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Material forces in finite elasto-plasticity with continuously distributed dislocations

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Abstract In this paper we propose a thermodynamically consistent model for elasto-plastic material with structural inhomogeneities such as dislocations, subjected to large deformations, in isothermal processes. The plastic measure of deformation is represented by a pair of plastic distortion, and plastic connection with non-zero torsion (in order to have the non-zero Burgers vector). The developments are focused on the balance equations (for material forces and for physical force system), derived from an appropriate principle of the virtual power formulated within the constitutive framework of finite elasto-plasticity and on constitutive restrictions imposed by the free energy imbalance. The presence of the material forces (microforce and microstress momentum) is a key point in the exposure, and viscoplastic (generally rate dependent) constitutive representation are derived.

Keywords Finite deformation · Plastic distortion · Configuration with torsion · Plastic connection · Stress momentum · Material forces · Free energy imbalance · Principle of virtual power · Dislocations

AMS Subject Classifications (2000) 74C99 · 74A20

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1 Introduction

The aim of the paper is to propose a *thermodynamically consistent model* for material with structural inhomogeneities such as dislocations, subjected to large deformations, in isothermal processes. *Continuously distributed dislocations* are modeled in Teodosiu (1970), Steinmann (2002, 1997), Cleja-Tigoiu (2002a), Gurtin (2004), by the non-zero Burgers vector, which is related to the non-zero *curl* of the plastic deformation component $\mathbf{F}^p$. In our model the plastic measure of deformation is represented by a pair $(\mathbf{F}^p, \overset{(p)}{\mathbf{\Gamma}})$. $\mathbf{F}^p$ is an invertible second order tensor, called *plastic distortion*, and $\overset{(p)}{\mathbf{\Gamma}}$ is an affine *plastic connection* with non-zero torsion, represented by a third order tensor. On the background of differential geometry concepts we introduce the *multiplicative decomposition of the deformation gradient* into elastic and plastic distortions, and the *decomposition of the motion connection* in terms of the elastic and plastic connections. Unlike **F**—the deformation gradient that is derived from the potential χ, which represents the motion function, the plastic distortion *cannot be derived from a certain potential*. The pair $(\mathbf{F}^p, \overset{(p)}{\mathbf{\Gamma}})$ defines an *anholonomic configuration* (see Acharya 2004; Bilby 1960; Schouten 1954), or a so called *configuration with torsion* $\mathcal{K}$.

In this paper a new appropriate *principle of the virtual power*, that can be applicable to the constitutive framework of finite elasto-plasticity, has been formulated. No Cosserat type kinematics is assumed in the model, and non-symmetric Cauchy stress $\mathbf{T}$ and the *stresses momentum* $\boldsymbol{\mu}$ (described by a third order tensorial field) are involved in the internal power $\mathcal{P}_{int}$. The pair $(\mathbf{T}, \boldsymbol{\mu})$ is power conjugated to the appropriate rate of the elastic deformation measure.

The principle of virtual power allows to formulate the *macroscopic balance laws* of macromomentum and angular momentum for ($\mathbf{T}$ and $\boldsymbol{\mu}$), which are similar to those derived in Fleck et al. (1994), and to derive the *micro-balance equations* (i.e. the microstructural material field equations) which are specific for the material force system $(\boldsymbol{\Upsilon}^p_{\mathcal{K}}, \boldsymbol{\mu}^p_{\mathcal{K}})$, like Gurtin (2003).

To complete the constitutive equations within the constitutive framework of finite elasto-plasticity, the *second law for isothermal processes* is formalized similar to Gurtin (2003, 2004). The second law leads to *local free energy inequality* or to *free energy imbalance* $\dot{\psi} - \mathcal{P}_{int} \leq 0$, where ψ is the free energy density. We determine the thermodynamic restrictions imposed by the requirement to have the free energy imbalance satisfied in *any virtual process*, for a given deformation state.

The *admissible virtual process* is defined based on the *kinematical relationships*, being consistent with them. The time derivatives of the appropriate distortions and connections put into evidence the presence of the velocity gradient $\mathbf{L}$ and of its gradient $\nabla_\chi \mathbf{L}$ in the deformed configuration, as well as of the rate of plastic distortion $\mathbf{L}^p$ and of its gradient $\nabla_{\mathcal{K}} \mathbf{L}^p$, written in the configuration with torsion, respectively.

The *constitutive hypotheses* concerning *microforces and micromomentum* (called the material forces) are motivated by *the dissipation inequality*. The material forces are represented by a non-dissipative part *energetic microforces*, derived from the free energy and a *dissipative part*, defined in a such way that the dissipation be positive. We emphasized the role played in the theory by the micro-stress $\boldsymbol{\Upsilon}^p_{\mathcal{K}}$, which is force conjugated to the rate of plastic distortion $\mathbf{L}^p$, and by the micro-stress momentum $\boldsymbol{\mu}^p_{\mathcal{K}}$, which generates work in conjunction with $\nabla_{\mathcal{K}} \mathbf{L}^p$.

We briefly recall different issues, involving continuously distributed dislocations results, which are closely related to the model proposed here.

A continuum theory for material with continuously distributed dislocations has been developed by Kondo and Yuki (1958), Bilby (1960), Kröner in (1963, 1992), Kröner and Lagoudas in (1992) (for elastic models), and mathematically founded by Noll (1967) and Wang (1967) (within the constitutive framework of simple materials), using the differential geometry concepts. The decomposition theorem of the connection with metric property into a Levy-Civita connection and contortion was studied in Schouten (1954), Kondo and Yuki (1958), and applied in finite elasto-plasticity by Steinmann (1994), Cleja-Tigoiu (2002a), see also Beju et al. (1983), Le and Stumpf (1996c)

A Cosserat theory for elasto-(visco)plastic single crystals, at finite deformations and based on the crystallographic slip mechanism of plastic deformation, was elaborated in Naghdi and Srinivasa (1994), Le and Stumpf (1996a), Steinmann (1994). In Forest et al. (1997) the natural Cosserat strains are considered for the development of the constitutive equations and evolution laws are proposed also for lattice torsion-curvature (second order) tensor.

Elasto-plastic model with dislocations was developed in Le and Stumpf (1996b) based on an appropriate principle of virtual work, and a thermodynamically consistent analysis of the anisotropic damage evolution was been performed by Stumpf and Hackle in (2003). Gurtin (2000) developed a gradient theory of single crystal plasticity that accounts for geometrically necessary dislocations.

Within the framework of finite elasto-plasticity the evolution equations to describing the irreversible behavior plays a fundamental role. Certain compatibility conditions concerning the evolution equation for the measure of continuously distributed dislocation could arrive, see for instance Acharia (2004), Cleja-Tigoiu (2002a), Gupta et al (2006), Cleja-Tigoiu et al. (in press). The compatibility conditions are viewed in Cleja-Tigoiu et al. (in press), say for the given plastic metric tensor, as partial differential equations for the torsion. We also mention here a second order theory, which allows for growth diffusion, developed in Epstein and Maugin (2000), especially for the evolution equation of the second order gradients that has been considered.

The boundary conditions (appropriate to plasticity that account for the dislocation) are discussed for instance by Gurtin and Needlemen (2005) and Gupta et al. (2006).

Maugin in (1999), combining the energy and momentum balance derives a so-called *pseudo-momentum* equation to derive material forces. Well-known examples of the material forces are driving forces on defects and the J-Integral in fracture mechanics.

We mention the proposed model in our paper Cleja-Ţigoiu (2002a), where the results from Cleja-Ţigoiu (1990), Cleja-Ţigoiu and Soós (1990), and Cleja-Ţigoiu (2001) have been extended to elasto-plastic materials with continuous distributed dislocations, based on the hypotheses listed below in an heuristic manner: (i) The crystalline body is not *homogeneous* and it has no relaxed (natural) global configuration. (ii) The local relaxed state is characterized by non-Euclidean and non-Riemannian metric space. (iii) Dynamical balance equations involve *non-symmetric Cauchy stress tensor* and *couple stresses*. (iv) The crystalline body behaves as an *elastic material element*, which means that the stress and the stress momentum are functions of the elastic distortion and elastic connection. (v) The *irreversible behavior* of the material is described by the evolution equations (of the rate independent type) for plastic distortion as well as for the gradient of plastic distortion.

In the present paper the developments are focused on the balance equations (micro and macro), derived from an appropriate principle of the virtual power formulated within the constitutive framework of finite elasto-plasticity and on the restriction on the constitutive equations imposed by the imbalanced free energy (i.e. *second law for isothermal processes*). The presence of the *material forces* is a key point in the exposure, and viscoplastic (generally rate dependent) constitutive representation are derived.

List of notations. Further the following notations will be used:

$\mathcal{E}$—the three dimensional Euclidean space, with the vector space of translations $\mathcal{V}$;
Lin—the set of the linear mappings from $\mathcal{V}$ to $\mathcal{V}$, i.e the set of second order tensor, $Skew \subset Lin$ the set of all skew-symmetric second order tensors;
$\mathbf{u} \times \mathbf{v}$ is the cross product, $\mathbf{u} \otimes \mathbf{v}$ and $\mathbf{u} \cdot \mathbf{v}$ denote the tensorial product and the scalar product of the vectors $\mathbf{u}, \mathbf{v} \in \mathcal{V}$.
$\mathbf{A}^s$ and $\mathbf{A}^a$ are the symmetrical and skew-symmetrical parts of the tensor $\mathbf{A}$, here $\mathbf{A}^T$ denotes the transpose of $\mathbf{A}$;
$\partial_{\mathbf{A}}\phi(x)$ denotes the partial differential of the function ϕ with respect to the field $\mathbf{A}$.
Curl of a second order tensor field $\mathbf{A}$ is defined by the second order tensor field

$$(\text{curl}\mathbf{A})(\mathbf{u} \times \mathbf{v}) := (\nabla\mathbf{A}(\mathbf{u}))\mathbf{v} - (\nabla\mathbf{A}(\mathbf{v}))\mathbf{u} \quad \forall \mathbf{u}, \mathbf{v} \in \mathcal{V}. \tag{1}$$

The component representation of the *curl* is given in a Cartesian basis by
$(\text{curl}\,\mathbf{A})_{pi} = \epsilon_{ijk}\dfrac{\partial A_{pk}}{\partial x^j}$, while the third order tensor field $\nabla\mathbf{A}$ is characterized by
$\nabla\mathbf{A} = \dfrac{\partial A_{ij}}{\partial x^k}\mathbf{i}^i \otimes \mathbf{i}^j \otimes \mathbf{i}^k$. Thus we have the formulae
$\nabla_\chi \mathbf{L} \equiv \dfrac{\partial}{\partial x^k}\left(\dfrac{\partial v_i}{\partial x^j}\right)\mathbf{i}^i \otimes \mathbf{i}^j \otimes \mathbf{i}^k$.
$Lin(\mathcal{V}, Lin) = \{\mathbf{N} : \mathcal{V} \longrightarrow Lin \quad \text{linear}\}-$ defines the third order tensors and it is given by $\mathbf{N} = N_{ijk}\mathbf{i}^i \otimes \mathbf{i}^j \otimes \mathbf{i}^k$.
The scalar product of two third order tensors is given by $\mathbf{N} \cdot \mathbf{M} = N_{ijk}M_{ijk}$.
Three configurations will be considered: k be a fixed reference configuration of the body $\mathcal{B}$, $k(\mathcal{B}) \subset \mathcal{E}$ with the vector space $\mathcal{V}_k$, $\chi(\cdot, t)$ the deformed configuration at time t, for any motion of the body $\mathcal{B}$, $\chi : \mathcal{B} \times R \longrightarrow \mathcal{E}$, there exists $\mathcal{K}$, time dependent (non-local) configuration with torsion, defined by the pair $(\mathbf{F}^p, \overset{(p)}{\mathbf{\Gamma}}_k)$, $\mathbf{F}^p$—plastic distortion and $\overset{(p)}{\mathbf{\Gamma}}_k$—plastic connection.
$\mathbf{F}-$ the deformation gradient is defined by

$$\mathbf{F}(\mathbf{Z}, t) = \nabla(\chi(\cdot, t) \circ k^{-1})(\mathbf{Z}), \quad \forall \quad \mathbf{Z} \in k(\mathcal{B}). \tag{2}$$

We recall here the *integrability theorem*: Let $\mathbf{F}$ be a function defined on a arcwise connected domain $\mathcal{U}$. $\mathbf{F}$ is a gradient of χ, i.e. $\mathbf{F}(\mathbf{x}) = \nabla\chi(\mathbf{x})$ for $\mathbf{x} \in \mathcal{U}$ if and only if

$$(\nabla\mathbf{F}(\mathbf{x})\mathbf{u})\mathbf{v} - (\nabla\mathbf{F}(\mathbf{x})\mathbf{v})\mathbf{u} = 0 \iff (\text{curl}\mathbf{F}(\mathbf{x}))(\mathbf{u} \times \mathbf{v}) = 0, \quad \forall \quad \mathbf{x} \in \mathcal{U}, \quad \forall \mathbf{u}, \mathbf{v} \in \mathcal{V}_k. \tag{3}$$

Dislocation means (1) non-zero curl or (2) non-zero plastic torsion

(1) $\text{curl}(\mathbf{F}^P) \neq 0$ or
$\exists\, \mathbf{u}, \mathbf{v}$ such that $(\nabla\mathbf{F}^p(\mathbf{u}))\mathbf{v} - (\nabla\mathbf{F}^p(\mathbf{v}))\mathbf{u} \neq 0$

(2) $\exists\, \mathbf{u}, \mathbf{v}$ such that $(\mathbf{S}^p\mathbf{u})\mathbf{v} = (\mathbf{\Gamma}^p\mathbf{u})\mathbf{v} - (\mathbf{\Gamma}^p\mathbf{v})\mathbf{u} \neq 0$.

Let us introduce the following notation for a third order field generated by a connection, say $\mathbf{\Gamma}$, and by second order tensors, for instance $\mathbf{F}_1, \mathbf{F}_2$,

$$(\mathbf{\Gamma}[\mathbf{F}_1, \mathbf{F}_2]\mathbf{u})\mathbf{v} = (\mathbf{\Gamma}(\mathbf{F}_1\mathbf{u}))\,\mathbf{F}_2\mathbf{v}, \quad \forall \mathbf{u}, \mathbf{v} \in \mathcal{V}_k. \tag{4}$$

2 Second order elastic and plastic pairs of deformations

Let us consider the body $\mathcal{B}$, k the initial configuration of the body (which will be not explicitly mentioned further) and $\chi(\cdot, t)$ the actual configuration attached to the function χ which defines the motion of the body.

Ax.1 (the existence of the second order pair of plastic deformations) For any motion χ of the body $\mathcal{B}$, at any material particle **X** and at any time t, there exists a pair $(\mathbf{F}^p, \overset{(p)}{\boldsymbol{\Gamma}})$ with $\mathbf{F}^p$ an invertible second order tensor, called plastic distortion and $\overset{(p)}{\boldsymbol{\Gamma}}$ a third order field, which represent an affine connection, called plastic connection. The pair $(\mathbf{F}^p, \overset{(p)}{\boldsymbol{\Gamma}})$ is invariant with respect to a change of frame in the actual configuration.

Definition The connection of the motion χ with respect to the reference configuration can be introduced by

$$\boldsymbol{\Gamma}\mathbf{u} = \mathbf{F}^{-1}\nabla\mathbf{F}\mathbf{u}, \quad \forall\ \mathbf{u} \in \mathcal{V}. \tag{5}$$

Ax.2 For any pair $(\mathbf{F}, \boldsymbol{\Gamma})$ of the deformation gradient and motion connection, there exists a *second order pair of elastic deformation*, where the *elastic distortion* is defined by

$$\mathbf{F}^e = \mathbf{F}(\mathbf{F}^p)^{-1}, \tag{6}$$

and the *elastic connection* is introduced in terms of the motion and plastic connections, both of them being related to the initial configuration, through the formula

$$\overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}\ \tilde{\mathbf{u}} = \mathbf{F}^p((\boldsymbol{\Gamma} - \overset{(p)}{\boldsymbol{\Gamma}})(\mathbf{F}^p)^{-1}\tilde{\mathbf{u}})(\mathbf{F}^p)^{-1}, \quad \forall \quad \tilde{\mathbf{u}} \in \mathcal{V}_{\mathcal{K}}. \tag{7}$$

Here $\mathcal{V}_{\mathcal{K}} := \mathbf{F}^p(\mathcal{V}_k)$.

The pull back to the reference configuration k leads to the following relationship between connections:

$$\overset{(e)}{\boldsymbol{\Gamma}}_{back} := (\mathbf{F}^p)^{-1}\ \overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}\ [\mathbf{F}^p, \mathbf{F}^p] = \boldsymbol{\Gamma} - \overset{(p)}{\boldsymbol{\Gamma}}, \tag{8}$$

derived from (7).

Definition The differential of any smooth tensor field $\bar{\mathbf{F}}$, defined on $k(\mathcal{B})$, with respect to the configuration with torsion $\mathcal{K}$ is given by

$$(\nabla_{\mathcal{K}}\bar{\mathbf{F}})\tilde{\mathbf{u}} = (\nabla\bar{\mathbf{F}})(\mathbf{F}^p)^{-1}\tilde{\mathbf{u}}, \quad \forall \quad \tilde{\mathbf{u}} \in \mathcal{V}_{\mathcal{K}} \tag{9}$$

Proposition 1

1. *The* multiplicative decomposition *of the deformation gradient* **F** *into the elastic and plastic distortions* $\mathbf{F}^e, \mathbf{F}^p$ *follows*

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p. \tag{10}$$

2. The connections $\boldsymbol{\Gamma}, \overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}$ and $\overset{(p)}{\boldsymbol{\Gamma}}$ *are related by*

$$\boldsymbol{\Gamma}\mathbf{u} = (\mathbf{F}^p)^{-1}(\overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}\ (\mathbf{F}^p\mathbf{u}))\mathbf{F}^p + \overset{(p)}{\boldsymbol{\Gamma}}\ \mathbf{u}, \quad \text{with}$$
$$\boldsymbol{\Gamma} = (\mathbf{F})^{-1}\nabla\mathbf{F} \tag{11}$$

$\forall\ \mathbf{u} \in \mathcal{V}$, *or using* (4)

$$\boldsymbol{\Gamma} = (\mathbf{F}^p)^{-1}(\overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}\ [\mathbf{F}^p, \mathbf{F}^p]) + \overset{(p)}{\boldsymbol{\Gamma}}. \tag{12}$$

We put into evidence the **rules of calculus** for the expressions of the differential of a smooth tensor field $\bar{\mathbf{F}}$, in various configurations, k, χ and $\mathcal{K}$, when we pass from one configuration to the other one:

$$\begin{aligned}(\nabla\bar{\mathbf{F}})\mathbf{u} &= (\nabla_\chi\bar{\mathbf{F}})\nabla(\chi \circ k^{-1})\mathbf{u} \equiv (\nabla_\chi\bar{\mathbf{F}})\mathbf{F}\mathbf{u},\\ (\nabla\bar{\mathbf{F}})\mathbf{u} &= (\nabla_{\mathcal{K}}\bar{\mathbf{F}})\mathbf{F}^p\mathbf{u},\\ (\nabla_\chi\bar{\mathbf{F}})\bar{\mathbf{u}} &= (\nabla\bar{\mathbf{F}})\mathbf{F}^{-1}\bar{\mathbf{u}} = (\nabla_{\mathcal{K}}\bar{\mathbf{F}})\mathbf{F}^p\mathbf{F}^{-1}\bar{\mathbf{u}}\\ &\equiv (\nabla_{\mathcal{K}}\bar{\mathbf{F}})(\mathbf{F}^e)^{-1}\bar{\mathbf{u}}, \quad \forall\mathbf{u} \in \mathcal{V}, \bar{\mathbf{u}} \in \mathcal{V}_\chi.\end{aligned} \tag{13}$$

Ax.3 The plastic connection $\overset{(p)}{\boldsymbol{\Gamma}}$ has the non-zero *Cartan torsion*, defined by the skew-symmetric part of the connection, as it follows

$$(\mathbf{S}^p\mathbf{v})\mathbf{u} \equiv (\overset{(p)}{\boldsymbol{\Gamma}}\ \mathbf{v})\mathbf{u} - (\overset{(p)}{\boldsymbol{\Gamma}}\ \mathbf{u})\mathbf{v}. \tag{14}$$

Ax.4 The plastic distortion and the plastic connection are compatible each other, in the sense that the *Frobenius integrability condition* is satisfied

$$\overset{(p)}{\boldsymbol{\Gamma}} = (\mathbf{F}^p)^{-1}\nabla\ \mathbf{F}^p. \tag{15}$$

2.1 Relationships between the connections attached to the plastic and elastic distortions

When we pass from one configuration, say from the initial configuration to another one, say $\mathcal{K}$, as a consequence of rule of calculus formulae (13), from (15) the relationship between the plastic connections follows

$$\overset{(p)}{\boldsymbol{\Gamma}} = -(\mathbf{F}^p)^{-1}\ \overset{(p)}{\boldsymbol{\Gamma}}_{\mathcal{K}}\ [\mathbf{F}^p, \mathbf{F}^p], \tag{16}$$

where the plastic connection with respect to $\mathcal{K}$ is introduced by

$$\overset{(p)}{\boldsymbol{\Gamma}}_{\mathcal{K}} = \mathbf{F}^p(\nabla_{\mathcal{K}}(\mathbf{F}^p)^{-1}). \tag{17}$$

Remark Consequently we defined two pairs $(\mathbf{F}^p, \overset{(p)}{\boldsymbol{\Gamma}})$ and $((\mathbf{F}^p)^{-1}, \overset{(p)}{\boldsymbol{\Gamma}}_{\mathcal{K}})$ of the appropriate plastic distortions, $\mathbf{F}^p$ and $(\mathbf{F}^p)^{-1}$, and the connections $\overset{(p)}{\boldsymbol{\Gamma}}$ and $\overset{(p)}{\boldsymbol{\Gamma}}_{\mathcal{K}}$, which are compatible.

The appropriate torsion (14), for the plastic connection defined in (15), becomes

$$(\mathbf{S}^p\mathbf{v})\mathbf{u} = (\mathbf{F}^p)^{-1}[((\nabla\,\mathbf{F}^p)\mathbf{v})\mathbf{u} - ((\nabla\,\mathbf{F}^p)\mathbf{u})\mathbf{v}]. \quad (18)$$

Let us introduce the *elastic connections* associated with respect to the specified configurations

$$\overset{(e)}{\mathbf{\Gamma}}_{\mathcal{K}} = (\mathbf{F}^e)^{-1}\nabla_{\mathcal{K}}\,\mathbf{F}^e, \quad \overset{(e)}{\mathbf{\Gamma}}_{\chi} = \mathbf{F}^e\nabla_{\chi}\,(\mathbf{F}^e)^{-1}. \quad (19)$$

From the above definitions together with (9) the following relationship can be put into evidence

$$\overset{(e)}{\mathbf{\Gamma}}_{\chi} = -\mathbf{F}^e\,\overset{(e)}{\mathbf{\Gamma}}_{\mathcal{K}}\,[(\mathbf{F}^e)^{-1}, (\mathbf{F}^e)^{-1}]. \quad (20)$$

On the other hand for the elastic connection, say $\overset{(e)}{\mathbf{\Gamma}}_{\mathcal{K}}$, the *torsion* can be similarly defined as the skew-symmetric part of the connection

$$(\mathbf{S}^e_{\mathcal{K}}\mathbf{v})\mathbf{u} \equiv (\overset{(e)}{\mathbf{\Gamma}}_{\mathcal{K}}\,\mathbf{v})\mathbf{u} - (\overset{(e)}{\mathbf{\Gamma}}_{\mathcal{K}}\,\mathbf{u})\mathbf{v}. \quad (21)$$

As a consequence of the multiplicative decomposition (10) and of the adopted definition for the connections, the relationships between appropriate elastic and plastic connections follows.

Due to the symmetry of the motion connection introduced in (5), provided by the fact that at any time t $\mathbf{F}$ is the gradient of an appropriate application (2),

$$(\mathbf{\Gamma}\mathbf{v})\mathbf{u} - (\mathbf{\Gamma}\mathbf{u})\mathbf{v} = 0. \quad (22)$$

As a consequence of (11) the torsion of the plastic connection with respect to the reference configuration and the torsion of the elastic connection, with respect to the so called configuration with torsion $\mathcal{K}$, are related one to another by

$$\overset{(p)}{\mathbf{S}} = -(\mathbf{F}^p)^{-1}(\overset{(e)}{\mathbf{S}}_{\mathcal{K}}\,[\mathbf{F}^p, \mathbf{F}^p]). \quad (23)$$

On the other hand, from (16) it follows

$$\overset{(p)}{\mathbf{S}} = -(\mathbf{F}^p)^{-1}(\overset{(p)}{\mathbf{S}}_{\mathcal{K}}\,[\mathbf{F}^p, \mathbf{F}^p]). \quad (24)$$

Thus the equality between the elastic torsion and plastic torsion is derived

$$\overset{(e)}{\mathbf{S}}_{\mathcal{K}} = \overset{(p)}{\mathbf{S}}_{\mathcal{K}} \equiv \mathbf{S}_{\mathcal{K}}. \quad (25)$$

Proposition 2 *As a consequence of the axioms* **Ax.3**, **Ax.4**, *of the definitions* (14), (7), *as well as of the relationship* (22), *it follows* $\mathbf{F}^p$, $\mathbf{F}^e$ *are not the gradients of certain mappings.*

2.2 Time-derivatives of the elastic connection with respect to relaxed configuration

When we take the time derivative of the motion connection (5), the rate of the total connection is expressed in term of the second order velocity gradient

$$\frac{d}{dt}(\mathbf{\Gamma}) = \mathbf{F}^{-1}(\nabla_{\chi}\mathbf{L})[\mathbf{F}, \mathbf{F}] \quad (26)$$

where the velocity gradient in the actual configuration is characterized by

$$\mathbf{L} := \nabla_{\chi}\mathbf{v}(\mathbf{x}, t), \quad \mathbf{L} = \dot{\mathbf{F}}(\mathbf{F})^{-1}, \quad (27)$$

where $\mathbf{v}$ is the vector field in the actual configuration.

As a consequence of the multiplicative decomposition (10) the kinematics relationships

$$\begin{aligned}\mathbf{L} &= \dot{\mathbf{F}}^e(\mathbf{F}^e)^{-1} + \mathbf{F}^e\mathbf{L}^p(\mathbf{F}^e)^{-1},\\ \mathbf{L}^e &= \dot{\mathbf{F}}^e(\mathbf{F}^e)^{-1}, \quad \mathbf{L}^p = \dot{\mathbf{F}}^p(\mathbf{F}^p)^{-1}\end{aligned} \quad (28)$$

follow. $\mathbf{L}^e, \mathbf{L}^p$ are the rates of elastic and plastic distortions in the deformed configuration and the configuration with torsion, respectively.

The rate of the plastic connection, relative to the reference configuration, can be derived from (15) under the form similar to (26)

$$\frac{d}{dt}(\overset{(p)}{\mathbf{\Gamma}}) = (\mathbf{F}^p)^{-1}(\nabla_{\mathcal{K}}\mathbf{L}^p)[\mathbf{F}^p, \mathbf{F}^p]. \quad (29)$$

Taking into account the relationship between the plastic connection when we pass from one configuration to the other one, the time derivative applied to (16) leads to

$$\begin{aligned}\frac{d}{dt}(\overset{(p)}{\mathbf{\Gamma}}_{\mathcal{K}})\tilde{\mathbf{u}} = \mathbf{L}^p(\overset{(p)}{\mathbf{\Gamma}}_{\mathcal{K}})\tilde{\mathbf{u}}\\ -\mathbf{F}^p\frac{d}{dt}(\overset{(p)}{\mathbf{\Gamma}})((\mathbf{F}^p)^{-1}\tilde{\mathbf{u}})(\mathbf{F}^p)^{-1}\\ -\overset{(p)}{\mathbf{\Gamma}}_{\mathcal{K}}\,(\mathbf{L}^p\tilde{\mathbf{u}}) - (\overset{(p)}{\mathbf{\Gamma}}_{\mathcal{K}}\,\tilde{\mathbf{u}})\mathbf{L}^p.\end{aligned} \quad (30)$$

When we replace (29) into the above relation the rate of plastic connection is calculated through

$$\begin{aligned}\frac{d}{dt}(\overset{(p)}{\mathbf{\Gamma}}_{\mathcal{K}})\tilde{\mathbf{u}} = \mathbf{L}^p(\overset{(p)}{\mathbf{\Gamma}}_{\mathcal{K}})\tilde{\mathbf{u}} - (\nabla_{\mathcal{K}}\mathbf{L}^p)\tilde{\mathbf{u}} - \overset{(p)}{\mathbf{\Gamma}}_{\mathcal{K}}\,(\mathbf{L}^p\tilde{\mathbf{u}})\\ -(\overset{(p)}{\mathbf{\Gamma}}_{\mathcal{K}}\,\tilde{\mathbf{u}})\mathbf{L}^p.\end{aligned} \quad (31)$$

When we take the time derivative in (12) we get

$$\frac{d}{dt}(\overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}})(\tilde{\mathbf{u}}) = \mathbf{F}^p\left(\frac{d}{dt}(\boldsymbol{\Gamma} - \overset{(p)}{\boldsymbol{\Gamma}})\right)((\mathbf{F}^p)^{-1}\tilde{\mathbf{u}})(\mathbf{F}^p)^{-1}$$
$$+\mathbf{L}^p\mathbf{F}^p((\boldsymbol{\Gamma} - \overset{(p)}{\boldsymbol{\Gamma}})(\mathbf{F}^p)^{-1}\tilde{\mathbf{u}})(\mathbf{F}^p)^{-1}$$
$$-\mathbf{F}^p((\boldsymbol{\Gamma} - \overset{(p)}{\boldsymbol{\Gamma}})(\mathbf{F}^p)^{-1}\mathbf{L}^p\tilde{\mathbf{u}})(\mathbf{F}^p)^{-1}$$
$$-\mathbf{F}^p((\boldsymbol{\Gamma} - \overset{(p)}{\boldsymbol{\Gamma}})(\mathbf{F}^p)^{-1}\tilde{\mathbf{u}})((\mathbf{F}^p)^{-1}\mathbf{L}^p)$$
$$\forall \quad \tilde{\mathbf{u}} \in \mathcal{V}_{\mathcal{K}}. \tag{32}$$

Using again (12) and the multiplicative decomposition (10) in relationship (32), the time derivative of the appropriate connections are related by

$$\frac{d}{dt}(\overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}})(\tilde{\mathbf{u}}) - \mathbf{L}^p(\overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}\,\tilde{\mathbf{u}}) + \overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}\,(\mathbf{L}^p\tilde{\mathbf{u}}) + (\overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}\,\tilde{\mathbf{u}})\mathbf{L}^p$$
$$\equiv \mathbf{F}^p\left(\frac{d}{dt}(\boldsymbol{\Gamma} - \overset{(p)}{\boldsymbol{\Gamma}})\right)((\mathbf{F}^p)^{-1}\tilde{\mathbf{u}})(\mathbf{F}^p)^{-1}. \tag{33}$$

Let us introduce a linear operator applied to the elastic connection $\overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}$, dependent on the rate of plastic distortion $\mathbf{L}^p$, by

$$(\mathcal{L}_{\mathbf{L}^p}[\overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}])\tilde{\mathbf{u}} := \frac{d}{dt}(\overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}})\tilde{\mathbf{u}} - \mathbf{L}^p(\overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}\,\tilde{\mathbf{u}}) + \overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}\,(\mathbf{L}^p\tilde{\mathbf{u}}) + (\overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}\,\tilde{\mathbf{u}})\mathbf{L}^p \tag{34}$$

for all $\tilde{\mathbf{u}}$. The expression of the above operator (34), introduced in the left hand side of (33), leads to the following equivalent formula

$$(\mathbf{F}^p)^{-1}((\mathcal{L}_{\mathbf{L}^p}[\overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}])[\mathbf{F}^p, \mathbf{F}^p]) = \frac{d}{dt}(\boldsymbol{\Gamma}) - \frac{d}{dt}(\overset{(p)}{\boldsymbol{\Gamma}}). \tag{35}$$

Using (26), (29) and the multiplicative decomposition (10), (35) becomes

$$(\mathcal{L}_{\mathbf{L}^p}[\overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}]) = (\mathbf{F}^e)^{-1}(\nabla_\chi \mathbf{L})[\mathbf{F}^e, \mathbf{F}^e] - \nabla_{\mathcal{K}}\mathbf{L}^p. \tag{36}$$

Definition Starting from the kinematic relationships derived above, for a given deformation state (i.e. $\mathbf{F}$, $\mathbf{F}^e$, $\mathbf{F}^p$, $\overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}$, $\overset{(p)}{\boldsymbol{\Gamma}}$ are considered to be given), we characterize a *virtual process* by

$\tilde{\mathbf{v}}$—the virtual velocity, $\tilde{\mathbf{L}}$—the virtual velocity gradient,

$\tilde{\mathbf{L}}^e$ and $\tilde{\mathbf{L}}^p$ the virtual rate of the elastic and plastic distortion, compatible with the kinematical relationships (27), (28), which means

$$\tilde{\mathbf{L}} := \nabla_\chi \tilde{\mathbf{v}}, \quad \text{and} \quad \tilde{\mathbf{L}} = \tilde{\mathbf{L}}^e + \mathbf{F}^e\tilde{\mathbf{L}}^p(\mathbf{F}^e)^{-1}. \tag{37}$$

Consequently, taking into account (34) and (36) the *virtual time-derivative of the elastic connection* with respect to the plastically deformed configuration can be introduced by the **Definition:**

$$\text{virt}\frac{d}{dt}(\overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}})(\tilde{\mathbf{u}}) = \mathbf{F}^{e\,-1}((\nabla_\chi\tilde{\mathbf{L}})[\mathbf{F}^e, \mathbf{F}^e])\tilde{\mathbf{u}} - (\nabla_{\mathcal{K}}\tilde{\mathbf{L}}^p)\tilde{\mathbf{u}}$$
$$+\tilde{\mathbf{L}}^p(\overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}\,\tilde{\mathbf{u}}) - (\overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}\,(\tilde{\mathbf{L}}^p\tilde{\mathbf{u}}))$$
$$-(\overset{(e)}{\boldsymbol{\Gamma}}\,\tilde{\mathbf{u}})\tilde{\mathbf{L}}^p, \quad \forall \quad \tilde{\mathbf{u}} \in \mathcal{V}_{\mathcal{K}}. \tag{38}$$

3 The macro and micro balance equations

The principle of the virtual power at any arbitrary fixed moment of the time t is built starting from the principle of the virtual power derived from Fleck et al. (1994) and using the result already proved by Cleja-Tigoiu in (2002a), relative to the expressions of the power expanded by an elasto-plastic material (without any relation with a principle of the virtual power).

First we recall the **definitions** for Piola-Kirchhoff stress tensor and the stress momentum as pulled back to the configuration with torsion, and Mandel's non-symmetric stress measure, all of them being expressed relative to $\mathcal{K}$

$$\boldsymbol{\Pi}_{\mathcal{K}} \equiv \boldsymbol{\Pi} = \det(\mathbf{F}^e)(\mathbf{F}^e)^{-1}\mathbf{T}^s(\mathbf{F}^e)^{-T}, \quad \det\mathbf{F}^e = \frac{\rho_{\mathcal{K}}}{\rho}$$
$$\boldsymbol{\mu}_{\mathcal{K}} = (\det\mathbf{F}^e)(\mathbf{F}^e)^T\boldsymbol{\mu}[(\mathbf{F}^e)^{-T}, (\mathbf{F}^e)^{-T}], \tag{39}$$
$$\frac{1}{\rho_{\mathcal{K}}}\boldsymbol{\Sigma}_{\mathcal{K}} = (\mathbf{F}^e)^T\frac{\mathbf{T}}{\rho}(\mathbf{F}^e)^{-T}, \qquad \mathbf{C}^e = (\mathbf{F}^e)^T\mathbf{F}^e,$$

associated to the non-symmetric Cauchy stress $\mathbf{T}$.

The virtual power at a fixed moment of time is written for any part $\mathcal{P} \subset \mathcal{B}$ and we accept that a surface element in the actual configuration, with unit normal $\mathbf{n}$, may transmit both force vector and couple vector. We start from the virtual power principle, **VPP-I**, formulated in Continuum Mechanics with couple stresses (see Fleck et al. (1994)).

PVP-I. In the deformed configuration, $\forall \quad \mathcal{P} \subset \mathcal{B}$ bounded by a smooth surface $\partial\mathcal{P}$, the virtual power at a fixed moment of time

$$\int_{\mathcal{P}_t} \rho\mathbf{a}\cdot\mathbf{w}d\mathbf{x} + \int_{\mathcal{P}_t}\mathbf{T}\cdot\nabla\mathbf{w}d\mathbf{x} + \int_{\mathcal{P}_t} 2\overset{\times}{\mathbf{T}}\cdot\boldsymbol{\theta}d\mathbf{x}$$
$$+\int_{\mathcal{P}_t}\mathbf{M}\cdot\nabla\boldsymbol{\theta}d\mathbf{x} = +\int_{\partial\mathcal{P}_t}\mathbf{Tn}\cdot\mathbf{w}d\sigma$$
$$+\int_{\partial\mathcal{P}_t}\mathbf{Mn}\cdot\boldsymbol{\theta}d\sigma + \int_{\mathcal{P}_t}\rho\mathbf{b}_f\cdot\mathbf{w}d\mathbf{x} \tag{40}$$
$$+\int_{\mathcal{P}_t}\rho\mathbf{b}_m\cdot\boldsymbol{\theta}d\mathbf{x}$$

holds for $\forall$ virtual velocity $\mathbf{w}$, and $\boldsymbol{\theta} = \frac{1}{2}\text{curl}\,\mathbf{w}$. Here $\mathbf{a}$ is the acceleration vector and the vector field $\overset{\times}{\mathbf{T}}$ is the coaxial vector associated with the skew-symmetric part of the Cauchy stress tensor $\mathbf{T}$

$$\mathbf{u}\times \overset{\times}{\mathbf{T}} = \mathbf{T}^a\mathbf{u}, \quad \forall\, \mathbf{u} \in \mathcal{V}_\chi. \tag{41}$$

$\mathbf{b}_f$, $\mathbf{b}_m$ are densities of the body forces and body couples, the vector force $\mathbf{Tn}$ and couple vector $\mathbf{Mn}$, acting on the surface of the normal characterized by the unit vector $\mathbf{n}$, $\mathbf{M}$ is the couple stress tensor. $\mathbf{T}$ and $\mathbf{M}$ are generally non-symmetric second order tensors.

Let us consider some simple properties, concerning the second order tensor fields with non-zero associated *curl*, which will be useful in defining an appropriate form of the virtual power principle.

Proposition 3 $\forall \mathbf{L} = \mathbf{W} \in Skew$ such that $curl\mathbf{L} = 0$, then

$$\begin{aligned}&(\nabla\mathbf{Lu})\mathbf{v} - (\nabla\mathbf{Lv})\mathbf{u} = 0 \quad \text{and}\\ &\exists \;\; \omega, \quad \psi \in \mathcal{V} \quad : \quad \mathbf{Wu} = \omega \times \mathbf{u},\\ &\forall \mathbf{u} \in \mathcal{V} \quad \omega = curl\psi.\end{aligned} \tag{42}$$

Proposition 4 $\forall \mathbf{L} = \mathbf{W} \in Skew$ such that $curl\mathbf{L} \neq 0$, then

$$\begin{aligned}&(\nabla\mathbf{L})\mathbf{u} + ((\nabla\mathbf{L})\mathbf{u})^T = 0 \quad \Longleftrightarrow\\ &\quad \exists \quad \Lambda \in Lin \quad \nabla\mathbf{Lu} = \Lambda\mathbf{u} \times \mathbf{I} \quad \text{and}\\ &\quad curl\mathbf{W} \equiv det\Lambda\mathbf{I} - \Lambda^T.\end{aligned} \tag{43}$$

Based on the two propositions derived above we propose a virtual power principle appropriate to finite elasto-plasticity with continuously distributed dislocation.

We pass to the virtual power principle, **PVP-II**, that can be derived from the previous one, when $\frac{1}{2}\text{curl}\,\mathbf{w}$—vector field replaced by $\{\nabla\mathbf{w}\}^a$—tensor field.

PVP-II. $\forall \quad \mathcal{P}$ and $\forall$ virtual velocity $\mathbf{w}$

$$\begin{aligned}&\int_{\mathcal{P}_t} \rho\mathbf{a}\cdot\mathbf{w}d\mathbf{x} + \int_{\mathcal{P}_t} \mathbf{T}\cdot\nabla\mathbf{w}d\mathbf{x} + \int_{\mathcal{P}_t} \mathbf{T}^*\cdot\{\nabla\mathbf{w}\}^a d\mathbf{x}\\ &\quad + \int_{\mathcal{P}_t} \boldsymbol{\mu}\cdot\nabla_\chi\{\nabla\mathbf{w}\}^a d\mathbf{x} = \int_{\mathcal{P}_t} \rho\mathbf{b}_f\cdot\mathbf{w}d\mathbf{x}\\ &\quad + \int_{\partial\mathcal{P}_t} \mathbf{Tn}\cdot\mathbf{w}d\sigma + \int_{\partial\mathcal{P}_t} \boldsymbol{\mu}\mathbf{n}\cdot\{\nabla\mathbf{w}\}^a d\sigma\\ &\quad + \int_{\mathcal{P}_t} \rho\mathbf{B}_m\cdot\{\nabla\mathbf{w}\}^a d\mathbf{x}\end{aligned} \tag{44}$$

$$\text{with} \quad \{\nabla\mathbf{w}\}^a = \frac{1}{2}(\nabla\mathbf{w} - \nabla\mathbf{w}^T). \tag{45}$$

In the above formulae $\mathbf{B}_m$ denotes the second order body couple density and $\boldsymbol{\mu}$ is a third order field associated with a second order tensor field $\mathbf{M}$ as it follows

$$\boldsymbol{\mu}\cdot\nabla_\chi\{\nabla\mathbf{w}\}^a = \frac{1}{2}\mathbf{M}\cdot\nabla\text{curl}\mathbf{w}. \tag{46}$$

The principle of the virtual power **PVP-III** follows from **PVP-II**, when $\{\nabla\mathbf{w}\}^a$ replaced by $\overline{\mathbf{W}} \in Skew$.

PVP-III $\forall \quad \mathcal{P}$ and $\forall$ virtual fields $\mathbf{w}$ and $\overline{\mathbf{W}}$ the equality

$$\begin{aligned}&\int_{\mathcal{P}_t} \rho\mathbf{a}\cdot\mathbf{w}d\mathbf{x} + \int_{\mathcal{P}_t} \mathbf{T}\cdot\nabla\mathbf{w}d\mathbf{x} + \int_{\mathcal{P}_t} \mathbf{T}^*\cdot\overline{\mathbf{W}}d\mathbf{x}\\ &\quad + \int_{\mathcal{P}_t} \boldsymbol{\mu}\cdot\nabla\overline{\mathbf{W}}d\mathbf{x} = \int_{\mathcal{P}_t} \rho\mathbf{b}_f\cdot\mathbf{w}d\mathbf{x}\\ &\quad + \int_{\partial\mathcal{P}_t} \mathbf{Tn}\cdot\mathbf{w}d\sigma + \int_{\partial\mathcal{P}_t} \boldsymbol{\mu}\mathbf{n}\cdot\overline{\mathbf{W}}d\sigma\\ &\quad + \int_{\mathcal{P}_t} \rho\mathbf{B}_m\cdot\overline{\mathbf{W}}d\mathbf{x}\end{aligned} \tag{47}$$

holds.

Finally a general form for the principle of the virtual power principle, **PVP-general**, can be put into evidence, when $\overline{\mathbf{W}} \in Skew$ in **PVP-III** is replaced by $\overline{\mathbf{L}}$, restricted to the conditions either $\overline{\mathbf{L}} = 0$ or $curl\overline{\mathbf{L}} \neq 0$.

Let us introduce the micro-stress $\boldsymbol{\Upsilon}^p_{\mathcal{K}}$ in the configuration with torsion, and the micro stress momentum $\boldsymbol{\mu}^p_{\mathcal{K}}$.

Ax.5 The **principle of the virtual power in finite elasto-plastic**, formulated $\forall$ part $\mathcal{P} \subset \mathcal{B}$ bounded by a smooth surface $\partial\mathcal{P}$

$$\begin{aligned}&\int_{\chi(\mathcal{P},t)} \rho\mathbf{a}\cdot\tilde{\mathbf{v}}dV + \int_{\chi(\mathcal{P},t)} \{\mathbf{T}\cdot\nabla_\chi\tilde{\mathbf{v}} + \mathbf{T}^*\cdot\widetilde{\mathbf{L}}^e\}dV\\ &\quad + \int_{\chi(\mathcal{P},t)} \boldsymbol{\mu}\cdot\nabla_\chi\widetilde{\mathbf{L}}^e dV\\ &\quad + \int_{\mathcal{K}_t(\mathcal{P})} \{\boldsymbol{\Upsilon}^p_{\mathcal{K}}\cdot\widetilde{\mathbf{L}}^p + \boldsymbol{\mu}^p_{\mathcal{K}}\cdot\nabla_{\mathcal{K}}\widetilde{\mathbf{L}}^p\}dV_{\mathcal{K}}\\ &\quad = \int_{\chi(\mathcal{P},t)} \rho\mathbf{b}_f\cdot\tilde{\mathbf{v}}dV\\ &\quad + \int_{\chi(\mathcal{P},t)} \rho\mathbf{B}_m\cdot\widetilde{\mathbf{L}}^e dV + \int_{\mathcal{K}_t} \tilde{\rho}\mathbf{B}^p_m\cdot\widetilde{\mathbf{L}}^p dV_{\mathcal{K}}\\ &\quad + \int_{\partial\chi(\mathcal{P},t)} \mathbf{t}(\mathbf{n})\cdot\tilde{\mathbf{v}}dV + \int_{\partial\chi(\mathcal{P},t)} \mathbf{M}(\mathbf{n})\cdot\widetilde{\mathbf{L}}^e dA\\ &\quad + \int_{\partial\mathcal{K}_t(\mathcal{P})} \mathbf{M}^p(\mathbf{n})\cdot\widetilde{\mathbf{L}}^p dA_{\mathcal{K}},\end{aligned} \tag{48}$$

holds for any all generalized virtual velocities and virtual rates $\tilde{\mathbf{v}}$, $\nabla_\chi\tilde{\mathbf{v}}$, $\widetilde{\mathbf{L}}^p$, $\nabla_{\mathcal{K}}\widetilde{\mathbf{L}}^p$, for all $\widetilde{\mathbf{L}}^e$, $\nabla_\chi\widetilde{\mathbf{L}}^e$.

Now we justify the postulated form for the principle of the virtual power (48), using the expression of the mechanical internal power as it follows from Cleja-Ţigoiu (2002a).

Proposition 5

1. *The density of the internal mechanical power produced by the non-symmetric Cauchy stress tensor can be written under the form*
$$\frac{1}{\rho}\mathbf{T}\cdot\mathbf{L} = \frac{1}{\rho}\mathbf{T}\cdot\mathbf{L}^e + \frac{1}{\rho_{\mathcal{K}}}\boldsymbol{\Sigma}_{\mathcal{K}}\cdot\mathbf{L}^p. \tag{49}$$
2. *The density of the internal mechanical power produced by the couple stresses can be expressed with the aid of the third order tensor of stress momentum defined in the actual configuration by $\boldsymbol{\mu}$ or in terms of the stress momentum pulled back to the configuration with torsion $\boldsymbol{\mu}_{\mathcal{K}}$*
$$\begin{aligned}\frac{1}{\rho}\boldsymbol{\mu}\cdot\nabla_{\chi}\mathbf{L} &= \frac{1}{\rho_{\mathcal{K}}}\boldsymbol{\mu}_{\mathcal{K}}\cdot(\mathbf{F}^e)^{-1}\nabla_{\chi}\mathbf{L}[\mathbf{F}^e,\mathbf{F}^e]\\ &= \frac{1}{\rho_{\mathcal{K}}}\boldsymbol{\mu}_{\mathcal{K}}\cdot(\mathcal{L}_{\mathbf{L}^p}[\overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}])\\ &\quad+\frac{1}{\rho_{\mathcal{K}}}\boldsymbol{\mu}_{\mathcal{K}}\cdot\nabla_{\mathcal{K}}\mathbf{L}^p.\end{aligned} \tag{50}$$

The last equality, that put in evidence the elastic and plastic part of the appropriate internal power, has been reformulated as a consequence of (36).

Remark We use Gurtin's argument, since there are no motivation to suppose that the terms which enter the previous formulae (49) and (50) refer to the same stress and the same momentum, and we replace in (49) $\boldsymbol{\Sigma}_{\mathcal{K}}$ by the microstress $\boldsymbol{\Upsilon}^p_{\mathcal{K}}$ and in (50) $\boldsymbol{\mu}_{\mathcal{K}}$, which is power conjugated with $\nabla_{\mathcal{K}}\mathbf{L}^p$, by $\boldsymbol{\mu}^p_{\mathcal{K}}$.

The *macro-balance equation* at any time t can be derived from (48) if we take $\widetilde{\mathbf{L}}^p = 0$ and $\nabla_{\mathcal{K}}\widetilde{\mathbf{L}}^p = 0$. For any $\widetilde{\mathbf{v}}$, a virtual velocity and for any second order tensor field $\widetilde{\mathbf{L}}^e$ with $curl(\widetilde{\mathbf{L}}^e) \neq 0$, i.e non-reducible to a gradient of an appropriate vector field, the *macro-balance equations* are derived. The *micro-balance equation* at any time t can be derived from (48) if $\widetilde{\mathbf{L}}^p$ and $\nabla_{\mathcal{K}}\widetilde{\mathbf{L}}^p$ are non-zero.

Theorem 1

1. *The impulse and momentum local balance equations can be written under the form*
$$\begin{aligned}\rho\mathbf{a} &= \mathrm{div}\,\mathbf{T} + \rho\mathbf{b}_f\\ \mathbf{T}^* &= \mathrm{div}\,\boldsymbol{\mu} + \rho\mathbf{B}_m, \quad \text{on} \quad \mathcal{P}_t\end{aligned} \tag{51}$$
with the appropriate boundary conditions on $\partial\mathcal{P}_t$
$$\mathbf{T}\mathbf{n} = \mathbf{t}(\mathbf{n}), \quad \textit{and} \quad \boldsymbol{\mu}\mathbf{n} = \mathbf{M}(\mathbf{n}). \tag{52}$$
2. *The micro balance equation is expressed by*
$$\begin{aligned}&\boldsymbol{\Upsilon}^p_{\mathcal{K}} - div\,\boldsymbol{\mu}^p_{\mathcal{K}} = \tilde{\rho}\mathbf{B}^p_m, \quad in\ \mathcal{K}(\mathcal{P},t),\\ &\boldsymbol{\mu}^p_{\mathcal{K}}\mathbf{n} = \mathbf{M}^p\mathbf{n} \quad on \quad \partial\mathcal{K}(\mathcal{P},t),\end{aligned} \tag{53}$$
micro-traction condition.

We remark that (51), $(53)_1$ together with the boundary conditions (52) and $(53)_2$ follow directly from the principle of the virtual power, stipulated in (48), without any additional assumptions.

Finally we **conclude** that the theory can be based either on the appropriate postulate of the variational principle for physical and material space or on the postulates of the physical and material balance laws.

4 Free energy imbalance

Ax.6 There exists a free energy density function ψ, invariant with respect to a change of frame in the actual configuration
$$\psi = \psi_{\mathcal{K}}(\mathbf{C}^e, \overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}, (\mathbf{F}^p)^{-1}, \overset{(p)}{\boldsymbol{\Gamma}}_{\mathcal{K}}), \tag{54}$$
represented in the configuration with torsion $\mathcal{K}$ by a function of the second order elastic pair $(\mathbf{C}^e, \overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}})$, and dependent on the plastic measure of deformation $((\mathbf{F}^p)^{-1}, \overset{(p)}{\boldsymbol{\Gamma}}_{\mathcal{K}})$.

Ax.7 The elasto-plastic behavior of the material is restricted to satisfy in $\mathcal{K}$ the *free energy imbalance*
$$-\dot{\psi}_{\mathcal{K}} + \frac{1}{\rho_{\mathcal{K}}}(\mathcal{P}_{int})_{\mathcal{K}} \geq 0 \tag{55}$$
for any *virtual (isothermic) processes*.

Proposition 6 In (55) *the internal power in the configuration with torsion can be calculated starting from the expression*
$$\begin{aligned}(\mathcal{P}_{int})_{\mathcal{K}} &= \frac{1}{\rho}(\mathbf{T}+\mathbf{T}^*)\cdot\mathbf{L}^e + \boldsymbol{\Upsilon}^p_{\mathcal{K}}\cdot\mathbf{L}^p\\ &\quad+\frac{1}{\rho_{\mathcal{K}}}\boldsymbol{\mu}_{\mathcal{K}}\cdot\mathcal{L}_{\mathbf{L}^p}[\overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}]\\ &\quad+\frac{1}{\rho_{\mathcal{K}}}\boldsymbol{\mu}^p_{\mathcal{K}}\cdot\nabla_{\mathcal{K}}\mathbf{L}^p,\end{aligned} \tag{56}$$

while *the time derivative of the free energy is expressed through*

$$\dot{\psi}_{\mathcal{K}} = \partial_{\mathbf{C}^e}\psi_{\mathcal{K}} \cdot \dot{\mathbf{C}}^e - (\mathbf{F}^p)^{-T}\partial_{(\mathbf{F}^p)^{-1}}\psi_{\mathcal{K}} \cdot \mathbf{L}^p + \partial_{\overset{(e)}{\Gamma}_{\mathcal{K}}}\psi_{\mathcal{K}} \cdot \left(\frac{d}{dt}\overset{(e)}{\Gamma}_{\mathcal{K}}\right) + \partial_{\overset{(p)}{\Gamma}_{\mathcal{K}}}\psi_{\mathcal{K}} \cdot \left(\frac{d}{dt}\overset{(p)}{\Gamma}_{\mathcal{K}}\right). \tag{57}$$

The formula (56) is derived from macro and micro balance equations, multiplied by $\mathbf{L}^e$ and by $\mathbf{L}^p$, respectively.

In order to pursuit the calculus

1. We eliminate the rate of the elastic distortion, which enters the expression (56) via formula (28), as well the gradient of the rate of elastic ditorsion via the formula (36). Then only $\mathbf{L}$ and $\mathbf{L}^p$ and their appropriate differentials, $\nabla_\chi \mathbf{L}$ and $\nabla_{\mathcal{K}}\mathbf{L}^p$, enter the internal power.
2. In (57) the rate of the elastic strain is replaced by

$$\dot{\mathbf{C}}^e = 2\,(\mathbf{F}^e)^T\{\mathbf{L}\}^s\mathbf{F}^e - 2\,\{\mathbf{C}^e\mathbf{L}^p\}^s, \tag{58}$$

using the elastic strain $\mathbf{C}^e$ expressed as a consequence of the multiplicative decomposition under the form

$$\mathbf{C}^e := (\mathbf{F}^e)^T\mathbf{F}^e = (\mathbf{F}^p)^{-T}\mathbf{C}(\mathbf{F}^p)^{-1}, \quad \text{where} \quad \mathbf{C} = \mathbf{F}^T\mathbf{F}. \tag{59}$$

In order to obtain the imbalanced energy condition, we pass to the virtual kinematic process, as it follows:

$$\begin{aligned}&\mathrm{virt}(\dot{\mathbf{C}}^e) = 2\,(\mathbf{F}^e)^T\{\widetilde{\mathbf{L}}\}^s\mathbf{F}^e - 2\,\{\mathbf{C}^e\cdot\widetilde{\mathbf{L}}^p\}^s, \quad \text{related to (58)}\\ &\mathrm{virt}(\mathcal{L}_{\mathbf{L}^p}[\overset{(e)}{\Gamma}_{\mathcal{K}}]) = (\mathbf{F}^e)^{-1}(\nabla_\chi\widetilde{\mathbf{L}})[\mathbf{F}^e,\mathbf{F}^e] - \nabla_{\mathcal{K}}\widetilde{\mathbf{L}}^p, \quad \text{related to (36).}\end{aligned} \tag{60}$$

Everywhere we replace $\mathbf{L}$, and $\nabla_\chi\mathbf{L}$, by the virtual $\widetilde{\mathbf{L}}$ $\nabla_\chi\widetilde{\mathbf{L}}$, $\mathbf{L}^p$, and $\nabla_{\mathcal{K}}\mathbf{L}^p$ are replaced by $\widetilde{\mathbf{L}}^p$ and $\nabla_\chi\widetilde{\mathbf{L}}^p$.

Proposition 7 *The* free energy imbalance *is satisfied for any* virtual process, *if the inequality written below*

$$\begin{aligned}&\left\{\frac{1}{\rho}(\mathbf{F}^e)^{-1}\{\mathbf{T}+\mathbf{T}^*\}^s(\mathbf{F}^e)^{-T} - 2\partial_{\mathbf{C}^e}\psi_{\mathcal{K}}\right\}\\ &\cdot[(\mathbf{F}^e)^T\{\widetilde{\mathbf{L}}\}^s\mathbf{F}^e - \{\mathbf{C}^e\cdot\widetilde{\mathbf{L}}^p\}^s]\\ &+\frac{1}{\rho}\{\mathbf{T}+\mathbf{T}^*\}^a\cdot(\widetilde{\mathbf{L}} - \mathbf{F}^e\widetilde{\mathbf{L}}^p(\mathbf{F}^e)^{-1})\\ &+\left\{\frac{1}{\rho_{\mathcal{K}}}\boldsymbol{\Upsilon}^p_{\mathcal{K}} + (\mathbf{F}^p)^{-T}\partial_{(\mathbf{F}^p)^{-1}}\psi_{\mathcal{K}}\right\}\cdot\widetilde{\mathbf{L}}^p\\ &+\frac{1}{\rho_{\mathcal{K}}}\boldsymbol{\mu}_{\mathcal{K}}\cdot[(\mathbf{F}^e)^{-1}(\nabla_\chi\widetilde{\mathbf{L}})[\mathbf{F}^e,\mathbf{F}^e] - \nabla_{\mathcal{K}}\widetilde{\mathbf{L}}^p]\\ &+\frac{1}{\rho_{\mathcal{K}}}\boldsymbol{\mu}^p_{\mathcal{K}}\cdot\nabla_{\mathcal{K}}\widetilde{\mathbf{L}}^p\\ &-\partial_{\Gamma^e{}_{\mathcal{K}}}\psi_{\mathcal{K}}\cdot\mathrm{virt}\left(\frac{d}{dt}\Gamma^e{}_{\mathcal{K}}\right) - \partial_{\overset{(p)}{\Gamma}_{\mathcal{K}}}\psi_{\mathcal{K}}\\ &\cdot\mathrm{virt}\left(\frac{d}{dt}\overset{(p)}{\Gamma}_{\mathcal{K}}\right) \geq 0\end{aligned} \tag{61}$$

holds for any $\widetilde{\mathbf{L}} \equiv \nabla_\chi\widetilde{\mathbf{v}}$, $\nabla_\chi\widetilde{\mathbf{L}}$, and for arbitrarily given $\widetilde{\mathbf{L}}^p$, $\nabla_{\mathcal{K}}\widetilde{\mathbf{L}}^p$.

The *virtual kinematic processes* are also characterized by the virtual variations of the fields via the formulae

$$\begin{aligned}\mathrm{virt}\left(\frac{d}{dt}(\overset{(e)}{\Gamma}_{\mathcal{K}})\right) &= \widetilde{\mathbf{L}}^p\,\overset{(e)}{\Gamma}_{\mathcal{K}} - \overset{(e)}{\Gamma}_{\mathcal{K}}\,\widetilde{\mathbf{L}}^p - \overset{(e)}{\Gamma}_{\mathcal{K}}\,[\mathbf{I},\widetilde{\mathbf{L}}^p]\\ &\quad+(\mathbf{F}^e)^{-1}(\nabla_\chi\widetilde{\mathbf{L}})[\mathbf{F}^e,\mathbf{F}^e] - \nabla_{\mathcal{K}}\widetilde{\mathbf{L}}^p\\ &\quad\text{related to (38)}\end{aligned} \tag{62}$$

and related to (31)

$$\mathrm{virt}\left(\frac{d}{dt}(\overset{(p)}{\Gamma}_{\mathcal{K}})\right) = \widetilde{\mathbf{L}}^p\,\overset{(p)}{\Gamma}_{\mathcal{K}} - \nabla_{\mathcal{K}}\widetilde{\mathbf{L}}^p - \overset{(p)}{\Gamma}\,\widetilde{\mathbf{L}}^p - \overset{(p)}{\Gamma}\,[\mathbf{I},\widetilde{\mathbf{L}}^p]. \tag{63}$$

5 Thermodynamic restrictions

We provide the thermomechanic restrictions on the constitutive description of elasto-plastic material, based on the *imbalanced free energy condition*. We require the *imbalanced condition* written in (61) to be satisfied for any *virtual process*, defined by the formulae (60)–(63), when $\widetilde{\mathbf{L}}$, $\nabla_\chi\widetilde{\mathbf{L}}$ are arbitrary, and for the given $\widetilde{\mathbf{L}}^p$, $\nabla_{\mathcal{K}}\widetilde{\mathbf{L}}^p$.

I. First step: we consider a virtual process in a such way to have $\widetilde{\mathbf{L}}^p = 0$, $\nabla_{\mathcal{K}}\widetilde{\mathbf{L}}^p = 0$.

Thus (61) holds for any $\widetilde{\mathbf{L}}$, $\nabla_\chi \widetilde{\mathbf{L}}$, if and only if the following *constitutive restrictions*

$$\begin{aligned}
&(\mathbf{F}^e)^{-1}\{\mathbf{T}+\mathbf{T}^*\}^s(\mathbf{F}^e)^{-T} = 2\rho\partial_{\mathbf{C}^e}\psi_{\mathcal{K}},\\
&\quad\{\mathbf{T}+\mathbf{T}^*\}^a = 0\\
&\frac{1}{\rho_{\mathcal{K}}}\boldsymbol{\mu}_{\mathcal{K}} = \partial_{\boldsymbol{\Gamma}^e{}_{\mathcal{K}}}\psi_{\mathcal{K}},
\end{aligned} \tag{64}$$

are satisfied.

II. Second step: we introduce the thermodynamic restriction (64) into *imbalanced free energy condition* (61) and we get the *dissipation inequality*

$$\begin{aligned}
&\left\{\frac{1}{\rho_{\mathcal{K}}}\boldsymbol{\Upsilon}^p_{\mathcal{K}} + (\mathbf{F}^p)^{-T}\partial_{(\mathbf{F}^p)^{-1}}\psi_{\mathcal{K}}\right\}\cdot\widetilde{\mathbf{L}}^p\\
&+\left\{\frac{1}{\rho_{\mathcal{K}}}\boldsymbol{\mu}^p_{\mathcal{K}} + \partial_{\overset{(p)}{\boldsymbol{\Gamma}}_{\mathcal{K}}}\psi_{\mathcal{K}}\right\}\cdot\nabla_{\mathcal{K}}\widetilde{\mathbf{L}}^p - \partial_{\overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}}\psi_{\mathcal{K}}\\
&\cdot\{\widetilde{\mathbf{L}}^p\,\overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}} - \overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}\,\widetilde{\mathbf{L}}^p - \overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}\,[\mathbf{I},\widetilde{\mathbf{L}}^p]\} - \partial_{\overset{(p)}{\boldsymbol{\Gamma}}_{\mathcal{K}}}\psi_{\mathcal{K}}\\
&\cdot\{\widetilde{\mathbf{L}}^p\,\overset{(p)}{\boldsymbol{\Gamma}}_{\mathcal{K}} - \overset{(p)}{\boldsymbol{\Gamma}}_{\mathcal{K}}\,\widetilde{\mathbf{L}}^p - \overset{(p)}{\boldsymbol{\Gamma}}_{\mathcal{K}}\,[\mathbf{I},\widetilde{\mathbf{L}}^p]\} \geq 0.
\end{aligned} \tag{65}$$

Let us introduce the free energy in the reference configuration

$$\begin{aligned}
\psi &= \psi_{\mathcal{K}}(\mathbf{C}^e, \overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}, (\mathbf{F}^p)^{-1}, \overset{(p)}{\boldsymbol{\Gamma}}_{\mathcal{K}})\\
&\equiv \overline{\psi}(\mathbf{C}, \overset{(e)}{\boldsymbol{\Gamma}}_{back}, \mathbf{F}^p, \overset{(p)}{\boldsymbol{\Gamma}}),
\end{aligned} \tag{66}$$

taking into account the relationships (59), (8) and (16).

Proposition 8 *When we pass to the free energy density in the reference configuration k, the dissipation inequality becomes*

$$\begin{aligned}
&\left\{\frac{1}{\rho_{\mathcal{K}}}\boldsymbol{\Upsilon}^p_{\mathcal{K}} - 2\mathbf{C}^e(\mathbf{F}^p)\partial_{\mathbf{C}}\bar{\psi}(\mathbf{F}^p)^T - \partial_{\mathbf{F}^p}\overline{\psi}(\mathbf{F}^p)^T\right\}\cdot\widetilde{\mathbf{L}}^p\\
&+\left\{\frac{1}{\rho_{\mathcal{K}}}\boldsymbol{\mu}^p_{\mathcal{K}} - (\mathbf{F}^p)^{-T}\partial_{\overset{(p)}{\boldsymbol{\Gamma}}}\overline{\psi}[(\mathbf{F}^p)^T, (\mathbf{F}^p)^T]\right\}\\
&\cdot\nabla_{\mathcal{K}}\widetilde{\mathbf{L}}^p \geq 0, \quad \forall\ \ \widetilde{\mathbf{L}}^p, \nabla_{\mathcal{K}}\widetilde{\mathbf{L}}^p.
\end{aligned} \tag{67}$$

In order to **prove** the above formulae we take into account the relationships between the partial derivatives of the free energy expressed relative to the initial configuration and to the configuration with torsion, derived from (66) together with (59), (8) and (16), under the form

$$\begin{aligned}
&\partial_{\mathbf{C}}\bar{\psi} = (\mathbf{F}^p)^{-1}\partial_{\mathbf{C}^e}\psi_{\mathcal{K}}(\mathbf{F}^p)^{-T}\\
&\partial_{\overset{(p)}{\boldsymbol{\Gamma}}}\bar{\psi} = -(\mathbf{F}^p)^T\partial_{\overset{(p)}{\boldsymbol{\Gamma}}_{\mathcal{K}}}\psi_{\mathcal{K}}[(\mathbf{F}^p)^{-T}, (\mathbf{F}^p)^{-T}],\\
&\partial_{\overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}}\psi_{\mathcal{K}} = (\mathbf{F}^p)^{-T}\partial_{\overset{(e)}{\boldsymbol{\Gamma}}_{back}}\bar{\psi}[(\mathbf{F}^p)^T, (\mathbf{F}^p)^T],\\
&\partial_{\mathbf{F}^p}\bar{\psi}(\mathbf{F}^p)^T\cdot\widetilde{\mathbf{L}}^p = -2\mathbf{C}^e\partial_{\mathbf{C}^e}\psi_{\mathcal{K}}\cdot\widetilde{\mathbf{L}}^p\\
&\quad - (\mathbf{F}^p)^{-T}\partial_{(\mathbf{F}^p)^{-1}}\psi_{\mathcal{K}}\cdot\widetilde{\mathbf{L}}^p\\
&\quad + \partial_{\overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}}\psi_{\mathcal{K}}\cdot\{\widetilde{\mathbf{L}}^p\,\overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}} - \overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}\,\widetilde{\mathbf{L}}^p - \overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}\,[\mathbf{I},\widetilde{\mathbf{L}}^p]\}\\
&\quad + \partial_{\overset{(p)}{\boldsymbol{\Gamma}}_{\mathcal{K}}}\psi_{\mathcal{K}}\cdot\{\widetilde{\mathbf{L}}^p\,\overset{(p)}{\boldsymbol{\Gamma}}_{\mathcal{K}} - \overset{(p)}{\boldsymbol{\Gamma}}_{\mathcal{K}}\,\widetilde{\mathbf{L}}^p - \overset{(p)}{\boldsymbol{\Gamma}}_{\mathcal{K}}\,[\mathbf{I},\widetilde{\mathbf{L}}^p]\}
\end{aligned} \tag{68}$$

The constitutive form of $\partial_{\mathbf{F}^p}\bar{\psi}$ following from (68) can be expressed through

$$\begin{aligned}
\partial_{\mathbf{F}^p}\bar{\psi} &= -2\mathbf{C}^e\partial_{\mathbf{C}^e}\psi_{\mathcal{K}} - (\mathbf{F}^p)^{-T}\partial_{(\mathbf{F}^p)^{-1}}\psi_{\mathcal{K}}\\
&\quad + \mathcal{D}\psi_{\mathcal{K}}(\overset{(e)}{\boldsymbol{\Gamma}})[\overset{(e)}{\boldsymbol{\Gamma}}] + \mathcal{D}\psi_{\mathcal{K}}(\overset{(p)}{\boldsymbol{\Gamma}})[\overset{(p)}{\boldsymbol{\Gamma}}].
\end{aligned} \tag{69}$$

Here the second order tensor denoted by $\mathcal{D}\psi_{\mathcal{K}}(\overset{(e)}{\boldsymbol{\Gamma}})[\overset{(e)}{\boldsymbol{\Gamma}}]$ is defined for any third order tensor field $\overset{(e)}{\boldsymbol{\Gamma}} \equiv \mathcal{X}$ as it follows

$$\begin{aligned}
\mathcal{D}\psi_{\mathcal{K}}(\mathcal{X})[\mathcal{X}] := \Bigg[&\frac{\partial\psi_{\mathcal{K}}}{\partial\mathcal{X}_{pjk}}\mathcal{X}_{sjk} - \mathcal{X}_{ijp}\frac{\partial\psi_{\mathcal{K}}}{\partial\mathcal{X}_{ijs}}\\
&- \mathcal{X}_{ipk}\frac{\partial\psi_{\mathcal{K}}}{\partial\mathcal{X}_{isk}}\Bigg]\mathbf{i}_p\otimes\mathbf{i}_s.
\end{aligned} \tag{70}$$

Based on the dissipation inequality, we formulate the *constitutive hypotheses*:

Ax.8 The microforces contain:

(1) a *dissipative part*,
(2) a non-dissipative part, which are derived from the free energy, the so-called *energetic microforces*,

and they are represented through

$$\begin{aligned}
&\frac{1}{\rho_{\mathcal{K}}}\boldsymbol{\Upsilon}^p_{\mathcal{K}} = 2\mathbf{C}^e(\mathbf{F}^p)\partial_{\mathbf{C}}\bar{\psi}(\mathbf{F}^p)^T + \partial_{\mathbf{F}^p}\bar{\psi}(\mathbf{F}^p)^T + Y_\gamma\widetilde{\mathbf{L}}^p\\
&\frac{1}{\rho_{\mathcal{K}}}\boldsymbol{\mu}^p_{\mathcal{K}} = (\mathbf{F}^p)^{-T}\partial_{\overset{(p)}{\boldsymbol{\Gamma}}}\bar{\psi}[(\mathbf{F}^p)^T, (\mathbf{F}^p)^T] + Y_\mu\nabla_{\mathcal{K}}\widetilde{\mathbf{L}}^p.
\end{aligned} \tag{71}$$

We **remark** that the non-dissipative parts of the microforces were derived from the free energy through (71), but only the last terms of the right sides of (71) are dissipative, because the free energy cannot depend on the rates.

Ax.9 The scalar constitutive functions Y_γ, Y_μ are defined in such a way to be compatible with the dissipation inequality

$$Y_\mu\nabla_{\mathcal{K}}\widetilde{\mathbf{L}}^p\cdot\nabla_{\mathcal{K}}\widetilde{\mathbf{L}}^p + Y_\gamma\widetilde{\mathbf{L}}^p\cdot\widetilde{\mathbf{L}}^p \geq 0. \tag{72}$$

Remark The presence of the non-dissipative part (the first term in the right-hand side, which has the significance of the Mandel's type stress) in the formula $(71)_1$ couples the macroscopic and microscopic forces.

Following Gurtin (2004), we introduce the *intensity for the accumulated effect* of the rate of plastic distortion and of the gradient of plastic distortion, through

$$d^p := \sqrt{\mathbf{L}^p \cdot \mathbf{L}^p + h^2 \nabla_{\mathcal{K}} \mathbf{L}^p \cdot \nabla_{\mathcal{K}} \mathbf{L}^p}, \tag{73}$$

and we define

$$Y_\gamma := Y(d^p), \quad Y_\mu := h^2 Y(d^p), \tag{74}$$

with h a length scale.

We resume the constitutive equations for macrostress and macrostress momentum, when $\mathbf{T}^* = -\{\mathbf{T}\}^a$,

$$\{\mathbf{T}\}^s = 2\rho \mathbf{F}^e \partial_{\mathbf{C}^e} \psi_{\mathcal{K}} (\mathbf{F}^e)^T, \quad \frac{1}{\rho_{\mathcal{K}}} \boldsymbol{\mu}_{\mathcal{K}} = \partial_{\boldsymbol{\Gamma}^e{}_{\mathcal{K}}} \psi_{\mathcal{K}}, \tag{75}$$

which can be equivalently represented as a consequence of the relationships derived in (68) through

$$\{\mathbf{T}\}^s = 2\rho \mathbf{F} \partial_{\mathbf{C}} \bar{\psi} \mathbf{F}^T, \quad \frac{1}{\rho} \boldsymbol{\mu} = \mathbf{F}^{-1} \partial_{\boldsymbol{\Gamma}^e{}_{\text{back}}} \bar{\psi} [\mathbf{F}^T, \mathbf{F}^T]. \tag{76}$$

The microforces and microstress momentum have been represented under the form of viscoplastic constitutive equations

$$\begin{aligned} \frac{1}{\rho_{\mathcal{K}}} \boldsymbol{\Upsilon}^p_{\mathcal{K}} &= 2\mathbf{C}^e (\mathbf{F}^p) \partial_{\mathbf{C}} \bar{\psi} (\mathbf{F}^p)^T + \partial_{\mathbf{F}^p} \bar{\psi} (\mathbf{F}^p)^T \\ &\quad + Y(d^p) \mathbf{L}^p \\ \frac{1}{\rho_{\mathcal{K}}} \boldsymbol{\mu}^p_{\mathcal{K}} &= (\mathbf{F}^p)^{-T} \partial_{\underset{\boldsymbol{\Gamma}}{(\mathrm{p})}} \bar{\psi} [(\mathbf{F}^p)^T, (\mathbf{F}^p)^T] \\ &\quad + h^2 Y(d^p) \nabla_{\mathcal{K}} \mathbf{L}^p, \end{aligned} \tag{77}$$

with $d^p := \sqrt{\mathbf{L}^p \cdot \mathbf{L}^p + h^2 \nabla_{\mathcal{K}} \mathbf{L}^p \cdot \nabla_{\mathcal{K}} \mathbf{L}^p}$. Thus the dissipation inequality (72) becomes

$$\begin{aligned} &Y(d^p) \mathbf{L}^p \cdot \mathbf{L}^p + h^2 Y(d^p) \nabla_{\mathcal{K}} \mathbf{L}^p \cdot \nabla_{\mathcal{K}} \mathbf{L}^p \\ &\equiv Y(d^p)(d^p)^2 \geq 0, \end{aligned} \tag{78}$$

under the supposition that $Y(d^p) \geq 0$.

Let us remark that the stress

$$\overline{\boldsymbol{\Upsilon}}^p_{\mathcal{K}} := \boldsymbol{\Upsilon}^p_{\mathcal{K}} - \boldsymbol{\Sigma}^p_{\mathcal{K}} \tag{79}$$

leads to the *viscoplastic constitutive equations*

$$\begin{aligned} \frac{1}{\rho_{\mathcal{K}}} \overline{\boldsymbol{\Upsilon}}^p_{\mathcal{K}} &= \partial_{\mathbf{F}^p} \bar{\psi} (\mathbf{F}^p)^T + Y(d^p) \mathbf{L}^p \\ \frac{1}{\rho_{\mathcal{K}}} \boldsymbol{\mu}^p_{\mathcal{K}} &= (\mathbf{F}^p)^{-T} \partial_{\underset{\boldsymbol{\Gamma}}{(\mathrm{p})}} \bar{\psi} [(\mathbf{F}^p)^T, (\mathbf{F}^p)^T] \\ &\quad + h^2 Y(d^p) \nabla_{\mathcal{K}} \widetilde{\mathbf{L}}^p. \end{aligned} \tag{80}$$

The micro balance Equation $(53)_1$ involves the *material forces* and finally it is written under the form

$$\overline{\boldsymbol{\Upsilon}}^p_{\mathcal{K}} = \boldsymbol{\Sigma}_{\mathcal{K}} + \operatorname{div} \boldsymbol{\mu}^p_{\mathcal{K}} + \tilde{\rho} \mathbf{B}^p_m, \quad \text{in } \mathcal{K}(\mathcal{P}, t). \tag{81}$$

6 Screw dislocations

We recall here the definitions of the *Burgers vector* $\mathbf{b}_{\mathcal{K}}$ adopted in our context of finite elasto-plasticity, first in terms of plastic distortion $\mathbf{F}^p$.

Definition For a given $\mathcal{A}_0$ surface with normal $\mathbf{N}$ bounded by C_0, a closed curve in the reference configuration, we introduce the definition

$$\begin{aligned} \mathbf{b}_{\mathcal{K}} &= \int_{C_0} \mathbf{F}^p \, d\mathbf{X} = \int_{\mathcal{A}_0} \operatorname{curl} (\mathbf{F}^p) \mathbf{N} dA \\ &= \int_{\mathcal{A}_{\mathcal{K}}} \boldsymbol{\alpha}_{\mathcal{K}} \mathbf{n}_{\mathcal{K}} dA_{\mathcal{K}}, \end{aligned} \tag{82}$$

where $\mathbf{N}$ is the unit normal on the surface in the reference configuration and Noll's dislocation (second order) tensor is given in terms of the plastic distortion

$$\boldsymbol{\alpha}_{\mathcal{K}} \equiv \frac{1}{\det \mathbf{F}^p} \operatorname{curl}(\mathbf{F}^p)(\mathbf{F}^p)^T. \tag{83}$$

The meaning of Burgers vector clearly appears from the approximate formula, derived from (82)

$$\mathbf{b}_{\mathcal{K}} \simeq \operatorname{curl} (\mathbf{F}^p) \mathbf{N} \operatorname{area}(\mathcal{A}_0), \tag{84}$$

with the functions calculated in an appropriate point.

A similar **definition** for the *Burgers vector* $\mathbf{b}_{\mathcal{K}}$ can be also done in terms of elastic distortion $\mathbf{F}^e$ and the approximate formula can be derived under the form $\mathbf{b}_{\mathcal{K}} \simeq \operatorname{curl} (\mathbf{F}^e)^{-1} \mathbf{n} \operatorname{area} (A_t)$, with the functions calculated in a fixed point inside A_t—an appropriate surface in the actual configuration.

The geometric dislocation tensor $\mathbf{G}$, which represents the Noll's second order dislocation tensor, $\mathbf{G} = \bar{\boldsymbol{\alpha}}_{\mathcal{K}} = \boldsymbol{\alpha}_{\mathcal{K}}$, is decomposed by Gurtin (2002) in pure *screw* and *edge dislocations*, corresponding to different slip system within the constitutive framework of Crystal plasticity.

Remark As it follows from the characteristics attributed to the *edge dislocation* (Hirth and Lothe 1982), the Burgers vector is defined by

$$\begin{aligned} &\mathbf{b} = (\operatorname{curl}(\mathbf{F}^p)) \, \mathbf{N}(\operatorname{area} \mathcal{A}_0) \quad \text{with} \\ &\operatorname{curl} (\mathbf{F}^p) = a^p (\mathbf{e}_3 \otimes \mathbf{e}_1) \end{aligned} \tag{85}$$

where the unit vectors $\mathbf{e}$ are chosen in a such way to have $\mathbf{e}_1 = \mathbf{N}$, $\mathbf{e}_3$ is the Burgers direction, $\mathbf{e}_3 = \dfrac{\mathbf{b}}{|\,\mathbf{b}\,|}$, $\mathbf{e}_2 \perp \mathbf{b}$.

Consequently the edge dislocation is characterized by the *curl of plastic distortion* of the form

$$\operatorname{curl} (\mathbf{F}^p) = \left(\frac{\partial F^p_{32}}{\partial x^3} - \frac{\partial F^p_{33}}{\partial x^2} \right) (\mathbf{e}_3 \otimes \mathbf{e}_1) \tag{86}$$

Let us **remark** that if there exists *a potential for plastic deformation*

$$F^p_{32} = \frac{\partial u^3}{\partial x^2}, \quad F^p_{33} = \frac{\partial u^3}{\partial x^3} \quad \text{with} \quad u^3 = u^3(x^2, x^3) \equiv \mathbf{u} \cdot \frac{\mathbf{b}}{|\,\mathbf{b}\,|}, \tag{87}$$

which can be associated to a certain *plane motion* of dislocation, then in order to have non-zero *curl* it is necessary to accept a non-simply arc wise connected physical domain in a certain neighborhood of the given material point.

Following again (Hirth and Lothe 1982) the *screw dislocation* is characterized by a Burgers vector with the property listed below

$$\mathbf{b} = (\text{curl } (\mathbf{F}^p))\, \mathbf{N}\, (\text{area} \mathcal{A}_0) \quad \text{for} \quad \text{curl } (\mathbf{F}^p) = a^p (\mathbf{b} \otimes \mathbf{b}), \tag{88}$$

$$\mathbf{N} = \mathbf{e}_3, \quad \mathbf{e}_3 = \frac{\mathbf{b}}{|\,\mathbf{b}\,|}.$$

Let us characterize the *plastic curl* in this case

$$\text{curl } (\mathbf{F}^p) = \left(\frac{\partial F^p_{31}}{\partial x^2} - \frac{\partial F^p_{32}}{\partial x^1} \right) (\mathbf{e}_3 \otimes \mathbf{e}_3) \tag{89}$$

If there exists a potential for plastic deformation, defining the only two components F^p_{31}, F^p_{32} of the plastic distortion supposed to be non-zero, then

$$F^p_{31} = \frac{\partial u^3}{\partial x^1}, \quad F^p_{32} = \frac{\partial u^3}{\partial x^2} \quad \text{with} \quad u^3 = u^3(x^1, x^2) \equiv \mathbf{u} \cdot \frac{\mathbf{b}}{|\,\mathbf{b}\,|}, \tag{90}$$

which means that an anti-plane motion of dislocation has been considered. Again, in order to have non-vanishing *curl*, the physical space in a certain neighborhood of the material point is locally a simply arc wise connected domain.

6.1 Characteristics of the plastic distortion in the case of screw dislocation

The *simplest form* of the plastic distortion, compatible with the characterization given for the *screw dislocation* can be represented by the matrix

$$\mathbf{F}^p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ F^p_{31} & F^p_{32} & 1 \end{pmatrix}, \tag{91}$$

in the Cartesian basis denoted by $\{\mathbf{e}_j\}_{\{j=1,3\}}$, where F^p_{31}, F^p_{32} are functions of (x^1, x^2), and

$$\frac{\partial F^p_{31}}{\partial x^2} - \frac{\partial F^p_{32}}{\partial x^1} \neq 0, \quad \det \mathbf{F}^p = 1. \tag{92}$$

We kept the same notation for the tensor and its matrix representation, in a certain mentioned basis.

The plastic metric tensor $\mathbf{C}^p$ associated to the plastic distortion (91) is given by

$$\mathbf{C}^p = \begin{pmatrix} 1 + (\gamma_1)^2 & \gamma_1\gamma_2 & \gamma_1 \\ \gamma_1\gamma_2 & 1 + \gamma_2^2 & \gamma_2 \\ \gamma_1 & \gamma_2 & 1 \end{pmatrix}, \tag{93}$$

with the notation $\quad F^p_{31} = \gamma_1, \quad F^p_{32} = \gamma_2.$

Let us introduce the function $\boldsymbol{\gamma}$

$$\boldsymbol{\gamma} = \gamma_1 \mathbf{e}_1 + \gamma_2 \mathbf{e}_2 \quad \text{with} \quad \gamma_1 = \gamma_1(x^1, x^2), \quad \gamma_2 = \gamma_2(x^1, x^2). \tag{94}$$

From (91) together with (94) we get

$$\mathbf{C}^p = \mathbf{I} + \boldsymbol{\gamma} \otimes \boldsymbol{\gamma} + \boldsymbol{\gamma} \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \boldsymbol{\gamma}. \tag{95}$$

Let us introduce a local basis $(\boldsymbol{\mu}, \boldsymbol{\nu}, \mathbf{e}_3)$

$$\boldsymbol{\nu} = \frac{\boldsymbol{\gamma}}{|\,\boldsymbol{\gamma}\,|}, \quad \text{for} \quad |\,\boldsymbol{\gamma}\,| \equiv \sqrt{\gamma_1^2 + \gamma_2^2}, \quad \boldsymbol{\mu} \in (\mathbf{e}_1, \mathbf{e}_2), \quad \text{such that} \quad \boldsymbol{\mu} \cdot \boldsymbol{\nu} = 0. \tag{96}$$

From (91) together with (94) we get

$$\mathbf{C}^p = \boldsymbol{\mu} \otimes \boldsymbol{\mu} + \mathbf{A}^p \quad \text{with} \quad \mathbf{A}^p = (1 + |\,\boldsymbol{\gamma}\,|^2)\boldsymbol{\nu} \otimes \boldsymbol{\nu} + |\,\boldsymbol{\gamma}\,| (\boldsymbol{\nu} \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \boldsymbol{\nu}) + \mathbf{e}_3 \otimes \mathbf{e}_3. \tag{97}$$

The second order tensor $\mathbf{A}^p$ has the matrix representation in the basis $\{\boldsymbol{\nu}, \mathbf{e}_3\}$

$$\mathbf{A}^p = \begin{pmatrix} 1 + |\,\boldsymbol{\gamma}\,|^2 & |\,\boldsymbol{\gamma}\,| \\ |\,\boldsymbol{\gamma}\,| & 1 \end{pmatrix}. \tag{98}$$

We define the positive square root tensor $(\mathbf{A}^p)^{1/2}$ and the symmetric and positive definite tensor $\mathbf{U}^p$.

Proposition 9 *$(\mathbf{A}^p)^{1/2}$ is defined by*

$$(\mathbf{A}^p)^{1/2} = \frac{1}{\sqrt{4 + |\,\boldsymbol{\gamma}\,|^2}} \left(\mathbf{A}^p + \hat{\mathbf{I}}_2 \right), \quad \hat{\mathbf{I}}_2 = \boldsymbol{\nu} \otimes \boldsymbol{\nu} + \mathbf{e}_3 \otimes \mathbf{e}_3 \tag{99}$$

or in a matrix representation

$$(\mathbf{A}^p)^{1/2} = \frac{1}{\sqrt{4 + |\,\boldsymbol{\gamma}\,|^2}} \begin{pmatrix} 2 + |\,\boldsymbol{\gamma}\,|^2 & |\,\boldsymbol{\gamma}\,| \\ |\,\boldsymbol{\gamma}\,| & 2 \end{pmatrix}. \tag{100}$$

The symmetric and positive definite tensor $\mathbf{U}^p$ can be expressed under the form

$$\mathbf{U}^p = \boldsymbol{\mu} \otimes \boldsymbol{\mu} + (\mathbf{A}^p)^{1/2}. \tag{101}$$

Finally, we prove the formula

$$
\begin{aligned}
\mathbf{U}^p \equiv (\mathbf{C}^p)^{1/2} &= \boldsymbol{\mu}\otimes\boldsymbol{\mu} \\
&+\frac{1}{\sqrt{4+\mid\boldsymbol{\gamma}\mid^2}}(2+\mid\boldsymbol{\gamma}\mid^2)\boldsymbol{\nu}\otimes\boldsymbol{\nu} \\
&+\frac{\mid\gamma\mid}{\sqrt{4+\mid\boldsymbol{\gamma}\mid^2}}(\boldsymbol{\nu}\otimes\mathbf{e}_3+\mathbf{e}_3\otimes\boldsymbol{\nu}) \\
&+\frac{2}{\sqrt{4+\mid\boldsymbol{\gamma}\mid^2}}\mathbf{e}_3\otimes\mathbf{e}_3 \\
&\text{where}\quad \boldsymbol{\nu}=\frac{\boldsymbol{\gamma}}{\mid\boldsymbol{\gamma}\mid},\quad \boldsymbol{\mu}\cdot\boldsymbol{\nu}=0,\quad \boldsymbol{\mu}\cdot\mathbf{e}_3=0.
\end{aligned}
\tag{102}
$$

Proof We write the Hamilton-Caley theorem for the tensor fields $(\mathbf{A}^p)^{1/2}$

$$\mathbf{A}^p-(\mathbf{A}^p)^{1/2}(\mathrm{tr}(\mathbf{A}^p)^{1/2})+\sqrt{(\det\mathbf{A}^p)}\hat{\mathbf{I}}_2=0 \tag{103}$$

When we apply the trace operator in (103) we get

$$(\mathrm{tr}(\mathbf{A}^p)^{1/2})^2=(\mathrm{tr}\mathbf{A}^p)+2\sqrt{(\det\mathbf{A}^p)} \tag{104}$$

and consequently

$$(\mathbf{A}^p)^{1/2}=\frac{1}{\mathrm{tr}(\mathbf{A}^p)^{1/2}}\left(\mathbf{A}^p+\sqrt{(\det\mathbf{A}^p)}\hat{\mathbf{I}}_2\right). \tag{105}$$

Using the explicit expression of the trace, the formula (99) follows at once.

Proposition 10

1. *The torsion of the plastic connection, attached to the plastic connection in the reference configuration , can be expressed in term of ω, by*

$$\mathbf{S}=\omega\mathbf{e}_3\otimes[\mathbf{e}_2\otimes\mathbf{e}_1-\mathbf{e}_1\otimes\mathbf{e}_2]. \tag{106}$$

2. *The second order torsion tensor is derived under the form*

$$\mathbf{N}=\omega\mathbf{e}_3\otimes\mathbf{e}_3,\quad\text{with}\quad \omega=\frac{\partial\gamma_2}{\partial x_1}-\frac{\partial\gamma_1}{\partial x_2}. \tag{107}$$

Proof By direct calculus we derive the expression of the plastic connection

$$
\begin{aligned}
\boldsymbol{\Gamma}^p \equiv (\mathbf{F}^p)^{-1}(\nabla\mathbf{F}^p) &= \mathbf{e}_3\otimes\mathbf{e}_1\otimes(\nabla\gamma_1) \\
&+\mathbf{e}_3\otimes\mathbf{e}_2\otimes(\nabla\gamma_2)
\end{aligned}
\tag{108}
$$

A full characterization of the edge dislocation as well as the appropriate compatibility condition can be found in Cleja-Ţigoiu et al. (2007, in press).

Concluding remarks.

In order to compare the presented here results with the existing in the literature results in the field, we put into evidence different internal power expressions, proposed by Gurtin.

a. The additive decomposition of the displacement vector $\nabla\mathbf{u}=\mathbf{H}^e+\mathbf{H}^p$ into elastic and plastic parts is accepted in Gurtin (2003), within the framework of small deformation theory. The power expended in terms of an appropriate force system is given under the form $\int_{\mathcal{P}}(\mathbf{T}\cdot\dot{\mathbf{H}}^e+\mathbf{T}^p\cdot\dot{\mathbf{H}}^p+\mathcal{S}^p\cdot\nabla\dot{\mathbf{H}}^p)d\mathbf{x}$. Here $\mathbf{T}^p$ a second-order microstress and $\mathcal{S}^p$ a polar (third-order) microstress that together perform work in the evolution of the defects through this structure, for any part $\mathcal{P}$ of the body.
b. In Gurtin (2004) the power expended within any part $\mathcal{P}$ has the form $\int_{\mathcal{P}}(\mathbf{T}\cdot\dot{\mathbf{E}}^e+\mathbf{T}^p\cdot\dot{\mathbf{H}}^p+\mathcal{S}^p\cdot\mathrm{curl}\dot{\mathbf{H}}^p)d\mathbf{x}$. Here $\mathbf{E}^e=\{\mathbf{H}^e\}^S$, and the microscopic stress performs work in conjunction with the rate of Burgers vector, which is characterized by $\mathbf{G}=\mathrm{curl}(\mathbf{H}^p)$
c. Within the constitutive framework of Crystal plasticity, based on the multiplicative decomposition of the deformation gradient, the tensor field $\mathbf{G}=1/(\det\mathbf{F}^p)\mathbf{F}^p\mathrm{curl}(\mathbf{F}^p)$ (i.e just Noll's dislocation density, in Noll (1967), here in terms of the plastic distortion $\mathbf{F}^p$) is considered by Gurtin (2000) to be a measure of geometrically necessary dislocations. In this case the internal power is written in the form $\int_{\mathcal{P}}(\mathbf{T}\cdot\mathbf{L}^e+\sum_\alpha(\pi^\alpha v^\alpha+\xi^\alpha\cdot\mathrm{grad}v^\alpha))d\mathbf{x}$, where the sum is espanded to the slip systems and for each α, π^α—internal microforces and ξ^α—microstresses are introduced as forces conjugated to slip and they produce work by slip and by slip gradient respectively.

We conclude

1. As a peculiar aspect of the models proposed by Gurtin in the mentioned papers, the microbalance equation *generates the viscoplastic yield conditions* or the viscoplastic flow rules. On the other hand *no yield criteria* has been introduced, and the irreversible behavior can develop at the very beginning.
2. Let us remark that the micro balance equation together with the viscopalstic constitutive equation for the microforces and microstress momentum generate an appropriate *flow rule* (see also the non-local yield condition in Gurtin (2000))

$$
\begin{aligned}
&2\mathbf{C}^e(\mathbf{F}^p)\partial_{\mathbf{C}}\bar{\psi}(\mathbf{F}^p)^T+\partial_{\mathbf{F}^p}\bar{\psi}(\mathbf{F}^p)^T+Y(d^p)\mathbf{L}^p \\
&\quad-\mathrm{div}_{\mathcal{K}}((\mathbf{F}^p)^{-T}\partial_{\underset{\boldsymbol{\Gamma}}{(\mathrm{p})}}\bar{\psi}[(\mathbf{F}^p)^T,(\mathbf{F}^p)^T] \\
&\quad+h^2Y(d^p)\nabla_{\mathcal{K}}\widetilde{\mathbf{L}}^p)=\tilde{\rho}\mathbf{B}^p_m,
\end{aligned}
\tag{109}
$$

with $d^p := \sqrt{\mathbf{L}^p \cdot \mathbf{L}^p + h^2 \nabla_{\mathcal{K}} \mathbf{L}^p \cdot \nabla_{\mathcal{K}} \mathbf{L}^p}$, for the plastic incompressible case $\rho_{\mathcal{K}} = \rho_0$.

3. Following the methodology developed in Cleja-Ţigoiu (2002b), we can derive from the proposed here model the behavior of the elasto-plastic material, in the case of small elastic strains but large elastic rotation $\mathbf{R^e}$. In this case $\mathbf{C}^e \simeq \mathbf{I} + 2\boldsymbol{\epsilon}$ with the elastic strain $\mid \boldsymbol{\epsilon} \mid << 1$, and the connection is generated by the gradient of the elastic rotation as well as the $\nabla\boldsymbol{\epsilon}$. Moreover, when we restrict ourselves to small elastic and plastic deformations, i.e. the small rotation and small strains, model within the constitutive framework adopted by Gurtin (2000) follows, but the microstress momentum are still presented.
4. The *plasticity and the damage defects* localize over the narrow region of the material. The plastic and the damage evolution process are inhomogeneous at the macroscale. The macroscopic inelastic deformation are sensitive to the structural defects within the volume element. Hence, *nonlocal theories are necessary* to adequately take into account mechanisms which take place in the neighborhood of the considered material points (see the comments by Brünig and Ricci (2005)). A nonlocal theory of an isotropically damaged materials can be developed based on our proposed model, in order to characterize the damage state configuration (identified with a configuration with torsion), independently of the current elastic deformation, i.e. for $\mathbf{C}^e = \mathbf{I}$. The damage produced by the microdefects can be identified with the presence of screw and/or edge dislocations.
5. The proposed description of the edge and screw dislocations (corresponding to deformation fields, which remain *incompatible* due to the non-zero *curl*$\mathbf{F}^p$) can be utilized in order to describe the appropriate fracture mode, by plane deformation and anti-plane deformation. The non-zero Burgers vector can be utilized in order to describe the cracks motion.
6. The quantities with the dimension of length appears as additional material parameters, in the expression of the free energy density, in the *intensity for the accumulated effect* of the rate of plastic distortion $\mathbf{L}^p$ and of the gradient of $\mathbf{L}^p$.
7. In the model (although at the general framework presented here there is not necessary) we can assume the existence of the viscoplastic (or an yield) function defined in the physical force system $(\mathbf{T}, \boldsymbol{\mu})$, such that the plastic (viscoplastic) behavior can develope if and only if during the deformation process the physical force system lays on the yield surface or it is situated outward the surface $f_{\mathcal{K}}(\mathbf{T}, \boldsymbol{\mu}) \geq 0$. Consequently the behavior would be elastic if the physical force system remains inside the yield surface.

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