

# A time-discrete model for dynamic fracture based on crack regularization

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**Abstract** We propose a discrete time model for dynamic fracture based on crack regularization. The advantages of our approach are threefold: first, our regularization of the crack set has been rigorously shown to converge to the correct sharp-interface energy Ambrosio and Tortorelli (Comm. Pure Appl. Math., 43(8): 999–1036 (1990); Boll. Un. Mat. Ital. B (7), 6(1):105–123, 1992); second, our condition for crack growth, based on Griffith’s criterion, matches that of quasi-static settings Bourdin (Interfaces Free Bound 9(3): 411–430, 2007) where Griffith originally stated his criterion; third, solutions to our model converge, as

the time-step tends to zero, to solutions of the correct continuous time model Larsen (Math Models Methods Appl Sci 20:1021–1048, 2010). Furthermore, in implementing this model, we naturally recover several features, such as the elastic wave speed as an upper bound on crack speed, and crack branching for sufficiently rapid boundary displacements. We conclude by comparing our approach to so-called “phase-field” ones. In particular, we explain why phase-field approaches are good for approximating free boundaries, but not the free discontinuity sets that model fracture.

**Keywords** Dynamic fracture · Phase field · Griffith’s criterion · Crack regularization · Variational fracture

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## 1 Introduction

Griffith’s criterion for quasi-static fracture evolution in perfectly brittle materials Griffith (1921) supposes that as a crack grows, the displacement field is instantly in a new equilibrium, with a resulting decrease in stored elastic energy that matches the surface energy of the crack increment. More precisely, at any given time, a crack propagates only if the rate of elastic energy decrease per unit surface area of increment equals the (quasi-static) *critical energy release rate*. The crack stalls if the elastic energy release rate is less than that critical rate. It is unstable if the release rate exceeds the critical rate Lawn (1993)[Sects. 1.2 and 4]. As

described in Griffith (1921)[Sect. 2], this critical rate is related to the surface energy increase due to the crack, with a proportionality constant that depends on the material point. For ideal isotropic linear materials, this critical energy release rate is linked to the *stress intensity factors* by Irwin's formula, and we refer to it as the *fracture toughness* in this context.

Making this principle mathematically precise began with Ambrosio and Braides (1995) in the static case and for gradient flows, and was extended to quasi-static evolutions by Francfort and Marigo (1998). They proposed minimizing the sum of the stored elastic energy and the surface energy of discontinuity sets, to obtain displacements that are stable in the sense of Griffith. That is, for displacements  $u \in SBV(\Omega)$ , the space of *special functions of bounded variation*, with  $\Omega \subset \mathbb{R}^N$  representing the reference configuration of a body ( $u$  taking real values, modeling antiplane displacement), they consider energy functionals of the form

$$E(u) := \frac{\mu}{2} \int_{\Omega} |\nabla u|^2 dx + G_c \mathcal{H}^{N-1}(S(u)).$$

Here  $\mu$  is the shear modulus of the considered material,  $G_c$  is its fracture toughness,  $S(u)$  denotes the discontinuity set of  $u$ ,  $\mathcal{H}^{N-1}$  is the  $N - 1$  dimensional Hausdorff measure, and the minimization is subject to a Dirichlet condition on a part  $\partial\Omega_D$  of its boundary while the remaining part  $\partial\Omega_N := \partial\Omega \setminus \partial\Omega_D$  remains traction-free. The idea is that, if  $u$  is a minimizer of  $E$  then it is stable in the sense of Griffith, as adding any increment to its crack set  $S(u)$  cannot reduce the stored elastic energy by more than the cost of the increment.

We introduce a discrete time model for brittle fracture dynamics, based on only two principles: First, the displacement  $u$  should follow elastodynamics off the crack, and second, the crack should grow based only on the stress at the crack "tip" together with the material properties at the tip, in a way that is consistent with Griffith's criterion for crack growth. Our guiding principle for crack growth is this: if the elastic field near the crack is such that the crack would grow based on Griffith's quasi-static criterion, then the crack should also grow, in the same infinitesimal way, if the stresses are due to elastodynamics.

Putting this criterion on a precise mathematical footing is considerably more difficult than the quasi-static formulation of Francfort and Marigo (1998). The reason is that Griffith's criterion in the quasi-static setting is in essence an energy comparison—for a

potential crack increment  $\Delta C$ , there is a corresponding and *instantaneous* decrease in stored elastic energy. The length, or surface energy, of the crack increment can be compared with the elastic decrease, and the increment that results in the best, and sufficiently large, elastic reduction is deemed to be the actual crack increment.

Extending this principle to dynamics is not straightforward, since crack increments do not result in instantaneous decreases in stored elastic energy (and the effect on kinetic energy is expected to be an increase). There is, however, a rigorous approximation procedure for quasi-static fracture based on a regularization of the crack Ambrosio and Tortorelli (1990, 1992), where the energy functional

$$E_\varepsilon(u, v) := \frac{\mu}{2} \int_{\Omega} (v^2 + \eta_\varepsilon) |\nabla u|^2 dx + G_c \int_{\Omega} \frac{(1-v)^2}{4\varepsilon} + \varepsilon |\nabla v|^2 dx \quad (1)$$

is shown to be an approximation of the total elastic energy plus the surface energy of a crack  $K$ :

$$E(u, K) := \frac{\mu}{2} \int_{\Omega \setminus K} |\nabla u|^2 dx + G_c \mathcal{H}^{N-1}(K).$$

Formally, when  $\varepsilon$  is "small", the second term in (1) forces  $v$  to be close to one except on a small set, an approximation of  $K$ , while the first term allows  $u$  to have large variation where  $v$  is close to zero. The parameter  $\eta_\varepsilon$  is such that  $\eta_\varepsilon \ll \varepsilon$  and can be seen as a vanishing residual stiffness of the cracks. Although it is not technically necessary from a mathematical standpoint (see Braides (1998)), it is used in the numerical implementation to prevent ill-posedness of the minimization of  $E_\varepsilon$  for a given  $\varepsilon$ .

It is tempting to interpret the regularization parameter  $\varepsilon$  as the critical length of a process zone, or  $E_\varepsilon$  as a gradient damage model; however, we stand clear from giving a mechanical interpretation to the regularized functional for a fixed  $\varepsilon > 0$ . Using the mathematical framework of  $\Gamma$ -convergence, one can construct an entire class of functionals similar to  $E_\varepsilon$ , and show that they all approximate  $E$  (see again Braides (1998)). In our view, the resemblance to a damage model is purely coincidental. As long as the  $\Gamma$ -convergence property is satisfied, the asymptotic behavior of all these functionals as  $\varepsilon \rightarrow 0$  is identical. In view of the finite elements discretization of the regularized energy, we also recall

that there is a comprehensive literature starting, with [Belletini and Coscia \(1994\)](#), proving that the convergence property is preserved by finite element discretization, provided that the mesh size  $h$  is such that  $h \ll \varepsilon$ .

In our approach, the energy corresponding to the phase-field function  $v$  is minimized at each discrete time (subject to irreversibility constraints as in [Bourdin \(2007\)](#); [Giacomini \(2005\)](#)), while the displacement  $u$  follows dynamics, affected by  $v$  only on a small set modeling the crack, where  $u$  is effectively allowed to be discontinuous. More precisely, given  $u(x, 0) = u_0(x)$ ,  $u_t(x, 0) = u_1(x)$ , we first solve for  $v(x, 0)$  by minimizing  $v \mapsto E_\varepsilon(u_0, v)$ . The dynamics are governed by

$$\rho u_{tt} - \mu \operatorname{div} \left[ \left( v^2 + \eta_\varepsilon \right) \nabla u \right] = f, \tag{2}$$

where  $f$  is the applied load. To find  $u(x, \Delta t)$ , we do one time step in (2) starting at time zero and using  $v = v(x, 0)$ . This gives  $u(x, \Delta t)$ , and  $v(x, \Delta t)$  is then found by minimizing  $v \mapsto E_\varepsilon(u(x, \Delta t), v)$ , etc. The corresponding continuous-time model is

$$\rho u_{tt} - \mu \operatorname{div} \left[ \left( v^2 + \eta_\varepsilon \right) \nabla u \right] = f,$$

together with

$$v \in \operatorname{argmin} E_\varepsilon(u, \cdot), v \text{ decreasing in time.}$$

In fact, we also get (see [Larsen et al. \(2010\)](#)) that the total energy (including the energy dissipated in forming the crack, and subtracting the work of external forces) is constant in time, and that solutions to our discrete-time model converge to solutions of this continuous-time model as the time step goes to zero. While in the quasi-static case, the  $(\Gamma)$  limit as  $\varepsilon \rightarrow 0$  of the regularized model is well known, its derivation in our model is a very complicated and largely open mathematical problem which we do not address in this study. For a discussion of corresponding sharp-interface models, see [Larsen \(2010\)](#).

The work presented here is quite theoretical by nature. The mode of operation is based on induction: we start from the rigorously derived and well studied variational approach to quasi-static fracture, and incorporate inertial effects. We conjecture that our model is a natural extension of the existing theory in the sense that it is compatible with the quasi-static one when appropriate, while addressing unphysical quasi-static evolutions arising from unstable evolutions (see the discussion in Sect. 3.2), and use numerical experiments to highlight this claim.

To be clear, we make no claims regarding the optimality of our numerical approach—our emphasis in this paper is not about implementation, but instead our goal is the introduction of a mathematically and theoretically sounder formulation of dynamic fracture based on Griffith’s criterion. In light of what we believe are very encouraging preliminary results, we will focus on a more efficient implementation in the future. Similarly, we are not concerned at this point about quantitative comparison with actual experiments. In summary, the focus of this paper is on the *verification* of the model, including comparison with quasi-static evolutions in the case of slow loading rates, rather than on the *validation* against real experiments or *comparison* with competing models, two important issues that will be the center of future works.

## 2 The discrete time model

Let  $\Omega \subset \mathbb{R}^N$  be a bounded connected domain with Lipschitz boundary. As in [Francfort and Larsen \(2003\)](#), we enforce boundary conditions by considering an extended domain  $\Omega'$ , bounded and open with Lipschitz boundary, such that  $\overline{\Omega} \subset \Omega'$ .

In this section, we will describe our algorithm for computing the evolution of cracks in the case of dynamic antiplane shear. As described in Sect. 1, the idea of this computational model is to describe the evolution of the system “away from the crack” by a PDE governing the dynamics of an elastic body, and combine this with an evolution law for the crack set based on minimizing an energy motivated by Griffith.

Specifically, given boundary conditions  $g \in H^1(\Omega')$ , a static model of antiplane shear with fracture is given by minimizing

$$E(u) := \frac{\mu}{2} \int_{\Omega} |\nabla u|^2 dx + G_c \mathcal{H}^{N-1}(S(u))$$

over  $u \in SBV(\Omega)$  with  $u = g$  in  $\Omega' \setminus \Omega$ ,  $\mu$  and  $G_c$  being the shear modulus and fracture toughness of the homogeneous, isotropic material considered. Following [Bourdin \(2007\)](#) and [Bourdin et al. \(2000\)](#), we define an Ambrosio-Tortorelli approximation of this static energy. We introduce

$$F(u, v) := \begin{cases} E(u) & \text{if } v \equiv 1, \\ +\infty & \text{otherwise.} \end{cases}$$

Then, for any  $\varepsilon > 0$  and  $\eta_\varepsilon \ll \varepsilon$ , define the functionals

$$E_\varepsilon(u, v) := \frac{\mu}{2} \int_{\Omega} (v^2 + \eta_\varepsilon) |\nabla u|^2 dx + G_c \int_{\Omega' \setminus \partial\Omega_N} \frac{(1-v)^2}{4\varepsilon} + \varepsilon |\nabla v|^2 dx$$

and

$$F_\varepsilon(u, v) := \begin{cases} E_\varepsilon(u, v) & \text{if } u \in W^{1,2}(\Omega' \setminus \partial\Omega_N; \mathbb{R}); v \in W^{1,2}(\Omega' \setminus \partial\Omega_N; \mathbb{R}), \\ +\infty & \text{otherwise.} \end{cases}$$

It is a well known result that

$$F_\varepsilon \xrightarrow{\Gamma} F \text{ as } \varepsilon \rightarrow 0,$$

which implies, using a classical argument for  $\Gamma$ -convergence Braides (2002); Dal Maso (1993), that the minimizers of  $F_\varepsilon$  converge to that of  $F$  as  $\varepsilon \rightarrow 0$ .

In the dynamic case, we have a time dependent loading given by  $g \in L^\infty([0, T]; L^\infty(\Omega')) \cap W^{1,1}([0, T]; W^{1,1}(\overline{\Omega}))$ . As in Bourdin (2007), we note that for fixed  $u$ ,  $F_\varepsilon(u, \cdot)$  is strictly convex and that the unique minimizing  $v$  can be directly computed. Thus, for any  $\varepsilon > 0$ , and given an initial condition  $u_\varepsilon(x; 0)$  such that  $u_\varepsilon(x; 0) = g(x; 0)$  in  $\Omega' \setminus \Omega$ , we find  $v_\varepsilon(x; 0)$  by

$$v_\varepsilon(x; 0) = \operatorname{argmin}_v F_\varepsilon(u_\varepsilon(x; 0), v).$$

Then, we advance the elastic dynamics by a finite difference discretization of the PDE

$$\rho(u_\varepsilon)_{tt} = \mu \operatorname{div} \left[ (v_\varepsilon^2 + \eta_\varepsilon) \nabla u_\varepsilon \right], \tag{3}$$

where  $\rho$  is the density of the material we consider.

In order to obtain a stable scheme, we use backward differences in time: we consider the time range  $(0, T)$  discretized into  $N$  intervals of length  $\delta_t = T/N$ , and set

$$(u_\varepsilon)_t(x; t) = \frac{u_\varepsilon(x; t) - u_\varepsilon(x; t - \delta_t)}{\delta_t},$$

and

$$(u_\varepsilon)_{tt}(x; t) = \frac{u_\varepsilon(x; t) - 2u_\varepsilon(x, t - \delta_t) + u_\varepsilon(x, t - 2\delta_t)}{\delta_t^2}.$$

After some simple algebra, the discrete version of (3) becomes then

$$\rho(u_\varepsilon)(x; t) - \mu \delta_t^2 \operatorname{div} \left[ (v_\varepsilon^2 + \eta_\varepsilon) \nabla u_\varepsilon(x; t) \right] = 2u_\varepsilon(x; t - \delta_t) - u_\varepsilon(x; t - 2\delta_t). \tag{4}$$

We repeat this process for each timestep. The steps of this computational model are summarized in Algorithm 1.

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**Algorithm 1** Time-discrete algorithm

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- Require:**  $u_\varepsilon(x; 0), (u_\varepsilon)_t(x; 0), \delta_t = T/N$   
 1: Set  $u_\varepsilon(x; -\delta_t) = \delta_t(u_\varepsilon(x; 0) - (u_\varepsilon)_t(x; 0))$ .  
 2: **for**  $i = 1$  to  $N$  **do**  
 3:   Set  $t = i\delta_t$ .  
 4:   Find  $u_\varepsilon$  solution of (4).  
 5:   Find  $v_\varepsilon(x; t) \in H^1(\Omega')$  minimizing  $F_\varepsilon(u_\varepsilon(x; t), v)$  under the irreversibility constraint  $v(x) = 0$  on  $\{x; v_\varepsilon(x; t - \delta_t) \leq \alpha_\varepsilon\}$   
 6: **end for**
- 

Note that at each time step, one now needs to solve two linear elliptic problems instead of only one for the  $v$ -minimization. The additional cost of such an implicit scheme is offset by an increased stability and scalability when implemented in parallel supercomputers (see for instance Keyes et al. (2006)).

A critical choice in the numerical implementation of the discrete model is the hyperbolic solver for the  $u$ -step (4). Our numerical implementation is derived from the quasi-static model, as presented in Bourdin et al. (2000); Bourdin (2007); Bourdin et al. (2008). It is based on linear finite elements, and the underlying infrastructure for the parallel implementation on supercomputers is provided by PETSc Balay et al. (2008, 2009). Linear finite elements may not be the most appropriate discretization scheme for the  $u$ -step. However, as the scope of this study is to give a quick insight on the properties of our model, we do not believe that this is a problem at this stage. Of course, a more suited numerical method (using a discontinuous Galerkin or a finite volume based discretization, for instance) will be studied in the future.

For the sake of simplicity, we restricted ourselves to the antiplane shear case. To use our model in the setting of two and three dimensional elasticity does not require major changes to the structure of the model or its implementation. This is due to the fact that the representation of the cracks is not geometry or mesh-based, as compared to approaches based on discontinuous, cohesive, or extended finite element methods. The chief issue is to prevent interpenetration of material across the crack surfaces under compressive stresses. This can be handled by suitable adjustments to the energy, as in Amor et al. (2009); Del Piero et al. (2007);

Freddi and Royer-Carfagni (2010). One can also deviate from ideally brittle linear elastic materials and account for different elastic and fracture response under traction and compression, as in Lancioni and Royer-Carfagni (2009), for instance.

### 3 Experiments

#### 3.1 Some properties of the hyperbolic solver

Before conducting dynamic fracture experiments, we seek to illustrate the adequacy of our numerical approach. To do this, we conducted a series of simple computations on an essentially unidimensional problem. Consider a rectangular domain  $\Omega = (0, 1) \times (0, .05)$ . To its left edge, we apply a time-dependent Dirichlet boundary condition  $g(y; t)$  given by

$$g(y; t) = \begin{cases} \frac{1}{2} (1 - \cos(\pi t/t_l)) & \text{if } t \leq t_l \\ g_0 & \text{if } t > t_l, \end{cases}$$

with  $g_0 = 1$  and  $t_l = .1$ . We prescribed a null displacement boundary condition on the right edge, while the upper and lower edge are left free. The material properties are  $\mu = 1$  (corresponding to an idealized homogeneous isotropic material with  $E = 2$  and  $\nu = 0$ ),  $\rho = 1$ . We set the fracture toughness  $G_c$  to 10, high enough that the domain remains purely elastic. For such a loading, the elasto-dynamic displacement consists in a single smooth transverse wave traveling at speed  $\omega = \sqrt{\mu/\rho} = 1$ , and reflecting on the left and right edges. Figure 1 represents the evolution of the bulk, surface and kinetic energies as a function of time for various values of the time step  $\delta_t$ . All computations were performed on a Delaunay-Voronoi mesh of size  $h = 2.5E-2$ , consisting of approximately 25,000 vertices and 50,000 elements. As we expected, the numerical scheme is dissipative (see how the total energy decays after the initial loading phase), and the amount of dissipation decreases with the time step. As  $\delta_t$  becomes small compared to  $h$ , the computed solution develops high frequency oscillations, leading to the small amplitude, high frequency oscillations of the individual components of the energy for  $\delta_t = 1E-4$ . This is also expected, as the smoothing effect of the second order term in (4) vanishes when  $\delta_t \rightarrow 0$ .

We then studied the dispersivity of our numerical scheme. Figure 2 compares the profile of the deformation field as the traveling wave crosses the midline

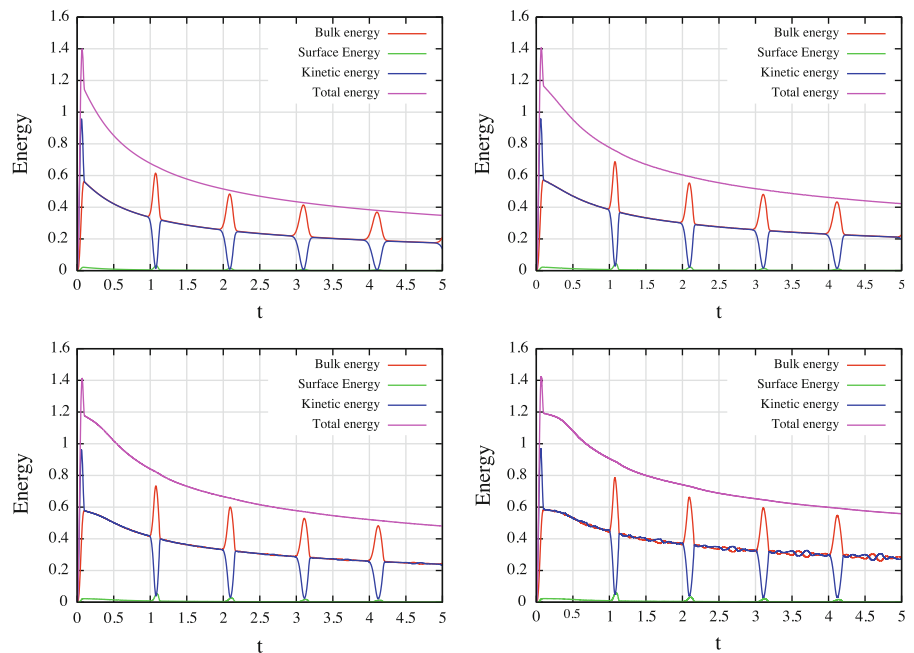
of the domain, traveling to the right, for the first, second and third time. The parameters correspond to that of Fig. 1 (bottom-left). That the profile spreads out is consistent with the presence of the smoothing term on the left hand side of (4), and the loss of bulk energy highlighted in Fig. 1 as time increases. Notice however how the peaks of the bulk and kinetic energies in Fig. 1 are 1 unit of time apart, which is consistent with a transverse wave speed of 1 and a domain length of 1.

#### 3.2 A tearing experiment

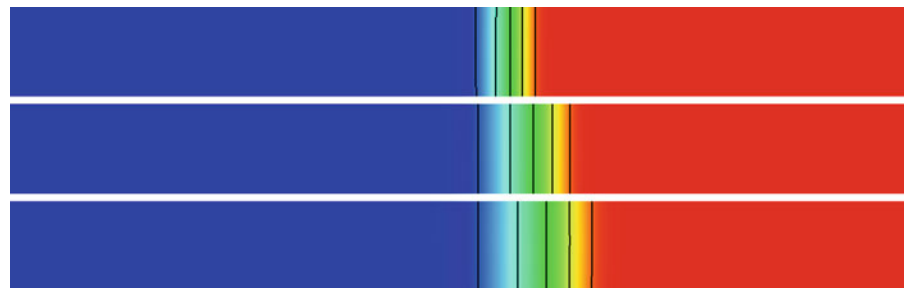
One of the strengths of the variational approach is its ability to consider *any* crack path, including kinking, branching, nucleation, etc. In some cases (typically a subset of crack evolution labelled as *unstable* in the classical Griffith-type analysis), this can lead to unphysical evolutions. An example of such an evolution is the antiplane tearing experiment presented in detail in Bourdin et al. (2008), and which we revisit here.

Consider a rectangular domain occupying the domain  $(0, L) \times (-H, H)$ , and assume  $H \ll L$ . To its left edge, one applies a monotonically increasing mode-III displacement  $g(x, y; t) = g_0(t) \operatorname{sgn}(y)$  along the  $z$ -axis,  $\operatorname{sgn}$  representing the sign function, and  $g_0(t)$  an arbitrary monotonically increasing function. Additionally, we consider only single cracks occupying the region  $(0, l(t)) \times \{0\}$  along the domain's symmetry axis. Figure 3(left), adapted from Bourdin et al. (2008), represents the evolution of the bulk, surface and total energies as a function of  $t$  for this problem. As the magnitude of the fixed displacement increases, the crack grows linearly until it reaches a critical length  $l_V \sim L/2$  then propagates *instantly* throughout the remaining half of the domain. The paradox here is not so much the instant propagation as it is the value of the threshold  $l_V$ . Indeed, the crack tip has to "know" when it has reached the mid-point. This is especially puzzling in the one dimensional limit (*i.e.* as  $H/L \rightarrow 0$ ), as the displacement field for  $0 \leq x \leq l(t)$  does not depend on  $L$  but  $l_V = L$ . Note that going back to the classical Griffith theory does not address this issue in a satisfying way. In the classical framework, the crack tip also propagates at a constant speed until it reaches a threshold  $l_G > l_V$ , corresponding to a displacement magnitude  $t_V$ . For  $t > t_V$ , Griffith's criterion *cannot be satisfied*. From the variational standpoint, the only local minimizers of  $E_\varepsilon$  for  $t > t_G$  corresponds again to a crack propagating instantly throughout the domain.

**Fig. 1** Evolution of the bulk, surface, kinetic and total energies for  $\delta_t = 1E-3$  (top left),  $\delta_t = 5E-4$  (top right),  $\delta_t = 2.5E-4$  (bottom left) and  $\delta_t = 1E-4$  (bottom right)



**Fig. 2** Profile of the deformation field at  $t =$  and  $t =$ , as the elastic wave crosses the *midline* of the domain traveling left to right. The color field corresponds to the value of  $u$ . In order to facilitate the comparison of the profile, a few level lines have been added



The paradox here is that this cannot be accomplished while preserving the total energy (see Fig. 3(right)). That the Ambrosio-Tortorelli–based approach approximates this crack evolution is also somewhat unexpected since there is no guarantee *a priori* that the local minimizers of the regularized energy converge to that of the variational one (for  $l_V \leq l \leq l_G$ ).

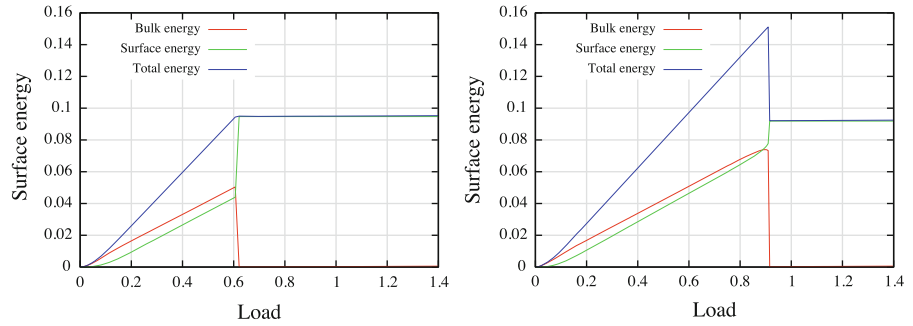
We implemented this problem using the dynamic model presented in Sect. 2. In order to avoid shocks in the loading, we consider a smooth ( $C^1$  in time) loading given by  $g(x, y; t) = g_0(t) \operatorname{sgn}(y)$  with

$$g_0(t) = \begin{cases} \frac{k}{2t_l} t^2 & \text{if } 0 \leq t \leq t_l, \\ kt - \frac{kt_l}{2} & \text{if } t_l \leq t \leq T, \end{cases} \quad (5)$$

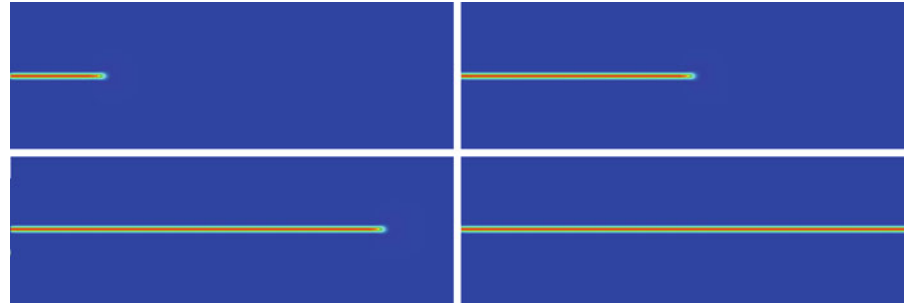
$t_l$  and  $T$  being two arbitrarily chosen thresholds with  $t_l \ll T$ .

In a first set of experiments, we prescribed the crack path along the symmetry axis of the domain by replacing the coupling term  $\operatorname{div} [(v^2 + \eta_\epsilon) \nabla u]$  in (4) with  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial y} [(v^2 + \eta_\epsilon) \frac{\partial u}{\partial y}]$ . It is easy to see that this modification of the energy allows only discontinuities of  $u$  with normal direction (0,1) (since  $\partial u / \partial x$  has to remain in  $L^2$ ). We fixed the dimensions of domain ( $L = 6$  and  $H = 1$ ), the material properties ( $E = 1$ ,  $\nu = 0$ , leading to  $\mu = .5$ ,  $G_c = 1.0E-2$ , and  $\rho = 1$ ). We generated a structured mesh by decomposing the domain into squares with size  $h = 2E-2$ , and subdividing each square into 4 similar triangular elements. The regularization parameters are  $\epsilon = 1E-2$ , and  $\eta_\epsilon = 1E-6$ . For this choice of  $h$ ,  $\epsilon$ , and  $G_c$ , the *effective toughness*, i.e. the surface energy associated with a discretized crack of length 1 is  $1.5E-2$  (see Bourdin et al. (2008) for a discussion of the effective toughness).

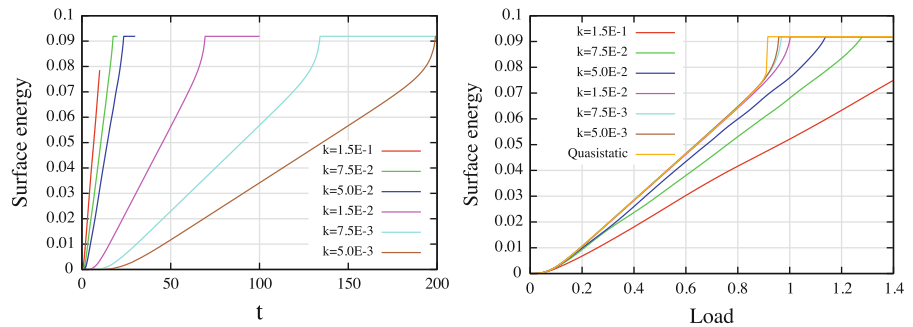
**Fig. 3** A mode-III tearing experiment: (left) global minimization leads to an unphysical evolution, (right) total energy is not conserved during “unstable” evolution



**Fig. 4** A typical crack evolution along the symmetry axis. The  $v$ -field indicates the position of the crack at  $t = 6$  ( $g_0 = .322$ ),  $t = 12$  ( $g_0 = .683$ ),  $t = 18$  ( $g_0 = 1.05$ ), and  $t = 18$  ( $g_0 = 1.4$ )



**Fig. 5** Evolution of the surface energy as a function of  $t$  (left) and of the loading  $g_0$  (right) and for various loading speeds  $k$



We varied the loading speed  $k$  and time  $T$  in such a way that  $g_0(T) = 1.5$ , and set  $t_l = T/20$ . We discretized the loading interval  $(0, T)$  in 6,000 time steps. Figure 4 represents a typical crack evolution in this setting (the specific computation corresponds to  $k = 5E-2$ ). The domain is colored using the  $v$ -field, the color blue is associated to  $v = 1$  (uncracked material) while the area where  $v = 0$  (associated to the crack location) is colored in red.

Figure 5 represents the evolution of the surface energy as a function of time (left) and the loading intensity  $g_0(t)$  (right) for various loading velocities  $k$  ranging from  $1.5E-1$  to  $5.0E-3$ . We observe that as the loading velocity decreases, the solution of the dynamic model converge to that of the quasi-static variational

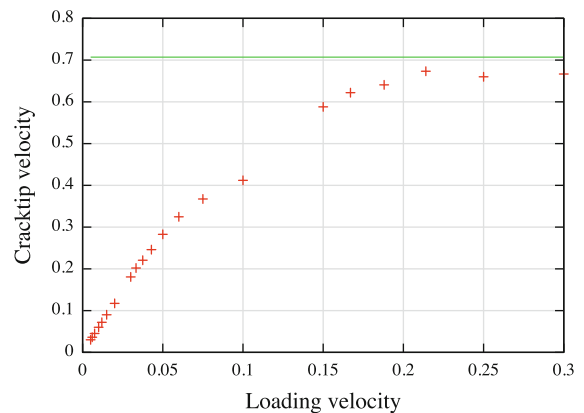
model, when the quasi-static model leads to progressive crack propagation. In other words, we recover a crack evolution similar to that of the quasi-static variational formulation, when the variational formulation satisfies Griffith’s criterion. Our formulation sets itself free of the paradoxes created by the quasi-static formulation while preserving its main strength, the ability to fully predict crack path, as we will see in the next experiments. As the loading speed  $k$  increases, a larger part of the total energy gets transferred into the kinetic energy, forcing a reduction of crack length (compare the position of the crack tips *i.e.* the magnitude of the surface energy for a given load in Fig. 5(right)).

Figure 6 represents the evolution of cracktip speed for loads in  $(.4,.8)$  as a function of the loading

velocity. It is obtained by dividing the variation of the surface energy by the effective fracture toughness. For low loading velocities, the cracktip velocity depends linearly on the loading speed. This is consistent with our previous observation that our dynamic model leads to evolutions compatible with the variational approach in the low loading velocity case. Indeed from the asymptotic analysis in Bourdin et al. (2008), we know that in the high aspect ratio, the surface energy is proportional to the opening magnitude. As the loading velocity increases, the crack velocity approaches the transverse wave speed of  $1/\sqrt{2}$  for the material we considered. Note that the existence of a maximum crack tip velocity is hinted in Fig. 5(left). As the loading speed increases, the slope of the surface energy which is proportional to the crack tip velocity seems to approach a high but finite value.

In a second set of experiments, we allow the crack to grow along arbitrary paths. Again, when the loading velocity is low enough ( $k < 3E-2$ ), a single crack propagates along the symmetry axis, and the crack position as a function of the loading intensity approaches that of the quasi-static case as  $k \rightarrow 0$ . As  $k$  increases, we observe a broader range of crack shapes. We first observe asymmetric paths where the crack starts propagating along the symmetry axis, then branches toward one of the lateral edges. Depending on the experimental conditions (including possible asymmetry of the mesh or accuracy of the linear solvers for instance), we obtain branching toward either edge. This is consistent with the fact that we do not expect the crack evolution to be unique. For faster loading velocities, we observe more complicated crack geometries involving multiple splits. Again, we stress that we did not include supplemental branching criteria to recover this expected crack behavior. As in the variational approach to quasi-static problems, our dynamic model leads to full crack path identification, without additional criteria.

Figure 7 represents the crack set at  $t = T$  for various loading speeds. The color coding is similar to Fig. 4. Here, we use a finer mesh ( $h = 1.0E-2$ ) and smaller time steps (the loading phase from  $g_0 = 0$  to  $g_0 = 1.5$  is discretized in 12,000 time steps), while the material properties remain unchanged. The regularization parameter  $\varepsilon$  is  $5.0E-3$  so that the effective toughness remains unchanged from the previous set of computations. As one would expect from a dynamic model, as the loading speed increases, one observes increasingly



**Fig. 6** Evolution of the cracktip velocity (obtained from the surface energy) as a function of the loading speed. The green line represents the transverse wave speed for the considered material

complicated crack patterns, including branching and splitting.

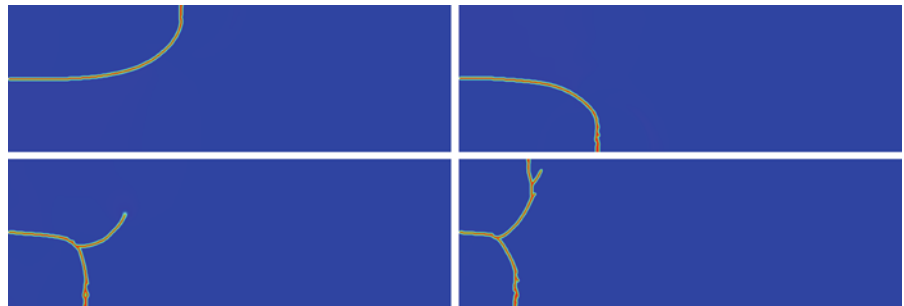
Lastly, Fig. 8, represents snapshots of a crack evolution at an even higher speed ( $k = .15$ ). Again, we observe multiple crack splitting and branching.

#### 4 Conclusions and comparison with phase-field approaches

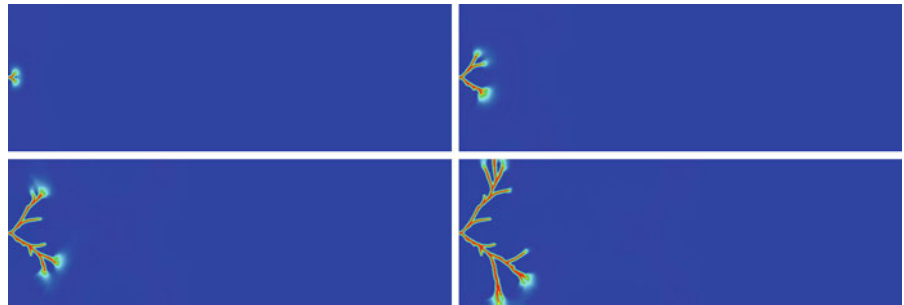
In the last few years, there have been several attempts at adapting the phase-field formalism to dynamic fracture problems. We cite in particular Aranson et al. (2000); Corson et al. (2009); Eastgate et al. (2002); Hakim and Karma (2009); Karma et al. (2001); Karma and Lobkovsky (2004); Marconi and Jagla (2005). There are two main differences between our approach and the aforementioned ones come from the form of the phase-field, and from the nature of the evolution. Our phase-field approach is derived from a rigorous approximation of free-discontinuity energies devised in Ambrosio and Tortorelli (1990, 1992), and the evolution of the crack regularization function  $v$  is given by a global minimization principle. In Aranson et al. (2000); Corson et al. (2009); Eastgate et al. (2002); Hakim and Karma (2009); Karma et al. (2001); Karma and Lobkovsky (2004), the authors introduce a phase function whose evolution is given by some variation of a gradient flow of a Ginsburg-Landau-type of free energy. This type of energy is sometimes referred to in the mathematics community as the Modica-Mortola energy Modica and Mortola (1977a,b). Its generic form is



**Fig. 7** Final crack path for various loading speeds:  $k = 3E-2$  (top left),  $k = 3.75E-2$  (top right),  $k = 4.28E-2$  (bottom left), and  $k = 5E-2$  (bottom right)



**Fig. 8** High speed loading leads to complex crack path. Snapshots of a branched crack evolution for  $k = .15$  at  $t = 1.20$  ( $g_0 = 2.16E-3$ ),  $t = 2.40$  ( $g_0 = 8.64E-3$ ),  $t = 3.60$  ( $g_0 = 1.94E-2$ ),  $t = 4.80$  ( $g_0 = 3.46E-2$ )

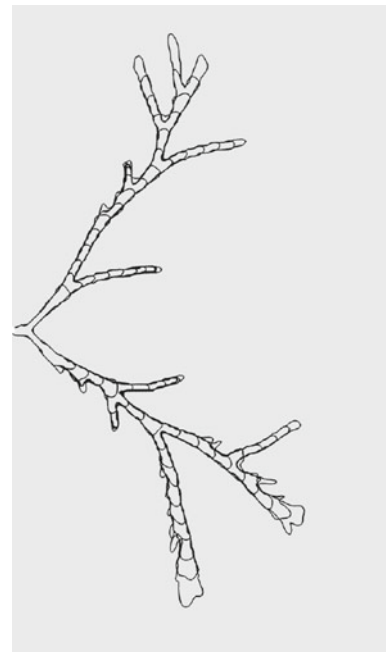


$$P_\epsilon(\phi) = \int_{\Omega} \left[ \frac{W(\phi)}{\epsilon} + \epsilon |\nabla \phi|^2 \right] dx, \tag{6}$$

where  $W$  is a double well function such that  $W(0) = W(1) = 0$  and  $W(x) > 0$  if  $x \notin \{0, 1\}$ . It is well known (see for instance Braides (1998)) that as  $\epsilon \rightarrow 0$ ,  $P_\epsilon \Gamma$ -converges to  $P$  defined by

$$P(\phi) = \begin{cases} c\mathcal{H}^{N-1}(S(\phi)) & \text{if } u \in BV(\Omega), \phi \in \{0, 1\} \text{ a.e.} \\ \infty & \text{otherwise,} \end{cases} \tag{7}$$

with  $c = \frac{1}{2} \int_0^1 \sqrt{W(t)} dt$ . That is,  $P(\phi) = c\mathcal{H}^{N-1}(\partial\{\phi = 0\})$ , or  $c$  times the perimeter of the set where  $\phi$  is zero. In the setting of an evolution problem, it is known that the gradient flow of (7) converges to the mean-curvature motion of almost all of its level lines, and one can estimate the normal velocity of the transition layers of  $\phi$  (see Bronsard and Kohn (1990)). This implies in particular that the phase field may continue to evolve away from the crack tip, even if the sample is brought back to equilibrium in its reference configuration. Roughly speaking, even if the field  $\phi$  in (6) initially represents an elongated domain (say for instance  $\phi(x) = 0$  if  $d(x, K) \leq \eta$ , where  $\eta$  is a given positive parameter and  $K$  is a given curve), given enough time,  $\phi$  will evolve into a “thick” domain instead of remaining focused along a curve. For the numerical



**Fig. 9** Evolution of the level line  $\cdot 4$  of  $v$  throughout the time evolution for the computation presented in Fig. 8. Note how the width of the area identified with the crack remains constant

implementation of dynamic fracture propagation, when crack tips propagate at a significant fraction of the wave speed, this effect is negligible. From a practical

standpoint, the actual form of the potential does not seem to have a strong impact on numerical results; however, from a mathematical standpoint, while  $P_\varepsilon$  can be used to represent the length of a discontinuity set in a *free boundary* problem, we have no hope of establishing the convergence of a regularized model based on a double-well potential to a free-discontinuity one. Moreover, in this paper, we are interested in accounting for the wide range of crack speeds, ranging from the infinitely slow quasi-static limit up to the transverse wave speed, and comparing dynamic simulations with quasistatic evolutions based on an energy balance. We need to be concerned with the long time evolution of the phase field and focus on our ability to maintain the same accuracy in our approximation of the surface energy at all times. With our approach, the phase field  $v$  does not evolve away from the crack tips, regardless of the curvature of the crack, as Fig. 9 illustrates.

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