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ORIGINAL ARTICLE

Fracture parameters of a penny-shaped crack at the interface of a piezoelectric bi-material system

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Abstract A penny-shaped crack at the interface of a piezoelectric bi-material system is considered. Analytical general solutions based on Hankel integral transforms are used to formulate the mixed-boundary value problem corresponding to an interfacial crack and the problem is reduced to a system of singular integral equations. The integral equations are further reduced to two systems of algebraic equations with the aid of Jacobi polynomials and Chebyshev polynomials. Thereafter, the exact expressions for the stress intensity factors and the electric displacement intensity factor at the tip of a crack are obtained. Selected numerical results are presented for various bi-material systems to portray the significant features of crack tip fracture parameters and their dependence on material properties, poling orientation and electric loading.

Keywords Cracks · Electric field · Fracture mechanics · Piezoelectricity · Stress intensity factors

1 Introduction

Piezoelectric actuators are used in many advanced engineering applications. Common examples include ink-jet printer heads, fuel injectors, ultra precision machine tool positioning devices, etc. This class of materials is generally brittle, has low fracture toughness and possesses micro-defects produced during the manufacturing process. These factors make fracture and reliability key issues in the design of piezoelectric devices for various engineering applications. Study of linear fracture mechanics of piezoelectric materials is quite well established and numerous papers have appeared in the literature. A comprehensive review of past studies is beyond the scope of this paper. It is suffice to refer to a few key papers to highlight the development over the past few decades (Parton 1976; Deeg 1980; McMeeking 1989, 2001; Sosa 1992; Pak 1992; Suo et al. 1992; Xu and Rajapakse 2001).

A majority of studies dealing with fracture of piezoelectric materials is based on two-dimensional models and homogeneous materials. In practical situations dealing with cylindrical actuators it is more common to encounter cracks that are circular, elliptic or arbitrary shaped and located at material interfaces. For example, ultra-thin electrodes are placed between thin piezoelectric layers with adjacent layers having opposite polarization in the case of commonly used cylindrical stack actuators (Fig. 1). Complex crack patterns exist in such devices that include electrode delamination cracks, branch cracks at electrode tips, delamination cracks between piezoelectric layers, electrode bridging cracks through a piezoelectric layer, etc. Needless to say, the development of mathematical models for such complex cases is a challenging task.

A few studies dealing with penny-shaped cracks in piezoelectric materials have appeared in the literature

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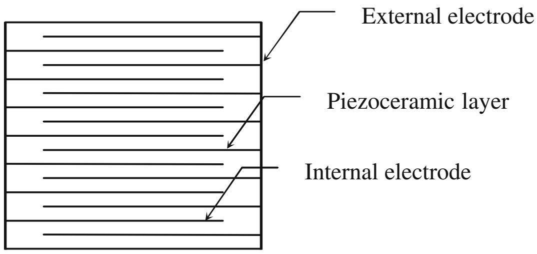

Fig. 1 Piezoelectric stack actuator

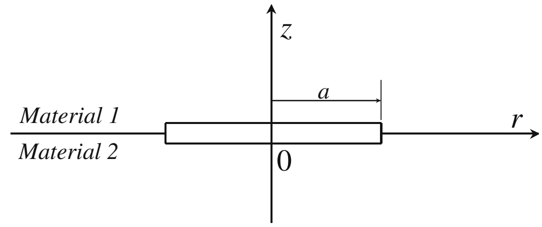

Fig. 2 A penny-shaped crack at the interface of a piezoelectric bi-material system

over the past 10 years. Wang (1994) and Kogan et al. (1996) analyzed the case of a penny-shaped crack in a transversely isotropic piezoelectric material. Huang (1997) also considered a penny-shaped crack in a transversely isotropic piezoelectric medium. Chen and Shioya (1999) presented a general three-dimensional analysis of a penny-shaped crack subjected to normal mechanical loads and electric charges symmetrically applied on the upper and lower surfaces. Wang (1992) and Wang and Huang (1995) considered an elliptical crack in a piezoelectric medium. Yang and Lee (2001, 2003) studied a penny-shaped crack in a piezoelectric strip and a cylinder. Three-dimensional cracks of different geometry were considered by Shang et al. (2003) by using the finite element method.

An interesting problem related to a penny-shaped crack is the case where such a crack lies in the interface between two piezoelectric materials. Such a case represents an idealized mathematical model for an interface crack in a stack actuator and could shed some insight into fracture mechanics of stack actuators. Bi-material crack problems in ideal elasticity have a rich history and a comprehensive review is beyond the scope of this paper. The studies by Erdogan (1965), Willis (1971), Erdogan and Arin (1972), Comninou (1977) and Rice (1988) are some key references. Two-dimensional interfacial cracks in dissimilar piezoelectric materials were considered by Beom and Atluri (1996), Gao and Wang (2000), Qin and Mai (2000), Ru (2000), Nishioka and Shen (2001) and Scherzer and Kuna (2004). However, the more practically useful case of an interfacial penny-shaped crack in a piezoelectric bi-material system is not yet solved.

The objective of this paper is to analyze by using analytical techniques the axi-symmetric problem of an interfacial penny-shaped crack in a piezoelectric bi-material system (Fig. 2). Analytical general solutions based on Hankel integral transforms are used to formulate the mixed-boundary value problem corresponding to an interfacial crack and the problem is reduced to a system of singular integral equations. The integral equations are further reduced to two systems of algebraic equations with the aid of Jacobi polynomials and Chebyshev polynomials. Exact expressions for the stress intensity factors and the electric displacement intensity factor of the crack tip are thereafter obtained. Selected numerical results are presented for various bi-material systems to portray the significant features of crack tip fracture parameters and their dependence on material properties and poling orientation.

2 Axisymmetric general solution

Consider a transversely isotropic piezoelectric material with a cylindrical polar coordinate system (r, θ, z) defined with (r, θ) as the plane of isotropy and z-axis as the poling direction.

In the case of axisymmetric deformations, the elastic displacement components (u_r, u_z) and the electric potential ϕ are functions of only r and z and $u_\theta \equiv 0$. Constitutive equations can be expressed in terms of the elastic displacements and electric potential as (Parton and Kudryavtsev, 1988)

$$
\begin{aligned}
\sigma_{rr} &= c_{11}\frac{\partial u_r}{\partial r} + c_{12}\frac{u_r}{r} + c_{13}\frac{\partial u_z}{\partial z} + e_{31}\frac{\partial \phi}{\partial z} \\
\sigma_{\theta\theta} &= c_{12}\frac{\partial u_r}{\partial r} + c_{11}\frac{u_r}{r} + c_{13}\frac{\partial u_z}{\partial z} + e_{31}\frac{\partial \phi}{\partial z} \\
\sigma_{xx} &= c_{13}\frac{\partial u_r}{\partial r} + c_{13}\frac{u_r}{r} + c_{33}\frac{\partial u_z}{\partial z} + e_{33}\frac{\partial \phi}{\partial z} \\
\sigma_{rz} &= c_{44}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) + e_{15}\frac{\partial \phi}{\partial r} \\
D_z &= e_{31}\frac{\partial u_r}{\partial r} + e_{31}\frac{u_r}{r} + e_{33}\frac{\partial u_z}{\partial z} - d_{33}\frac{\partial \phi}{\partial z} \\
D_r &= e_{15}\left(\frac{u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) - d_{11}\frac{\partial \phi}{\partial r}
\end{aligned}
\tag{1}
$$

where $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}$ and σ_{rz} denote stresses; D_r and D_z denote electric displacements; $c_{11}, c_{12}, c_{13}, c_{33}$ and c_{44} denote the elastic moduli; e_{31}, e_{33} and e_{15} denote the piezoelectric constants; and d_{11} and d_{33}denote the dielectric permeability coefficients.

In the absence of body forces and body electric charges, the equilibrium equations can be expressed in terms of the elastic displacements and electric potential as

$$\begin{aligned}
& c_{11}\left(\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}\right)+c_{44}\frac{\partial^2 u_r}{\partial z^2} \\
& \quad +(c_{13}+c_{44})\frac{\partial^2 u_z}{\partial r\partial z}+(e_{31}+e_{15})\frac{\partial^2 \phi}{\partial r\partial z}=0 \\
& (c_{13}+c_{44})\left(\frac{\partial^2 u_r}{\partial r\partial z}+\frac{1}{r}\frac{\partial u_r}{\partial z}\right)+c_{44}\left(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}\right) \\
& \quad +c_{33}\frac{\partial^2 u_z}{\partial z^2}+e_{15}\left(\frac{\partial^2 \phi}{\partial r^2}+\frac{1}{r}\frac{\partial \phi}{\partial r}\right)+e_{33}\frac{\partial^2 \phi}{\partial z^2}=0 \\
& (e_{15}+e_{31})\left(\frac{\partial^2 u_r}{\partial r\partial z}+\frac{1}{r}\frac{\partial u_r}{\partial z}\right)+e_{15}\left(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}\right) \\
& \quad +e_{33}\frac{\partial^2 u_z}{\partial z^2}-d_{11}\left(\frac{\partial^2 \phi}{\partial r^2}+\frac{1}{r}\frac{\partial \phi}{\partial r}\right)-d_{33}\frac{\partial^2 \phi}{\partial z^2}=0
\end{aligned} \tag{2}$$

In order to solve the equation (2), the following generalized displacement potential relations are defined (Rajapakse and Zhou 1997).

$$u_r=\sum_{i=1}^{3}\frac{\partial \Phi_i(r,z)}{\partial r},\ u_z=\sum_{i=1}^{3}\lambda_{1i}\frac{\partial \Phi_i(r,z)}{\partial z},$$
$$\phi=-\sum_{i=1}^{3}\lambda_{2i}\frac{\partial \Phi_i(r,z)}{\partial z} \tag{3}$$

where $\Phi_i(r,z)(i=1,2,3)$ denotes generalized displacement potential functions and λ_{1i} and λ_{2i} $(i=1,2,3)$ are unknown constants. Substitution of Eq. (3) into Eq. (2) yields,

$$\frac{\partial^2 \Phi_i(r,z)}{\partial r^2}+\frac{1}{r}\frac{\partial \Phi_i(r,z)}{\partial r}+\frac{\partial^2 \Phi_i(r,z)}{\partial (z_i)^2}=0,\quad (i=1,2,3) \tag{4}$$

where,

$$z_i=\frac{z}{\sqrt{n_i}}=\mu_i z,\quad (i=1,2,3) \tag{5}$$

and $n_i(i=1,2,3)$ are the roots of the following equation.

$$L_1(n_i)^3+L_2(n_i)^2+L_3 n_i+L_4=0 \tag{6}$$

where

$$\begin{aligned}
L_1 &= c_{11}c_{44}d_{11}+c_{11}(e_{15})^2 \\
L_2 &= 2c_{13}e_{15}e_{31}-c_{44}(e_{31})^2-2c_{11}e_{15}e_{33} \\
&\quad +2c_{13}(e_{15})^2+(c_{13})^2 d_{11}-c_{11}c_{33}d_{11} \\
&\quad +2c_{13}c_{44}d_{11}-c_{11}c_{44}d_{33} \\
L_3 &= 2c_{33}e_{15}e_{31}+c_{33}(e_{31})^2-2c_{13}e_{15}e_{33} \\
&\quad -2c_{13}e_{31}e_{33}-2c_{44}e_{31}e_{33} \\
&\quad +c_{11}(e_{33})^2+c_{33}(e_{15})^2+c_{33}c_{44}d_{11} \\
&\quad -(c_{13})^2 d_{33}+c_{11}c_{33}d_{33}-2c_{13}c_{44}d_{33} \\
L_4 &= -c_{44}c_{33}d_{33}-c_{44}(e_{33})^2
\end{aligned} \tag{7}$$

The constants λ_{1i} and λ_{2i} $(i=1,2,3)$ are determined from the following relationship.

$$\begin{aligned}
n_i &= \frac{c_{44}+(c_{13}+c_{44})\lambda_{1i}-(e_{31}+e_{15})\lambda_{2i}}{c_{11}} \\
&= \frac{c_{33}\lambda_{1i}-e_{33}\lambda_{2i}}{c_{44}\lambda_{1i}+c_{13}+c_{44}-e_{15}\lambda_{2i}} \\
&= \frac{e_{33}\lambda_{1i}+d_{33}\lambda_{2i}}{e_{15}\lambda_{1i}+e_{15}+e_{31}+d_{11}\lambda_{2i}}
\end{aligned} \tag{8}$$

The solution for $\Phi_i(r,z)$ can be expressed in the following form by using Hankel integral transforms.

$$\Phi_i(r,z)=\int_0^\infty \frac{1}{\xi}[A_i(\xi)\exp(-\xi\mu_i z)+B_i(\xi)\exp(\xi\mu_i z)]J_0(\xi r)\mathrm{d}\xi \tag{9}$$

where $A_i(\xi)$ and $B_i(\xi)$denotes a set of arbitrary functions to be determined from the boundary and continuity conditions, and $J_0()$ is the Bessel function of the first kind of order zero (Abramowitz and Stegun 1964).

In view of Eqs. (9) and (3), the general solutions for displacements and electric field can be expressed as,

$$\begin{aligned}
u_r(r,z) &= \sum_{i=1}^{3}\frac{\partial \Phi_i(r,z)}{\partial r} \\
&= -\sum_{i=1}^{3}\left[\int_0^\infty\left[A_i(\xi)\exp(-\xi\mu_i z)\right.\right. \\
&\qquad \left.\left.+B_i(\xi)\exp(\xi\mu_i z)\right]J_1(\xi r)\mathrm{d}\xi\right] \\
u_z(r,z) &= \sum_{i=1}^{3}\lambda_{1i}\frac{\partial \Phi_i(r,z)}{\partial z} \\
&= -\sum_{i=1}^{3}\left[\lambda_{1i}\mu_i\int_0^\infty (A_i(\xi)\exp(-\xi\mu_i z)\right. \\
&\qquad \left.-B_i(\xi)\exp(\xi\mu_i z))\,J_0(\xi r)\mathrm{d}\xi\right]
\end{aligned}$$

$$\phi(r,z) = -\sum_{i=1}^{3} \lambda_{2i} \frac{\partial \Phi_i(r,z)}{\partial z} = \sum_{i=1}^{3} \left[\lambda_{2i}\mu_i \int_0^\infty (A_i(\xi)\exp(-\xi\mu_i z) - B_i(\xi)\exp(\xi\mu_i z))\, J_0(\xi r)\mathrm{d}\xi \right] \quad (10)$$

The general solutions for relevant components of stress and electric displacements can be expressed as,

$$\sigma_{zz} = \sum_{i=1}^{3} \left[\int_0^\infty \xi H_{1i} \Big(A_i(\xi)\exp(-\xi\mu_i z) + B_i(\xi)\exp(\xi\mu_i z) \Big) J_0(\xi r)\mathrm{d}\xi \right]$$
$$\sigma_{rz} = \sum_{i=1}^{3} \left[\int_0^\infty \xi H_{2i} \Big(A_i(\xi)\exp(-\xi\mu_i z) - B_i(\xi)\exp(\xi\mu_i z) \Big) J_1(\xi r)\mathrm{d}\xi \right] \quad (11)$$
$$D_z = \sum_{i=1}^{3} \left[\int_0^\infty \xi H_{3i} \Big(A_i(\xi)\exp(-\xi\mu_i z) + B_i(\xi)\exp(\xi\mu_i z) \Big) J_0(\xi r)\mathrm{d}\xi \right]$$

where

$$H_{1i} = (c_{33}\lambda_{1i} - e_{33}\lambda_{2i})(\mu_i)^2 - c_{13}$$
$$H_{2i} = (c_{44}\lambda_{1i} + c_{44} - e_{15}\lambda_{2i})\mu_i \quad (i = 1,2,3) \quad (12)$$
$$H_{3i} = (d_{33}\lambda_{2i} + e_{33}\lambda_{1i})(\mu_i)^2 - e_{31}$$

3 Formulation of crack problem

Consider now a piezoelectric bi-material system with an electrically impermeable interface penny-shaped crack as shown in Fig. 2. The radius of the crack is denoted by a and the piezoelectric general solutions of the top and bottom regions are given by Eqs. (10) and (11). In the ensuing analysis a superscript $j\,(j = 1,2)$ is used to identify the quantities associated with the top and lower half spaces. It is assumed that axis-symmetric normal traction $p_1(r)$, radial traction $p_2(r)$ and electric charge $p_3(r)$ act on the surfaces of the crack. The boundary and continuity conditions of the system shown in Fig. 2 can be expressed as,

$$\begin{aligned} \sigma_{zz}^{(1)}(r,0^+) &= \sigma_{zz}^{(2)}(r,0^-) = p_1(r), \quad 0 \le r < a \\ \sigma_{rz}^{(1)}(r,0^+) &= \sigma_{rz}^{(2)}(r,0^-) = p_2(r), \quad 0 \le r < a \\ D_z^{(1)}(r,0^+) &= D_z^{(2)}(r,0^-) = p_3(r), \quad 0 \le r < a \end{aligned} \quad (13a)$$

$$\begin{aligned} u_r^{(1)}(r,0^+) &= u_r^{(2)}(r,0^-), \quad r \ge a \\ u_z^{(1)}(r,0^+) &= u_z^{(2)}(r,0^-), \quad r \ge a \\ \phi^{(1)}(r,0^+) &= \phi^{(2)}(r,0^-), \quad r \ge a \end{aligned} \quad (13b)$$

$$\begin{aligned} \sigma_{zz}^{(1)}(r,0^+) &= \sigma_{zz}^{(2)}(r,0^-), \quad 0 \le r < \infty \\ \sigma_{rz}^{(1)}(r,0^+) &= \sigma_{rz}^{(2)}(r,0^-), \quad 0 \le r < \infty \\ D_z^{(1)}(r,0^+) &= D_z^{(2)}(r,0^-), \quad 0 \le r < \infty \end{aligned} \quad (13c)$$

In addition, the electroelastic field in the top and bottom half spaces should satisfy the regularity conditions at infinity. For the top half space therefore $B_i^{(1)}(\xi) \equiv 0$ and $A_i^{(2)}(\xi) \equiv 0$ for the lower half space.

In view of the stress and electric displacement continuity conditions expressed by Eq. (13c) the following relationship can be established by using Eq. (11).

$$\mathbf{B}^{(2)}(\xi) = \Omega \mathbf{A}^{(1)}(\xi) \quad (14)$$

where

$$\Omega = \left[\mathbf{M}^{(2)}\right]^{-1} \mathbf{H}^{(1)}$$

$$\mathbf{H}^{(1)} = \begin{bmatrix} H_{11}^{(1)} & H_{12}^{(1)} & H_{13}^{(1)} \\ H_{21}^{(1)} & H_{22}^{(1)} & H_{23}^{(1)} \\ H_{31}^{(1)} & H_{32}^{(1)} & H_{33}^{(1)} \end{bmatrix}$$

$$\mathbf{M}^{(2)} = \begin{bmatrix} H_{11}^{(2)} & H_{12}^{(2)} & H_{13}^{(2)} \\ -H_{21}^{(2)} & -H_{22}^{(2)} & -H_{23}^{(2)} \\ H_{31}^{(2)} & H_{32}^{(2)} & H_{33}^{(2)} \end{bmatrix}$$

$$\mathbf{A}^{(1)}(\xi) = \begin{Bmatrix} A_1^{(1)}(\xi) \\ A_2^{(1)}(\xi) \\ A_3^{(1)}(\xi) \end{Bmatrix} \quad \mathbf{B}^{(2)}(\xi) = \begin{Bmatrix} B_1^{(2)}(\xi) \\ B_2^{(2)}(\xi) \\ B_3^{(2)}(\xi) \end{Bmatrix} \quad (15)$$

A set of new unknown functions, $g_i\,(i = 1,2,3)$, are now introduced to reduce the mixed boundary conditions of Eqs. (13a) and (13b) to a system of integral equations.

$$\begin{aligned} g_1(r) &= \frac{\partial}{\partial r}\left\{ u_z^{(1)}(r,0^+) - u_z^{(2)}(r,0^-) \right\} \\ g_2(r) &= \frac{1}{r}\frac{\partial}{\partial r}\left\{ r u_r^{(1)}(r,0^+) - r u_r^{(2)}(r,0^-) \right\} \\ g_3(r) &= \frac{\partial}{\partial r}\left\{ \phi^{(1)}(r,0^+) - \phi^{(2)}(r,0^-) \right\} \end{aligned} \quad (16)$$

Substitution of Eq. (10) into (16), yields

$$g_1(r)=\sum_{i=1}^{3}Q_{1i}\int_0^\infty \xi A_i^{(1)}(\xi)J_1(r\xi)\mathrm{d}\xi$$
$$g_2(r)=\sum_{i=1}^{3}Q_{2i}\int_0^\infty \xi A_i^{(1)}(\xi)J_0(r\xi)\mathrm{d}\xi \qquad (17)$$
$$g_3(r)=\sum_{i=1}^{3}Q_{3i}\int_0^\infty \xi A_i^{(1)}(\xi)J_1(r\xi)\mathrm{d}\xi$$

where

$$\mathbf{Q}=\begin{bmatrix}\lambda_{11}^{(1)}\mu_1^{(1)}+\sum_{i=1}^{3}\lambda_{1i}^{(2)}\mu_i^{(2)}\Omega_{i1} & \lambda_{12}^{(1)}\mu_2^{(1)}+\sum_{i=1}^{3}\lambda_{1i}^{(2)}\mu_i^{(2)}\Omega_{i2} & \lambda_{13}^{(1)}\mu_3^{(1)}+\sum_{i=1}^{3}\lambda_{1i}^{(2)}\mu_i^{(2)}\Omega_{i3}\\ -1+\sum_{i=1}^{3}\Omega_{i1} & -1+\sum_{i=1}^{3}\Omega_{i2} & -1+\sum_{i=1}^{3}\Omega_{i3}\\ -\lambda_{21}^{(1)}\mu_1^{(1)}-\sum_{i=1}^{3}\lambda_{2i}^{(2)}\mu_i^{(2)}\Omega_{i1} & -\lambda_{22}^{(1)}\mu_2^{(1)}-\sum_{i=1}^{3}\lambda_{2i}^{(2)}\mu_i^{(2)}\Omega_{i2} & -\lambda_{23}^{(1)}\mu_3^{(1)}-\sum_{i=1}^{3}\lambda_{2i}^{(2)}\mu_i^{(2)}\Omega_{i3}\end{bmatrix} \qquad (18)$$

The arbitrary functions $A_i^{(1)}(\xi)(i=1,2,3)$ can be expressed in terms of $g_i(r)(i=1,2,3)$ by taking inverse Hankel transform of (17). According to the continuity conditions of (13b), $g_i(r)=0$, $(i=1,2,3)$ for $r>a$. Therefore,

$$\mathbf{A}^{(1)}(\xi)=\mathbf{R}\left\langle\int_0^a sg_1(s)J_1(s\xi)\mathrm{d}s,\ \int_0^a sg_2(s)J_0(s\xi)\mathrm{d}s,\ \int_0^a sg_3(s)J_1(s\xi)\mathrm{d}s\right\rangle^{\mathrm{T}} \qquad (19)$$

where $\mathbf{R}=\mathbf{Q}^{-1}$.

The boundary conditions of Eq. (13a) can be expressed in the following form by substituting Eq. (19) into (11).

$$\int_0^\infty \xi J_0(r\xi)\mathrm{d}\xi\int_0^a[F_{11}J_1(s\xi)g_1(s)+F_{12}J_0(s\xi)g_2(s)+F_{13}J_1(s\xi)g_3(s)]s\mathrm{d}s=p_1(r) \qquad (20a)$$

$$\int_0^\infty \xi J_1(r\xi)\mathrm{d}\xi\int_0^a[F_{21}J_1(s\xi)g_1(s)+F_{22}J_0(s\xi)g_2(s)+F_{23}J_1(s\xi)g_3(s)]s\mathrm{d}s=p_2(r) \qquad (20b)$$

$$\int_0^\infty \xi J_0(r\xi)\mathrm{d}\xi\int_0^a[F_{31}J_1(s\xi)g_1(s)+F_{32}J_0(s\xi)g_2(s)+F_{33}J_1(s\xi)g_3(s)]s\mathrm{d}s=p_3(r) \qquad (20c)$$

$$(0\le r<a)$$

where $\mathbf{F}=\mathbf{H}^{(1)}\mathbf{R}$.

In order to avoid divergent integrals, Eq. (20) is now integrated with respect to r to yield the following equations.

$$\int_0^\infty J_1(r\xi)\mathrm{d}\xi\int_0^a\left[F_{11}J_1(s\xi)g_1(s)+F_{12}J_0(s\xi)g_2(s)+F_{13}J_1(s\xi)g_3(s)\right]s\mathrm{d}s=\frac{1}{r}\left(\int_0^r sp_1(s)\mathrm{d}s+C_1\right) \qquad (21a)$$

$$-\int_0^\infty J_0(r\xi)\mathrm{d}\xi\int_0^a\left[F_{21}J_1(s\xi)g_1(s)+F_{22}J_0(s\xi)g_2(s)+F_{23}J_1(s\xi)g_3(s)\right]s\mathrm{d}s=\int_0^r p_2(s)\mathrm{d}s+C_2 \qquad (21b)$$

$$\int_0^\infty J_1(r\xi)\mathrm{d}\xi\int_0^a\left[F_{31}J_1(s\xi)g_1(s)+F_{32}J_0(s\xi)g_2(s)+F_{33}J_1(s\xi)g_3(s)\right]s\mathrm{d}s=\frac{1}{r}\left(\int_0^r sp_3(s)\mathrm{d}s+C_3\right)$$
$$(0\le r<a) \qquad (21c)$$

where $C_i(i=1,2,3)$ are arbitrary constants.

After changing the order of integration, Eq. (21) can be written as

$$\int_0^a \{h_{11}(r,s)[(F_{11}g_1(s)+F_{13}g_3(s)]+h_{12}(r,s) \times F_{12}g_2(s)\}s\mathrm{d}s = \frac{1}{r}\left(\int_0^r sp_1(s)\mathrm{d}s + C_1\right) \quad (22a)$$

$$-\int_0^a \{h_{21}(r,s)(F_{21}g_1(s)+F_{23}g_3(s)]+h_{22}(r,s) \times F_{22}g_2(s)\}s\mathrm{d}s = \int_0^r p_2(s)\mathrm{d}s + C_2 \quad (22b)$$

$$\int_0^a [h_{11}(r,s)(F_{31}g_1(s)+F_{33}g_3(s)]+h_{12}(r,s) \times F_{32}g_2(s)]s\mathrm{d}s = \frac{1}{r}(\int_0^r sp_3(s)\mathrm{d}s + C_3) \quad (0\le r<a) \quad (22c)$$

where

$$h_{11}(r,s) = \int_0^\infty J_1(\xi r)J_1(\xi s)\mathrm{d}\xi = \frac{2}{\pi}\begin{cases}\frac{1}{s}[K(s/r)-E(s/r)] & s<r\\ \frac{1}{r}[K(r/s)-E(r/s)] & s>r\end{cases}$$

$$h_{22}(r,s) = \int_0^\infty J_0(\xi r)J_0(\xi s)\mathrm{d}\xi = \frac{2}{\pi}\begin{cases}\frac{1}{r}K(s/r) & s<r\\ \frac{1}{s}K(r/s) & s>r\end{cases}$$

$$h_{12}(r,s) = \int_0^\infty J_1(\xi r)J_0(\xi s)\mathrm{d}\xi = \begin{cases}\frac{1}{r} & s<r\\ 0 & s>r\end{cases}$$

$$h_{21}(r,s) = \int_0^\infty J_0(\xi r)J_1(\xi s)\mathrm{d}\xi = \begin{cases}0 & s<r\\ \frac{1}{s} & s>r\end{cases} \quad (23)$$

where E and K are the complete elliptic integrals of the second and first kind respectively (Abramowitz and Stegun 1965) and

$$K(b) = \int_0^{\pi/2}\frac{\mathrm{d}\theta}{(1-b\sin^2\theta)^{1/2}} \quad E(b) = \int_0^{\pi/2}(1-b\sin^2\theta)^{1/2}\mathrm{d}\theta \quad 0\le b\le 1 \quad (24)$$

Differentiating Eq. (22) with respect to r yields,

$$\frac{1}{\pi}\int_0^a \frac{1}{(s-r)}[F_{11}g_1(s)+F_{13}g_3(s)]\mathrm{d}s + \int_0^a \kappa_{11}(r,s)[F_{11}g_1(s) + F_{13}g_3(s)]\mathrm{d}s + F_{12}g_2(r) = p_1(r)$$

$$\frac{1}{\pi}\int_0^a \frac{1}{(s-r)}F_{22}g_2(s)\mathrm{d}s + \int_0^a \kappa_{22}(r,s)F_{22}g_2(s)\mathrm{d}s - F_{21}g_1(r) - F_{23}g_3(r) = p_2(r)$$

$$\frac{1}{\pi}\int_0^a \frac{1}{(s-r)}[F_{31}g_1(s)+F_{33}g_3(s)]\mathrm{d}s + \int_0^a \kappa_{11}(r,s)[F_{31}g_1(s) + F_{33}g_3(s)]\mathrm{d}s + F_{32}g_2(r) = p_3(r)$$

$$(0\le r<a) \quad (25a\text{–}c)$$

where

$$\kappa_{11}(r,s) = \frac{1}{\pi}\left[\frac{2sM_1(r,s)}{s^2-r^2} - \frac{1}{(s-r)}\right],$$

$$\kappa_{22}(r,s) = \frac{1}{\pi}\left[\frac{2rM_2(r,s)}{s^2-r^2} - \frac{1}{(s-r)}\right]$$

$$M_1(r,s) = \begin{cases}\frac{r}{s}E\left(\frac{s}{r}\right) + \frac{s^2-r^2}{rs}K\left(\frac{s}{r}\right) & s<r\\ E\left(\frac{r}{s}\right) & s>r\end{cases}$$

$$M_2(r,s) = \begin{cases}\frac{s}{r}E\left(\frac{s}{r}\right) & s<r\\ \frac{s^2}{r^2}E\left(\frac{r}{s}\right) - \frac{s^2-r^2}{r^2}K\left(\frac{r}{s}\right) & s>r\end{cases} \quad (26)$$

For the problem shown in Fig. 1, the following limits must be satisfied on physical grounds.

$$\begin{aligned}&\left[u_z^{(1)}(r,0^+) - u_z^{(2)}(r,0^-)\right]\to 0,\\ &\left[\phi^{(1)}(r,0^+) - \phi^{(2)}(r,0^-)\right]\to 0 \qquad r\to a\\ &\left[u_r^{(1)}(r,0^+) - u_r^{(2)}(r,0^-)\right]\to 0 \qquad r\to a\\ &u_r^{(1)}(r,0^+)\to 0\\ &u_r^{(2)}(r,0^+)\to 0 \qquad r\to 0\\ &\frac{\partial}{\partial r}\left[u_z^{(1)}(r,0^+) - u_z^{(2)}(r,0^-)\right]\to 0,\\ &\frac{\partial}{\partial r}\left[\phi^{(1)}(r,0^+) - \phi^{(2)}(r,0^-)\right]\to 0 \quad r\to 0\end{aligned} \quad (27)$$

In view of the Eq. (27), the system of Eq. (25) has to be solved under the following conditions.

$$\int_0^a g_1(r)\mathrm{d}r = 0, \quad \int_0^a g_3(r)\mathrm{d}r = 0, \quad \int_0^a rg_2(r)\mathrm{d}r = 0 \quad (28)$$

Now normalize the integration interval of the Eq. (25) by defining,

$$s = \frac{1+t}{2}a, \quad r = \frac{1+x}{2}a \quad (29)$$

In view of the Eq. (29) the Eq. (25) can be expressed as,

$$\frac{1}{\pi}\int_{-1}^{1}\frac{1}{(t-x)}[F_{11}G_1(t)+F_{13}G_3(t)]\mathrm{d}t + \int_{-1}^{1}K_{11}(x,t)[F_{11}G_1(t)+F_{13}G_3(t)]\mathrm{d}t + F_{12}G_2(x) = P_1(x)$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{1}{(t-x)}F_{22}G_2(t)\mathrm{d}t + \int_{-1}^{1}K_{22}(x,t)F_{22}G_2(t)\mathrm{d}t - F_{21}G_1(x) - F_{23}G_3(x) = P_2(x)$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{1}{(t-x)}[F_{31}G_1(t)+F_{33}G_3(t)]\mathrm{d}t + \int_{-1}^{1}K_{11}(x,t)[F_{31}G_1(t)+F_{33}G_3(t)]\mathrm{d}t + F_{32}G_2(x) = P_3(x)$$

$$(|x|<1) \qquad \text{(30a–c)}$$

where

$$G_1(t)=g_1\left(\frac{1+t}{2}a\right),\quad G_2(t)=g_2\left(\frac{1+t}{2}a\right),\quad G_3(t)=g_3\left(\frac{1+t}{2}a\right)$$

$$P_1(x)=p_1\left(\frac{1+x}{2}a\right),\quad P_2(x)=p_2\left(\frac{1+x}{2}a\right),\quad P_3(x)=p_3\left(\frac{1+x}{2}a\right)$$

$$K_{11}(x,t)=\frac{a}{2}\kappa_{11}\left(\frac{1+x}{2}a,\frac{1+t}{2}a\right),\quad K_{22}(x,t)=\frac{a}{2}\kappa_{22}\left(\frac{1+x}{2}a,\frac{1+t}{2}a\right) \qquad (31)$$

According to the Eqs. (30a) and (30c),

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{(t-x)}+\pi K_{11}(x,t)\right]G_0(t)dt = F_{32}P_1(x)-F_{12}P_3(x) \qquad (32)$$

where $G_0(t)=[(F_{11}F_{32}-F_{31}F_{12})G_1(t)+(F_{13}F_{32}-F_{33}F_{12})G_3(t)]$.

It is known from the Eqs. (28) and (31) that the unknown function $G_0(t)$must satisfy the following condition.

$$\int_{-1}^{1}G_0(t)dt=0 \qquad (33)$$

Using the numerical method developed by Erdogan (1975), $G_0(t)$ can be expressed as

$$G_0(t)=\frac{\Psi(t)}{\sqrt{1-t^2}} \qquad (34)$$

Expanding $\Psi(t)$ in terms of Chebyshev polynomials of the first kind, T_n, yields,

$$\Psi(t)=\sum_{n=0}^{\infty}B_nT_n(t) \qquad (35)$$

where B_n denote a set of constants.

From Eqs. (33)–(35), it can be inferred that

$$B_0=0 \qquad (36)$$

Consider the following identity for Chebyshev polynomials,

$$\frac{1}{\pi}\int_{-1}^{1}\frac{T_n(t)}{\sqrt{1-t^2}}\frac{\mathrm{d}t}{t-x}=U_{n-1}(x)\quad(|t|<1,|x|<1) \qquad (37)$$

where U_n denote Chebyshev polynomials of the second kind.

The integral Eq. (32) can be reduced to the following equation system by using Eq. (37).

$$\sum_{n=1}^{\infty}B_nU_{n-1}(x)+\sum_{n=1}^{\infty}B_nV_n(x) = F_{32}P_1(x)-F_{12}P_3(x) \qquad (38)$$

where

$$V_n(x)=\int_{-1}^{1}\frac{K_{11}(x,t)T_n(t)}{\sqrt{1-t^2}}\mathrm{d}t \qquad (39)$$

and B_n can be obtained by solving Eq. (38) and the continuity condition (28).

Combining Eqs. 30(a)–(c), yields,

$$-\gamma_{11}G_2(x)+\frac{\gamma_{12}}{\pi}\int_{-1}^{1}\frac{G(t)}{t-x}dt+\gamma_{12}\int_{-1}^{1}K_{11}(x,t)\times G(t)\mathrm{d}t=\left(F_{33}-\frac{F_{23}}{F_{21}}F_{31}\right)P_1(x)-\left(F_{13}-\frac{F_{23}}{F_{21}}F_{11}\right)P_3(x)$$

$$\gamma_{22}G(x)+\frac{\gamma_{21}}{\pi}\int_{-1}^{1}\frac{G_2(t)}{t-x}\mathrm{d}t+\gamma_{21}\int_{-1}^{1}K_{22}(x,t)\times G_2(t)\mathrm{d}t=P_2(x)\quad |x|<1 \qquad \text{(40a,b)}$$

where

$$\gamma_{22}=-F_{21},\quad \gamma_{21}=F_{22},\quad \gamma_{12}=F_{11}F_{33}-F_{31}F_{13},$$

$$\gamma_{11} = \frac{F_{23}}{F_{21}}(F_{12}F_{31} - F_{32}F_{11}) - (F_{12}F_{33} - F_{32}F_{13}) \tag{41}$$

$$G(t) = G_1(t) + \frac{F_{23}}{F_{21}}G_3(t)$$

Eq. (40) can be further simplified to obtain the following system of singular integral equation.

$$\begin{aligned} &-\gamma\varphi_j(x) + \frac{\varsigma_j}{\pi i}\int_{-1}^{1}\frac{\varphi_j(t)}{t-x}\mathrm{d}t + \int_{-1}^{1}[f_{1j}(x,t) \\ &\times Re(\varphi_j(t)) + f_{2j}(x,t)\mathrm{Im}(\varphi_j(t))]\mathrm{d}t \qquad (42)\\ &= \frac{\sqrt{v_1}}{\gamma_{12}}\left[\left(F_{33} - \frac{F_{23}}{F_{21}}F_{31}\right)P_1(x)\right. \\ &\left.-\left(F_{13} - \frac{F_{23}}{F_{21}}F_{11}\right)P_3(x)\right] - i\frac{\varsigma_j\sqrt{v_2}}{\gamma_{21}}P_2(x) \\ &|x| < 1 \end{aligned}$$

where

$$\varphi_j(x) = \sqrt{v_2}G_2(x) + i\varsigma_j\sqrt{v_1}G(x), \quad v_1 = \frac{\gamma_{22}}{\gamma_{21}}$$

$$v_2 = \frac{\gamma_{11}}{\gamma_{12}}, \gamma = \sqrt{v_1 v_2}$$

$$\begin{aligned} f_{1j}(x,t) &= -\frac{1}{\Delta_j}[b_2K_{11}(x,t)\sqrt{v_1} + i\varsigma_j a_1 \qquad (43)\\ &\quad \times K_{22}(x,t)\sqrt{v_2}] \quad \varsigma_1 = 1, \quad \varsigma_2 = -1 \\ f_{2j}(x,t) &= \frac{1}{\Delta_j}[a_2K_{11}(x,t)\sqrt{v_1} - ib_1\varsigma_j \\ &\quad \times K_{22}(x,t)\sqrt{v_2}], \Delta_j = \varsigma_j(a_1a_2 + b_1b_2) \end{aligned}$$

$$a_1 = \mathrm{Re}(\sqrt{v_1}), a_2 = \mathrm{Re}(\sqrt{v_2})$$
$$b_1 = \mathrm{Im}(\sqrt{v_1}), b_2 = \mathrm{Im}(\sqrt{v_2}), j = 1, 2$$

Eq. (42) can be solved by expressing $\varphi_j(x)$ in terms of Jacobi polynomials, $P_n^{(\alpha_j,\beta_j)}(x)$, in the following form (Erdogan and Arin 1972, Erdogan 1969, Erdogan et al. 1973).

$$\varphi_j(x) = \sum_{n=0}^{\infty}\tau_{jn}w_j(x)P_n^{(\alpha_j,\beta_j)}(x) \tag{44}$$

where τ_{jn} are unknown complex constants and $w_j(x)$ is a weight function defined as,

$$w_j(x) = (1-x)^{\alpha_j}(1+x)^{\beta_j} \tag{45a}$$

and

$$\alpha_j = -\frac{1}{2} - i\varepsilon_j, \quad \beta_j = -\frac{1}{2} + i\varepsilon_j,$$
$$\varepsilon_1 = \frac{1}{2\pi}\log\left(\frac{1+\gamma}{1-\gamma}\right), \quad \varepsilon_2 = \frac{1}{2\pi}\log\left(\frac{1-\gamma}{1+\gamma}\right) \tag{45b}$$

Substitution of Eq. (44) into (42) yields (Erdogan and Arin 1972),

$$\begin{aligned} &\sum_{n=0}^{\infty}\frac{\sqrt{(1-\gamma^2)}}{2i}\varsigma_j\tau_{jn}P_{n-1}^{(-\alpha_j,-\beta_j)}(x) \\ &+\int_{-1}^{1}[f_{1j}(x,t)\mathrm{Re}\sum_{n=0}^{\infty}\tau_{jn}w_j(t)P_n^{(\alpha_j,\beta_j)}(t) \\ &+f_{2j}(x,t)\mathrm{Im}\sum_{n=0}^{\infty}\tau_{jn}w_j(t)P_n^{(\alpha_j,\beta_j)}(t)]\mathrm{d}t \\ &= \frac{\sqrt{v_1}}{\gamma_{12}}\left[\left(F_{33} - \frac{F_{23}}{F_{21}}F_{31}\right)P_1(x)\right. \\ &\left.-\left(F_{13} - \frac{F_{23}}{F_{21}}F_{11}\right)P_3(x)\right] \\ &-i\varsigma_j\frac{\sqrt{v_2}}{\gamma_{21}}P_2(x) \quad (|x| < 1) \end{aligned} \tag{46}$$

Now using the orthogonality relations

$$\int_{-1}^{1}P_k^{(-\alpha_j,-\beta_j)}(x)P_m^{(-\alpha_j,-\beta_j)}(x)\varpi_j(x) \times\mathrm{d}x = \begin{cases} 0 & k \neq m \\ \vartheta_{jk} & k = m \end{cases}$$

with $\varpi_j(x) = (1-x)^{-\alpha_j}(1+x)^{-\beta_j}$

$$\vartheta_{jk} = \frac{2^{1-\alpha_j-\beta_j}\Gamma(k+1-\alpha_j)\Gamma(k+1-\beta_j)}{(2k-\alpha_j-\beta_j+1)\Gamma(k+1-\alpha_j-\beta_j)k!} \tag{47}$$

and multiplying both sides of Eq. (46) by $P_k^{(-\alpha_j,-\beta_j)}(x)$ ($k = 0, 1, 2, \ldots$) and then integrating with respect to x, the Eq. (46) can be reduced to the following infinite system of linear algebraic equations.

$$\begin{aligned} &-\frac{\sqrt{1-\gamma^2}}{2}\varsigma_j i\vartheta_{jk}\tau_{jk+1} + \sum_{n=0}^{\infty}(\Lambda_{jkn}\mathrm{Re}(\tau_{jn}) \\ &+\Delta_{jkn}\mathrm{Im}(\tau_{jn}) = \eta_{jk} \end{aligned} \tag{48}$$

where

$$\Lambda_{jkn} = \int_{-1}^{1} P_k^{(-\alpha_j,-\beta_j)}(x)\varpi_j(x)\mathrm{d}x \int_{-1}^{1}\{f_{1j}(x,t)\mathrm{Re}[w_j(t)P_n^{(\alpha_j,\beta_j)}(t)] + f_{2j}(x,t)\mathrm{Im}[w_j(t)P_n^{(\alpha_j,\beta_j)}(t)]\}\mathrm{d}t$$

$$\Delta_{jkn} = \int_{-1}^{1} P_k^{(-\alpha_j,-\beta_j)}(x)\varpi_j(x)\mathrm{d}x \int_{-1}^{1}\{-f_{1j}(x,t)\mathrm{Im}[w_j(t)P_n^{(\alpha_j,\beta_j)}(t)] + f_{2j}(x,t)\mathrm{Re}[w_j(t)P_n^{(\alpha_j,\beta_j)}(t)]\}\mathrm{d}t$$

$$\eta_{jk} = \int_{-1}^{1}\left\{\frac{\sqrt{\upsilon_1}}{\gamma_{12}}\left[\left(F_{33}-\frac{F_{23}}{F_{21}}F_{31}\right)P_1(x) - \left(F_{13}-\frac{F_{23}}{F_{21}}F_{11}\right)P_3(x)\right] - i\varsigma_j\frac{\sqrt{\upsilon_2}}{\gamma_{21}}P_2(x)\right\}P_k^{(-\alpha_j,-\beta_j)}(x)\varpi_j(x)\mathrm{d}x \tag{49}$$

Therefore τ_{jk} can be obtained from Eq. (48) and (28).

4 Field intensity factors

Noting that Eqs. (32) and (42) are valid for $x > 1$ as well as for $0 \le x < 1$, and $\varphi(x) = 0$ for $x > 1$, the interfacial stresses and normal electric displacement can be expressed as

$$F_{32}\sigma_z^{(1)}(x,0) - F_{12}D_z^{(1)}(x,0) = \frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{(t-x)} + \pi K_{11}(t,x)\right]G_0(t)\mathrm{d}t \quad (x > 1)$$

$$\frac{\sqrt{\upsilon_1}}{\gamma_{12}}\left[\left(F_{33}-\frac{F_{23}}{F_{21}}F_{31}\right)\sigma_z^{(1)}(x,0) - \left(F_{13}-\frac{F_{23}}{F_{21}}F_{11}\right)D_z^{(1)}(x,0)\right] - i\varsigma_j\frac{\sqrt{\upsilon_2}}{\gamma_{21}}\sigma_{rz}^{(1)}(x,0) = \frac{\varsigma_j}{\pi i}\int_{-1}^{1}\frac{\varphi_j(t)}{t-x}\mathrm{d}t + \int_{-1}^{1}[f_{1j}(x,t)\mathrm{Re}(\varphi_j(t)) + f_{2j}(x,t)\mathrm{Im}(\varphi_j(t))]\mathrm{d}t \tag{50a,b}$$

In the neighborhood of $x = 1$, the second integral in Eqs. (50a) and (50b) are bounded, and the first term can be evaluated by using Eqs. (34), (35) and (44), and the following identity.

$$\frac{\varsigma_j}{\pi i}\int_{-1}^{1}P_n^{(\alpha_j,\beta_j)}(t)w_j(t)\frac{1}{t-x}dt = -\varsigma_j(1+\gamma) \times[-w_j(x)P_n^{(\alpha_j,\beta_j)}(x) + \mathrm{X}_{jn}^{\infty}(x)] \tag{51}$$

where $\mathrm{X}_{jn}^{\infty}(x)$ is the principal part of $w_jP_n^{(\alpha,\beta)}$ at infinity.

It can be seen from Eqs. (50b) together with Eqs. (42), (44) and (45) that the order of singularity of the crack tip field is generally complex-valued resulting in an oscillatory field. For certain bi-material systems, however, α_j and $\beta_j$are real and this leads to a real-valued singularity with a non-oscillatory field. In the case of a homogeneous material system, α_j and β_j are equal to −1/2 and the crack tip field is non-oscillatory. Xu and Rajapakse (2000) give a comprehensive treatment of singularities in multi-material piezoelectric wedges and junctions.

Define the stress intensity factors k_1 and k_2 and the electric displacement intensity factor k_d by

$$F_{32}k_1 - F_{12}k_d = \lim_{r\to a}\sqrt{2(r-a)}[F_{32}\sigma_z^{(1)}(r,0) - F_{12}D_z^{(1)}(r,0)] \tag{52a}$$

and

$$\frac{\sqrt{\upsilon_1}}{\gamma_{12}}\left[\left(F_{33}-\frac{F_{23}}{F_{21}}F_{31}\right)k_1 - \left(F_{13}-\frac{F_{23}}{F_{21}}F_{11}\right)k_d\right] - i\varsigma_j\frac{\sqrt{\upsilon_2}}{\gamma_{21}}k_2 = \lim_{r\to a}\sqrt{2(r-a)}\left(\frac{r-a}{2a}\right)^{i\varepsilon_j} \left\{\frac{\sqrt{\upsilon_1}}{\gamma_{12}}\left[\left(F_{33}-\frac{F_{23}}{F_{21}}F_{31}\right)\sigma_z^{(1)}(r,0) - \left(F_{13}-\frac{F_{23}}{F_{21}}F_{11}\right)D_z^{(1)}(r,0)\right] - i\varsigma_j\frac{\sqrt{\upsilon_2}}{\gamma_{21}}\sigma_{rz}^{(1)}(r,0)\right\}$$
$$= \lim_{x\to 1}2^{-2i\varepsilon_j}\sqrt{a}\,\frac{\frac{\sqrt{\upsilon_1}}{\gamma_{12}}\left[\left(F_{33}-\frac{F_{23}}{F_{21}}F_{31}\right)\sigma_z^{(1)}(x,0) - \left(F_{13}-\frac{F_{23}}{F_{21}}F_{11}\right)D_z^{(1)}(x,0)\right] - i\varsigma_j\frac{\sqrt{\upsilon_2}}{\gamma_{21}}\sigma_{rz}^{(1)}(x,0)}{(x-1)^{\alpha}} \tag{52b}$$

Eqs. (34), (35), (44), (50), (51) and (52) yields

$$F_{32}k_1 - F_{12}k_d = -\sqrt{\frac{a}{2}}\sum_{n=0}^{\infty}B_n \tag{53a}$$

$$\frac{\sqrt{\upsilon_1}}{\gamma_{12}}\left[\left(F_{33}-\frac{F_{23}}{F_{21}}F_{31}\right)k_1 - \left(F_{13}-\frac{F_{23}}{F_{21}}F_{11}\right)k_d\right] - i\varsigma_j\frac{\sqrt{\upsilon_2}}{\gamma_{21}}k_2 \tag{53b}$$

$$= i2^{\alpha_j} \varsigma_j \sqrt{a} \sqrt{1-\gamma^2} \sum_{n=0}^{\infty} \tau_{jn} P_n^{(\alpha_j, \beta_j)}(1)$$

The field intensity factors (k_1, k_2, k_d) of an interfacial crack can be obtained by solving the complex-valued Eqs. (53a) and (53b). It can be seen from Eqs. (53a) and (53b) that mixed mode fracture together with electric singularity can exist under Mode 1 loading or vice versa.

5 Numerical results and discussion

Selected numerical results are presented in this section for a few bi-material systems with a penny-shaped interface crack to examine the salient features of fracture parameters. The properties of the materials used in the numerical study are listed in Table 1 and the loading on the crack faces are:

$$\sigma_{zz}(r, 0) = p_1(r) = -p_0, \quad \sigma_{rz}(r, 0) = p_2(r) = 0,$$
$$D_z(r, 0) = p_3(r) = -D_0 \tag{54}$$

Accuracy of the present scheme is first confirmed by comparing with the solution for an interfacial penny shaped crack in an ideal elastic bi-material system (Erdogan and Arin 1972). Same elastic properties as in Erdogan and Arin (1972) are used in the calculations and the piezoelectric and dielectric constants are set to negligibly small values. The solution for the normalized mode I stress intensity factor, $k_1/p_0\sqrt{a}$, obtained from the present analysis is 0.6715 and the solution reported by Erdogan and Arin (1972) is 0.6716.

The solutions for piezoelectric bi-material systems are now considered. Three material combinations, namely, PZT-5H/PZT-4, PZT-5/PZT-4 and PZT-7/PZT-4 are considered with PZT-4 occupying the lower half space in all three cases. The crack-tip stress and electric displacement intensity factors are shown in Figs. 3(a) and 3(b) respectively for varying electric charge loading ratio, D_0/p_0. Note that solid lines in Fig. 3 correspond to the situation where the poling direction of both upper and lower half spaces are along the positive z-axis whereas the dashed lines for the case where the poling direction of the upper half space is along the positive z-axis and the lower half space is along the negative z-axis such as the case of adjacent layers in a stack actuator.

It is found that both stress intensity factor and electric displacement intensity factor vary linearly with the intensity of the applied electric charge loading. Both field intensity factors depend significantly on the material type and the poling orientation of the bi-material system. A positive electric charge loading has a shielding effect on a crack in all three bi-material systems when the poling directions are opposite to each other whereas a negative electric charge loading enhances crack growth in such cases. This behaviour is reversed for two bi-material systems (PZT-5H/PZT-4 and PZT-5/PZT-4) when the poling directions of the upper and lower half spaces are aligned in the same direction. A crack in PZT-5H/PZT-4 and PZT-5/PZT-4 systems with opposite polarization directions and subjected to positive charge loading is more stable when compared to a crack in an identical bi-material system with the same poling directions. A crack in PZT-7/PZT-4 system is most unstable under negative charge loading when compared to the other two bi-material systems. Electric displacement intensity factor is negative for negative charge loading and shows negligible dependence on the material types of the bi-material system but more dependence on poling orientation of the materials. The potential for dielectric breakdown at a crack tip is higher under both positive and negative electric charge loading when the two materials of the bi-material system have opposite polarization.

Figure 3(c) and (d) show the mode 1 stress and electric displacement factors of three bi-material systems made out of the same material except the poling directions of the upper and lower half spaces are either identical or opposite. The case where the poling direction is identical is therefore same as the case of a penny-shaped crack in a homogeneous full space. It can be seen from Fig. 3(c) that stress intensity factor is practically independent of the magnitude of electric charge loading and the materials of the system including their poling orientation. The solution is approximately 6% lower than the solution for an ideal elastic material (Erdogan and Arin 1972). It is also noted that stress intensity factor in Fig. 3(c) is smaller than that in Fig. 3(a). The solutions for electric displacement shown in Fig. 3(d) behave similar to Fig. 3(b) but have slightly lower absolute values. It is found that for the three bi-material systems considered in the present study, k_{II} is nearly zero and negligible k_{d} exists under Mode I loading.

Table 1 Material properties

	C_{11} (GPa)	C_{12} (GPa)	C_{13} (GPa)	C_{33} (GPa)	C_{44} (GPa)
PZT-5H	126	79.5	84.1	117	23
PZT-5	121	75.4	75.2	111	21.1
PZT-7	130	83	83	119	25
PZT-4	139	77.8	74.3	115	25.6
	e_{31} (C/m^2)	e_{33} (C/m^2)	e_{15} (C/m^2)	$d_{11}(\times 10^{-10}$F/m)	$d_{33}(\times 10^{-10}$F/m)
PZT-5H	-6.55	23.3	17	153.8	127.6
PZT-5	-54	15.8	12.3	81.7	73.46
PZT-7	-10.3	14.7	13.5	171	186
PZT-4	-5.2	15.1	12.7	64.605	56.1975

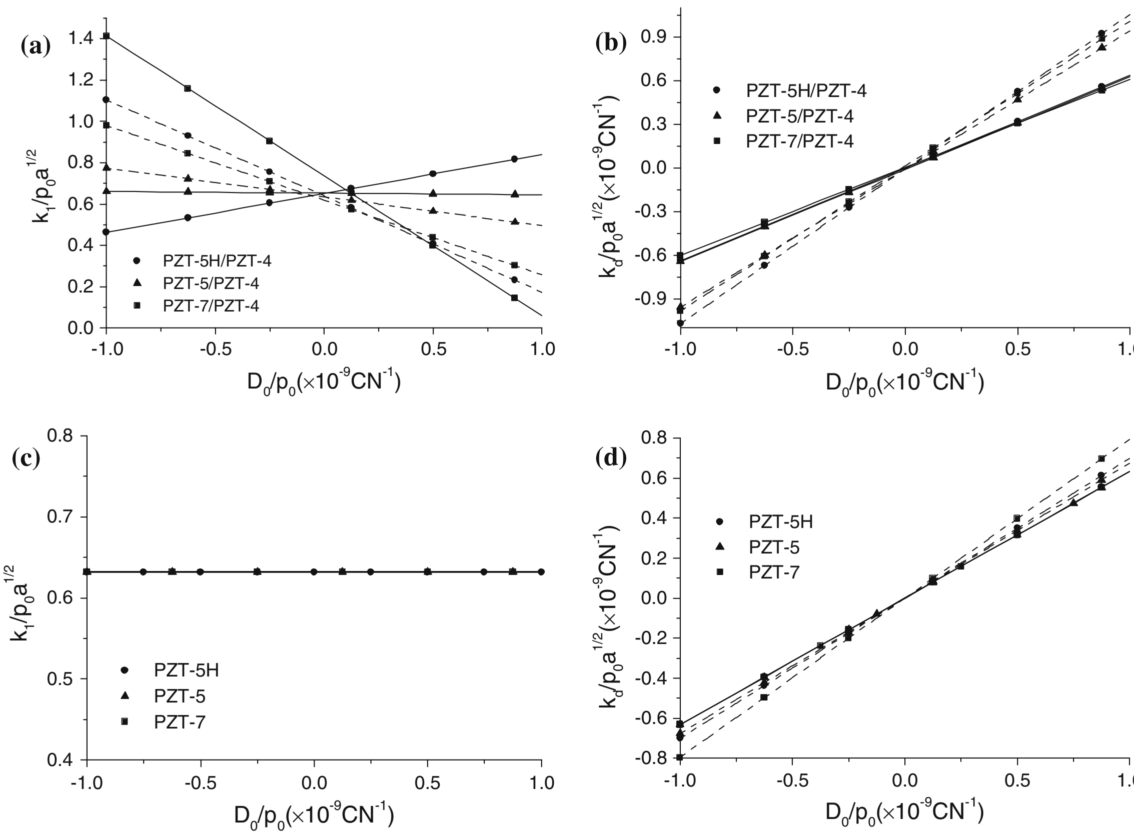

Fig. 3 Variation of normalized stress and electric displacement intensity factors with applied electric charge intensity, D_0/p_0

6 Conclusions

An integral equation formulation is successfully developed to analyze the case of a penny-shaped crack at the interface of a piezoelectric bi-material system. A numerical scheme based on Jacobi polynomials is used to reduce the integral equation system to a system of linear simultaneous algebraic Eqs. Numerical solution of the algebraic system is stable and accurate for a wide range of bi-material systems. Stress and electric displacement factors vary linearly with electric charge loading. Positive electric loading retards crack growth in bi-material systems with opposite poling directions. In the case of bi-material systems with identical poling direction, positive loading may or may not retard crack growth depending on the type of material in the system. Mode 1 stress intensity factor of a bi-material system with identical materials and same or opposite poling directions is independent of the material type, poling orientation and magnitude of electric charge loading. Electric displacement intensity factor show negligible dependence on the material composition of a bi-material system and a crack in a system with opposite poling directions has a higher chance for dielectric breakdown under both positive and negative charge loading.

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