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ON THE PLASTIC ZONE SIZE AND THE CRACK TIP OPENING DISPLACEMENT OF AN INTERFACE CRACK BETWEEN TWO DISSIMILAR MATERIALS

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Abstract. In the paper, the elastic-plastic fracture behavior of an interface crack between two dissimilar materials is investigated. The mixed-mode Dugdale model is applied to examine the plastic zone size and the crack tip opening displacement. In numerical examples, the plastic zone size and the crack tip opening displacement of an interface crack under uniform loads are studied in detail. Two formulae are proposed to calculate the plastic zone size and the crack tip opening displacement of an interface crack under small scale yielding conditions.

Keywords: Interface crack, singularity integral equation, dislocation, crack tip opening displacement (CTOD), plastic zone size, elastic and plastic fracture mechanics.

1. Introduction. The elastic problems of cracks were studied first by some pioneer researchers (Williams, 1959; Erdogan, 1963; England, 1965; Rice and Sih, 1965.) in the 1950s and the 1960s. Rice and Sih (1965) and Erdogan and Gupta (1971) defined the complex stress intensity factor. For a uniform remote loading, the complex stress intensity factor of an interface crack between two dissimilar semi-infinite planes (shown in Fig.1) can be written as

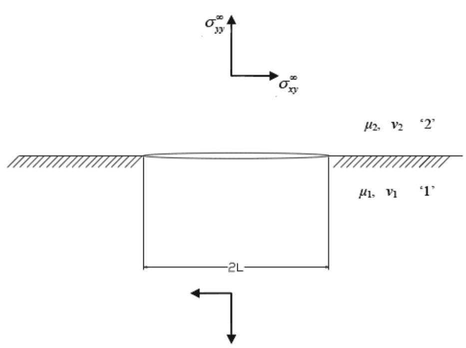

Fig 1. Remote tensile and shear loading of an interface crack.

$$K = K_1 + iK_2 = \left(\sigma_{yy}^{\infty} + i\,\sigma_{xy}^{\infty}\right)\sqrt{\pi L}\,\left(1-2i\,\varepsilon\right)\left(2L\right)^{i\varepsilon}. \quad (1)$$

Here, L is the half crack length, $\sigma_{yy}^{\infty} + i\sigma_{xy}^{\infty}$ is the uniform remote stress loads, and

$$\varepsilon = \frac{1}{2\pi}\log\left(\frac{1+\beta}{1-\beta}\right) , \quad (2)$$

$$\beta = \frac{\mu_2(\kappa_1 - 1) - \mu_1(\kappa_2 - 1)}{\mu_2(\kappa_1 + 1) + \mu_1(\kappa_2 + 1)}, \tag{3}$$

where β is Dundurs' parameters, $\kappa_i = 3 - 4\nu_i$ in plane strain and $\kappa_i = (3 - \nu_i)/(1 + \nu_i)$ in plane stress, μ_i is the shear modulus of material i and ν_i is the Poisson's ratio of material i.

The complex stress intensity factor is often applied to judging the initial of crack advance in the interface. Ikeda et al (1998) suggested an ellipse law as the following equation

$$\left(\frac{K_1^*}{K_{1C}}\right)^2 + \left(\frac{K_2^*}{K_{2C}}\right)^2 = 1. \tag{4}$$

Here, $K_1^* + iK_2^* = Kl^{-i\varepsilon}$, l is a reference length, K_{1C} and K_{2C} are the critical values in mode I and mode II, separately. The difficulty in the criterion is that the reference length l is arbitrary, and the complex stress intensity factors, K_1^* and K_2^*, depend on its value.

In our current work, we focus on evaluating the plastic zone size and the crack tip opening displacement of an interface crack between two dissimilar semi-finite plates under a normal load on the crack surfaces.

2. Formulation and model.

2.1. Stress intensity factors. When continuously distributed edge dislocations are used to model an interface crack between two semi-infinite bi-material plates under loads, $\tilde{\sigma}_{yy}(x)$ and $\tilde{\sigma}_{xy}(x)$, on the crack faces (shown in Fig. 2),

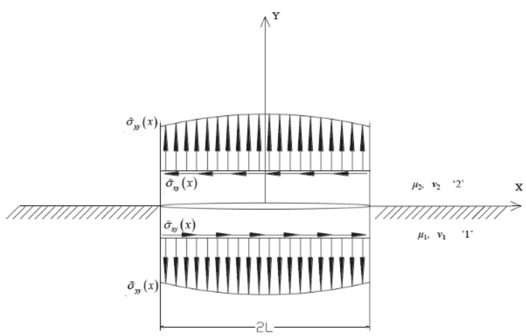

Fig 2. An interface crack between two semi-infinite bi-material plates under loads, $\tilde{\sigma}_{yy}(x)$ and $\tilde{\sigma}_{xy}(x)$, on the crackfaces.

the governing integral equation can be written as (Hills et al. 1996)

$$-\beta B(x) - \frac{i}{\pi}\int_{-L}^{+L}\frac{B(\xi)}{x-\xi}d\xi = F(x) \qquad |x| < L. \tag{5}$$

Here, β is Dundurs' parameter, $B(\xi)$ is the dislocation density and can be expressed as

$$B(\xi) = B_x(\xi) + iB_y(\xi). \tag{6}$$

$$F(x)=\frac{1}{C}\left(\tilde{\sigma}_{yy}(x)-i\,\tilde{\sigma}_{xy}(x)\right), \tag{7}$$

where

$$C=\frac{2\mu_1(1+\alpha)}{(\kappa_1+1)\left(1-\beta^2\right)}=\frac{2\mu_2(1-\alpha)}{(\kappa_2+1)\left(1-\beta^2\right)} . \tag{8}$$

Here, α is another Dundurs' parameter and can expressed as

$$\alpha=\frac{\mu_2(\kappa_1+1)-\mu_1(\kappa_2+1)}{\mu_1(\kappa_2+1)+\mu_2(\kappa_1+1)} . \tag{9}$$

Let $\xi=Ls$ and $x=Lt$, equation (5) can be re-written as

$$-\beta B(t)-\frac{i}{\pi}\int_{-1}^{+1}\frac{B(s)}{t-s}ds=F(t) \qquad |t|<1 . \tag{10}$$

The dislocation density, $B(s)$, can be written as

$$B(s)=\omega(s)\Phi(s) . \tag{11}$$

Here,

$$\omega(s)=(1-s)^a(1+s)^b , \tag{12}$$

and

$$a=-\frac{1}{2}+i\,\varepsilon, b=-\frac{1}{2}-i\,\varepsilon . \tag{13}$$

The boundary function $\Phi(s)$ can be expressed by an infinite series of Jacobi Polynomials, $P_n^{(a,b)}$, as the following equation

$$\Phi(s)=\sum_{n=0}^{\infty}c_n P_n^{(a,b)}(s), \tag{14}$$

and

$$\begin{aligned}P_n^{(a,b)}(s)&=\frac{1}{2^n}\sum_{m=0}^{n}\binom{n+a}{m}\binom{n+b}{n-m}(s-1)^{n-m}(s+1)^m\\&=\frac{1}{2^n}\sum_{m=0}^{n}\frac{\Gamma(n+a+1)\Gamma(n+b+1)}{\Gamma(m+1)\Gamma(n+a-m+1)\Gamma(b+m+1)\Gamma(n-m+1)}(s-1)^{n-m}(s+1)^m\end{aligned} . \tag{15}$$

Substituting equation (14) to equation (11), then, substituting equation (11) to equation (10), and using the relation (Krenk 1975)

$$-\beta\omega(t)P_n^{(a,b)}(t)-\frac{i}{\pi}\int_{-1}^{+1}\frac{\omega(s)P_n^{(a,b)}(s)}{t-s}ds=\frac{i}{2}\sqrt{1-\beta^2}\,P_{n-1}^{(-a,-b)}(t), \tag{16}$$

equation (10) can be re-expressed as

$$\frac{i}{2}\sqrt{1-\beta^2}\sum_{n=0}^{\infty}c_n P_{n-1}^{(-a,-b)}(t)=F(t) \qquad |t|<1 . \tag{17}$$

Multiplying both sides of equation (17) by $\omega^{-1}(t)P_k^{(-a,-b)}(t)$, where $\omega^{-1}(t)=(1-t)^{-a}(1+t)^{-b}$, then integrating equation (17) over [-1 1], one can get

$$\frac{i}{2}\sqrt{1-\beta^2}\sum_{n=0}^{\infty}c_n\int_{-1}^{+1}\omega^{-1}(t)P_k^{(-a,-b)}(t)P_{n-1}^{(-a,-b)}(t)dt=\int_{-1}^{+1}F(t)\omega^{-1}(t)P_k^{(-a,-b)}(t)dt\,. \tag{18}$$

Using the orthogonality relation

$$\int_{-1}^{+1}\omega^{-1}(t)P_k^{(-a,-b)}(t)P_{n-1}^{(-a,-b)}(t)dt=\begin{cases}0, & n-1\neq k\\ \Theta_k^{(-a,-b)}, & n-1=k\end{cases},\qquad k=0,1\cdots, \tag{19}$$

equation (18) can be written as

$$c_{k+1}=\frac{\int_{-1}^{+1}F(t)\omega^{-1}(t)P_k^{(-a,-b)}(t)dt}{\frac{i}{2}\sqrt{1-\beta^2}\Theta_k^{(-a,-b)}}\qquad k=0,1\cdots. \tag{20}$$

Here,

$$\Theta_k^{(-a,-b)}=\frac{2^{-a-b+1}}{2k-a-b+1}\sqrt{\frac{\Gamma(k-a+1)\Gamma(k-b+1)}{k!\Gamma(k-a-b+1)}}\,. \tag{21}$$

The no-net-dislocation condition leads

$$\int_{-1}^{+1}B(s)ds=\sum_{n=0}^{\infty}c_n\int_{-1}^{+1}\omega(s)P_n^{(a,b)}(s)ds=0\,. \tag{22}$$

Noticing $P_0^{(a,b)}(s)=1$ and the orthogonality relation in equation (19), equation (22) can be re-written as

$$c_0\int_{-1}^{+1}\omega(s)P_0^{(a,b)}(s)ds=c_0\Theta_0^{(a,b)}=0\,, \tag{23}$$

or

$$c_0=0\,. \tag{24}$$

The crack surface displacements near the right crack tip can be written as (Hills et al. 1996)

$$g(r)+ih(r)=K_R\sqrt{\frac{2r}{\pi}}\frac{r^{-i\varepsilon}}{C(1-2i\varepsilon)\sqrt{1-\beta^2}}\,. \tag{25}$$

Here, K_R is the stress intensity factor at the right crack tip, and $r(=L-x)$ is the distance from a point on the crack surfaces to the right crack tip. $g(r)=u_y(r,0^+)-u_y(r,0^-)$ and $h(r)=u_x(r,0^+)-u_x(r,0^-)$, where u_y and u_x are the normal and the tangential components of the crack surface displacements. The dislocation density near the right crack tip can be expressed as

$$B(r) = B_x(r) + iB_y(r) = \frac{\partial\left(h(r) + ig(r)\right)}{\partial r} = \frac{i\bar{K}_R r^{i\varepsilon}}{\sqrt{2r\pi C}\sqrt{1-\beta^2}}. \tag{26}$$

With the aid of equation (11), the complex conjugate stress intensity factor at the right crack tip can be written as

$$\bar{K}_R = \lim_{r\to 0}\left(-i\sqrt{2r\pi}\sqrt{1-\beta^2}Cr^{-i\varepsilon}\omega(r)\Phi(r)\right). \tag{27}$$

Substituting equations (12-14) to equation (27), and let $r = L(1-s)$, equation (27) can be expressed as

$$\begin{aligned}\bar{K}_R &= \lim_{r\to 0, s\to 1}\left(-i\sqrt{2r\pi}\sqrt{1-\beta^2}Cr^{-i\varepsilon}\left(\frac{r}{L}\right)^{-\frac{1}{2}+i\varepsilon}(1+s)^{-\frac{1}{2}-i\varepsilon}\sum_{n=0}^{\infty}c_n P_n^{(a,b)}(s)\right) \\ &= -i\sqrt{\pi L}\sqrt{1-\beta^2}(2L)^{-i\varepsilon}C\sum_{n=0}^{\infty}c_n P_n^{(a,b)}(1)\end{aligned}. \tag{28}$$

Similarly, the complex conjugate stress intensity factor at the left crack tip can be obtained according to equations in this section by exchanging material 1 and material 2.

2.2. The current model with plastic zone correction. The current physical problem (an interface crack) is shown in Fig. 3.

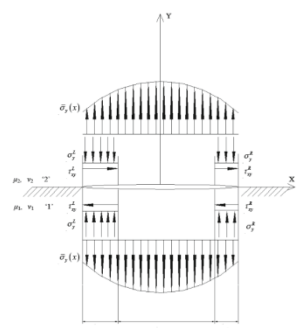

Fig 3. The current model with plastic zone correction.

Two long, slim plastic zones are assumed at both crack tips. The crack length is taken as $2L + \rho_L + \rho_R$, where ρ_L and ρ_R are the plastic zone lengths at the left and right crack tips, respectively. The plane stress condition is considered. The stresses applied in the plastic zones include the normal stress, σ_y, and the shear stress, τ_{xy}, and they satisfy the Von Mises yield criterion having the form as

$$\sqrt{\sigma_y^2 + 3\tau_{xy}^2} = \sigma_{ys}. \tag{29}$$

where σ_{ys} is the lower yielding stress of the two solids.

The dislocation express for the model (shown in Fig. 3) can be written as

$$\begin{aligned}
&-\beta B(x)-\frac{i}{\pi}\int_{-L-\rho_L}^{L+\rho_R}\frac{B(\xi)}{x-\xi}d\xi=\frac{1}{C}\left(\tilde{\sigma}_{yy}(x)-i\tilde{\sigma}_{xy}\right) && -L-\rho_L<x<L+\rho_R\\
&\tilde{\sigma}_{yy}(x)=-\bar{\sigma}_y(x) && |x|<L\\
&\tilde{\sigma}_{yy}(x)=-\bar{\sigma}_y(x)+\sigma_y^L,\tilde{\sigma}_{xy}(x)=-\tau_{xy}^L && -L-\rho_L<x<-L\\
&\tilde{\sigma}_{yy}(x)=-\bar{\sigma}_y(x)+\sigma_y^R,\tilde{\sigma}_{xy}(x)=-\tau_{xy}^R && L<x<L+\rho_R
\end{aligned} \quad .(30)$$

The lengths of plastic zones (ρ_L and ρ_R), the normal stresses (σ_y^L and σ_y^R) and the shear stresses (τ_{xy}^L and τ_{xy}^R) in the plastic zones can be determined when the stress singularity vanishes:

$$K_L+K_\rho^L=0,\ K_R+K_\rho^R=0\,. \tag{31}$$

Here, K_L and K_R are the stress intensity factors caused by the applied load, $\bar{\sigma}_y$, at the left and right crack tips, respectively. K_ρ^L and K_ρ^R are the stress intensity factors caused by the stresses in the plastic zones, at the left and right crack tips, respectively. Specially, in the case of homogenous materials and uniform loads, the shear stresses in the plastic zones are zero and the current model reduces to the Dugdale model.

The crack tip opening displacements, δ_L , and, δ_R , at the left and right crack tips can be obtained by

$$\delta_L=\left|\int_{-L-\rho_L}^{-L}B_y(\xi)d\xi\right|,\ \delta_R=\left|\int_{L}^{L+\rho_R}B_y(\xi)d\xi\right|\,. \tag{32}$$

3. Numerical examples and discussion. In this section, the plastic zone size and the crack tip opening displacement of an interface crack between two dissimilar semi-infinite planes under uniform tensile loads on the crack surfaces are discussed in detail. The plane stress case is considered. The length of the crack is taken as 2*L*. The effect of the normalized uniform loading, σ_0/σ_{ys} , and the elastic modulus ratio, E_2/E_1, on the normalized plastic zone size, ρ/ρ_{ref} , and the normalized crack tip opening displacement, δ/δ_{ref} , is investigated. The Poisson's ratios of the two materials are chosen as 0.3. Here, σ_0 is the uniform applied loading on the crack surfaces, σ_{ys} is the lower yielding stress of the two solids, E_1 and E_2 are the elastic modulus of material 1and material 2, respectively, ρ is the plastic zone size, δ is the crack tip opening displacement, ρ_{ref} and δ_{ref} are the corresponding values of the plastic zone size and the crack tip opening displacement of the same size crack in pure homogeneous material '1', and can be expressed as

$$\rho_{ref}=\frac{\pi^2 L}{8}\left(\frac{\sigma_0}{\sigma_{ys}}\right)^2-\frac{L}{24}\left(\frac{\pi\sigma_0}{2\sigma_{ys}}\right)^4, \tag{33}$$

and

$$\delta_{ref} = \frac{\pi\sigma_0^2 L}{\sigma_{ys} E_1}\left(1+\frac{1}{6}\left(\frac{\pi\sigma_0}{2\sigma_{ys}}\right)^2\right). \tag{34}$$

The numerical results are shown in Fig. 4 and Fig. 5.

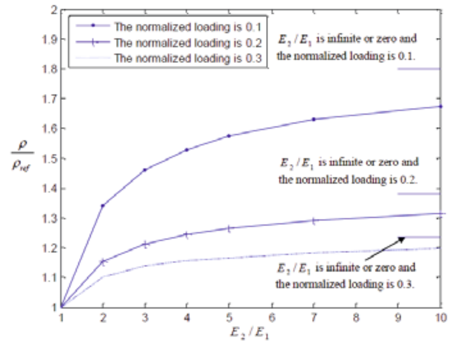

Fig 4. The normalized plastic zones size, ρ/ρ_{ref} Vs the elastic modulus ratio, E_2/E_1.

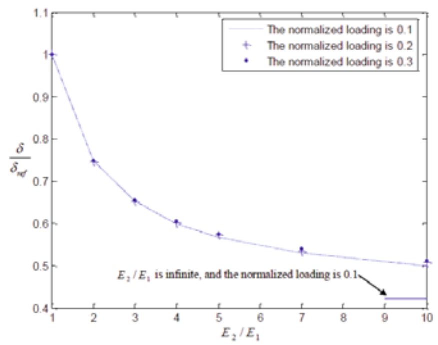

Fig 5. The normalized crack tip opening displacement, δ/δ_{ref} Vs the elastic modulus ratio, E_2/E_1.

From Fig. 4, one can observe that the plastic zone size reaches the minimum value when the crack is embedded in the homogenous materials. When the elastic modulus ratio, E_2/E_1, is taken as, for example, n and $1/n$, the corresponding normalized plastic zone sizes have the same value. This can be explained from the physical viewpoint. When the two materials are exchanged, the value of elastic modulus ratio, E_2/E_1, varies from n to $1/n$, but the plastic zone size should not have any change. For small plastic deformations (the normalized uniform loading is 0.1), the relationship between the normalized plastic zone size and the elastic modulus ratio can be expressed as the following fitting formula approximately

$$\frac{\rho}{\rho_{ref}} = 1.8 - 0.8\left(\frac{E_2}{E_1}\right)^{-0.79} \qquad \frac{E_2}{E_1} \geq 1. \tag{35}$$

From Fig. 4, one also observes that the values of the normalized plastic zone size decrease with increasing the normalized loading from 0.1 to 0.3.

From Fig. 5, it is observed that the value of the normalized crack tip opening displacement decreases with increasing the value of the elastic modulus ratio, E_2/E_1. For small plastic deformations (the normalized loading is 0.1), the relationship between the normalized crack tip opening displacement and the elastic modulus ratio can be expressed as the following fitting formula approximately

$$\frac{\delta}{\delta_{ref}} = 0.58\left(\frac{E_2}{E_1}\right)^{-0.86} + 0.42 \qquad \frac{E_2}{E_1} \geq 1 \ . \tag{36}$$

From Fig. 5, one also finds that the values of the normalized crack tip opening displacement are almost invariable with increasing the normalized loading from 0.1 to 0.3.

4. Conclusions.

In the present paper, the mixed-mode Dugdale model is applied to investigate the plastic zone size and the crack tip opening displacement of an interface crack between two semi-infinite dissimilar plates under a normal load on the crack surfaces. In the numerical example, the effect of the normalized uniform loading and the elastic modulus ratio on the normalized plastic zone size and the normalized crack tip opening displacement is studied in detail. The numerical results show that the plastic zone size reaches the minimum value when the crack is embedded in the homogenous materials. Two empirical functions (equations (35-36)) are obtained to calculate the plastic zone size and the crack tip opening displacement for small plastic deformations.

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