - 1)}{CVq_{\text{ult}}(B,D,c,\phi,\gamma) \cdot CFs}$$

$$R := 1 - p_{\text{norm}}(-\beta, 0, 1)$$

Results

$CV_{q_{ult}}(B,D,c,\phi,\gamma) = 0.434$
$R = 0.950013$
$1 - R = 0.049987$
$CF_{\text{s}} = 3.499$
$\mu_{q_{ult}}(B,D,c,\phi,\gamma) = 1.374 \times 10^3$
$q_{\text{ult}}(B,D,c,\phi,\gamma) = 1.208 \times 10^3$

$\beta = 1.645$
$B = 1.274$
$I = 9$
$D := 0.7$ notes
$Q := 180$ this program solves for case 2, design of
$c := 0$ **stripfootings**
$\phi := 30$ Q = actual design load, kN/m
$\gamma := 18$ D = depth of footing
$CV_c := 0$
$CV_\phi := 0.12$
$CV_\gamma := 0.1$
$F_s := 3$

$$\phi := (\phi + 0.001) \cdot \frac{\pi}{180}$$

$$N_q(\phi) := \left(\tan\left(\frac{\phi}{2} + \frac{\pi}{4}\right)\right)^2 \cdot \exp(\pi \cdot \tan(\phi))$$

$$N_\gamma(\phi) := 2 \cdot (N_q(\phi) + 1) \cdot \tan(\phi)$$

$$N_c(\phi) := (N_q(\phi) - 1) \cdot \cot(\phi)$$

$S_c := 1$
$S_q := 1$
$S_\gamma := 1$
$V_c := (CV_{oc})^2$
$V_\phi := (CV_{\phi\cdot\phi})^2$
$V_\gamma := (CV_{\gamma\cdot\gamma})^2$

$$q_{ult}(B, D, c, \phi, \gamma) := c \cdot N_c(\phi) \cdot S_c + \gamma \cdot D \cdot N_q(\phi) \cdot S_q + 0.5 \cdot B \cdot \gamma \cdot N_\gamma(\phi) \cdot S_\gamma$$

$$\sigma_{q_{ult}}(B, D, c, \phi, \gamma) := \left[V_c \cdot \left(\frac{d}{dc} q_{ult}(B, D, c, \phi, \gamma)\right)^2 + V_\phi \left(\frac{d}{d\phi} q_{ult}(B, D, c, \phi, \gamma)\right)^2 + V_\gamma \cdot \left(\frac{d}{d\gamma} q_{ult}(B, D, c, \phi, \gamma)\right)^2\right]^{0.5}$$

$$\mu_{q_{ult}}(B, D, c, \phi, \gamma) := q_{ult}(B, D, c, \phi, \gamma) + 0.5 \cdot \left(V_c \cdot \frac{d^2}{dc^2} q_{ult}(B, D, c, \phi, \gamma) + V_\phi \frac{d^2}{d\phi^2} q_{ult}(B, D, c, \phi, \gamma) + V_\gamma \cdot \frac{d^2}{d\gamma^2} q_{ult}(B, D, c, \phi, \gamma)\right)$$

$$CV_{q_{\text{ult}}}(B, D, c, \phi, \gamma) := \frac{\sigma_{q_{\text{ult}}}(B, D, c, \phi, \gamma)}{\mu_{q_{\text{ult}}}(B, D, c, \phi, \gamma)}$$

$$B := \text{root}\left(q_{\text{ult}}(B, D, c, \phi, \gamma) \cdot \frac{B}{Fs} - Q, B, 0.3, 100\right)$$

$$q_{\text{app}} := \frac{Q}{B}$$

$$P_{\text{f}} := p_{\text{norm}}(q_{\text{app}}, \mu_{q_{\text{ult}}}(B, D, c, \phi, \gamma), \sigma_{q_{\text{ult}}}(B, D, c, \phi, \gamma))$$

$$R = 1 - P_{\text{f}}$$

$$\beta := q_{\text{norm}}(\text{R}, 0, 1)$$

$$CF_{\text{s}} := \frac{\mu_{q_{\text{ult}}}(B, D, c, \phi, \gamma)}{q_{\text{app}}}$$

$$F_{\text{s}} := \frac{q_{\text{ult}}(B, D, c, \phi, \gamma)}{q_{\text{app}}}$$

Results

$CV_{q_{\text{ult}}}(B, D, c, \phi, \gamma) = 0.428$
$R = 0.949836$
$1 - R = 0.050164$
$CF_{\text{s}} = 3.372$
$F_{\text{s}} = 3$
$\mu_{q_{\text{ult}}}(B, D, c, \phi, \gamma) = 523.416$

$q_{\text{ult}}(B, D, c, \phi, \gamma) = 465.703$
$\beta = 1.643$
$B = 1.16$

$B := 1$	notes
$D := 0.5$	this program solves for case 3, analysis of
$Q := 180$	**stripfootings**
$c := 0$	Q =actual design load, kN/m
$\phi := 30$	D = depth of footing
$\gamma := 19.61$	
$CV_{\text{c}} := 0$	
$CV_{\phi} := 0.2$	
$CV_{\gamma} := 0$	

$$\phi := (\phi + 0.001) \cdot \frac{\pi}{180}$$

$$N_{\mathrm{q}}(\phi) := \left(\tan\left(\frac{\phi}{2} + \frac{\pi}{4}\right)\right)^2 \cdot \exp(\pi \cdot \tan(\phi))$$

$$N_{\gamma}(\phi) := 2 \cdot (\mathrm{Nq}(\phi) + 1) \cdot \tan(\phi)$$

$$N_{\mathrm{c}}(\phi) := (N_{\mathrm{q}}(\phi) - 1) \cdot \cot(\phi)$$

$$S_{\mathrm{c}\ :=1} \quad S_{\mathrm{q}\ :=1} \quad S_{\gamma} := 1$$

$$V_c := (CV_{c \cdot c})^2$$

$$V_{\phi} := (CV_{\phi \cdot \phi})^2$$

$$V_{\gamma} := (CV_{\gamma \cdot \gamma})^2$$

$$q_{\mathrm{app}} := \frac{Q}{B}$$

$$q_{\mathrm{ult}}(B, D, c, \phi, \gamma) := \mathrm{c} \cdot N_{\mathrm{c}}(\phi) \cdot S_{\mathrm{c}} + \gamma \cdot \mathrm{D} \cdot N_{\mathrm{q}}(\phi) \cdot S_{\mathrm{q}} + 0.5 \cdot B \cdot \gamma \cdot N\gamma(\phi) \cdot S\gamma$$

$$\mathrm{V}_{q_{\mathrm{ult}}} := V_c \cdot \left(\frac{\mathrm{d}}{\mathrm{d}c} q_{\mathrm{ult}}(B, D, c, \phi, \gamma)\right)^2 + V_{\phi}\left(\frac{\mathrm{d}}{\mathrm{d}\phi} q_{\mathrm{ult}}(B, D, c, \phi, \gamma)\right)^2 + V_{\gamma} \cdot \left(\frac{\mathrm{d}}{\mathrm{d}\gamma} q_{\mathrm{ult}}(B, D, c, \phi, \gamma)\right)^2$$

$$\sigma_{q_{\mathrm{ult}}} := V_{q_{\mathrm{ult}}}^{0.5}$$

$$\mu_{q_{\mathrm{ult}}} := q_{\mathrm{ult}}(B, D, c, \phi, \gamma) + 0.5 \cdot \left(V_{\mathrm{c}} \cdot \frac{\mathrm{d}^2}{\mathrm{dc}^2} q_{\mathrm{ult}}(B, D, c, \phi, \gamma) + V_{\phi} \cdot \frac{\mathrm{d}^2}{\mathrm{d}\phi^2} q_{\mathrm{ult}}(B, D, c, \phi, \gamma) + V_{\gamma} \cdot \frac{\mathrm{d}^2}{\mathrm{d}\gamma^2} q_{\mathrm{ult}}(B, D, c, \phi, \gamma)\right)$$

$$CV_{q_{\mathrm{ult}}} := \frac{\sigma_{q_{\mathrm{ult}}}}{\mu_{q_{\mathrm{ult}}}}$$

$$CF_{\mathrm{s}} := \frac{\mu_{q_{\mathrm{ult}}}}{q_{\mathrm{app}}}$$

$$F_s := \frac{q_{ult}(B, D, c, \phi, \gamma)}{q_{app}}$$

$$\beta := \frac{CF_s - 1}{CV_{q_{ult}} \cdot CF_s}$$

$$R := 1 - p_{norm(-\beta,\ 0,\ 1)}$$

Results

$$CV_{q_{ult}} = 0.587$$

$$R = 0.872$$

$$1 - R = 0.128$$

$$CF_s = 3.005$$

$$F_s = 2.223$$

$$\mu_{q_{ult}} = 540.825$$

$$q_{ult}(B, D, c, \phi, \gamma) = 400.132$$

$$\beta = 1.136$$

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