Foundations of Physics Letters, Vol. 19, No. 4, August 2006 (©2006)
DOI: 10.1007/s10702-006-0796-9

PREQUANTUM CLASSICAL STATISTICAL FIELD THEORY: COMPLEX REPRESENTATION, HAMILTON-SCHRÖDINGER EQUATION, AND INTERPRETATION OF STATIONARY STATES

Andrei Khrennikov

International Center for Mathematical Modeling in Physics, Engineering, and Cognitive Sciences
University of Växjö, S-35195, Sweden
E-mail:Andrei.Khrennikov@msi.vxu.se

Received 11 January 2006; revised 6 March 2006

We develop a prequantum classical statistical model in that the role of hidden variables is played by classical (vector) fields. We call this model Prequantum Classical Statistical Field Theory (PCSFT). The correspondence between classical and quantum quantities is asymptotic, so we call our approach asymptotic dequantization. We construct the complex representation of PCSFT. In particular, the conventional Schrödinger equation is obtained as the complex representation of the system of Hamilton equations on the infinite-dimensional phase space. In this note we pay the main attention to interpretation of so called pure quantum states (wave functions) in PCSFT, especially stationary states. We show, see Theorem 2, that pure states of QM can be considered as labels for Gaussian measures concentrated on one dimensional complex subspaces of phase space that are invariant with respect to the Schrödinger dynamics. "A quantum system in a stationary state ψ" in PCSFT is nothing else than a Gaussian ensemble of classical fields (fluctuations of the vacuum field of a very small magnitude) which is not changed in the process of Schrödinger's evolution. We interpret in this way the problem of *stability of hydrogen atom.* One of unexpected consequences of PCSFT is the infinite dimension of physical space on the prequantum scale.

Key words: prequantum classical statistical field theory, completeness of QM, hidden variables, interpretation of pure quantum states, stationary states, stability of hydrogen atom, infinite dimension of physical space.

0894-9875/06/0800-0299/0 © 2006 Springer Science+Business Media, Inc.

1. INTRODUCTION

The problem of *completeness of QM* has been an important source of investigations on quantum foundations, see, e.g., for recent debates Ref. [1]-[8]. Now days this problem is typically regarded as the problem of *hidden variables.* This problem is not of purely philosophic interest. By constructing a model that would provide a finer description of physical reality than given by the quantum wave function ψ we obtain at least theoretical possibility to go *beyond quantum mechanics.* In principle, we might find effects that are not described by quantum mechanics. One of the main barriers on the way beyond quantum mechanics are various "NO-GO" theorems (e.g., theorems of von Neumann, Kochen-Specker, Bell,...). Therefore in considering a prequantum classical statistical model one should take into account all known "NO-GO" theorems.

In a series of papers [7] we showed that in principle all distinguishing features of quantum probabilities (e.g., *interference, Born's rule*, representation of random variables by noncommuting operators) can be obtained in classical (but contextual) probabilistic framework. In [8] there was proposed to represent physical contexts by special ensembles of classical fields. It was shown that it is possible to represent quantum mechanics as an asymptotic projection of classical statistical mechanics on *infinite-dimensional phase space* $\Omega = H \times H$, where H is Hilbert space.

By realizing Hilbert space H as the $L_2(\mathbf{R}^3)$-space we obtain the representation of prequantum classical phase space as the space of classical (real vector) fields $\psi(x) = (q(x), p(x))$ on $\mathbf{R}^3$. We call this approach to the problem of hidden variables *Prequantum Classical Statistical Field Theory,* PCSFT. In this model quantum states are just labels for Gaussian ensembles of classical fields. Such ensembles (Gaussian measures ρ) are characterized by zero mean value and very small dispersion:

$$\int_{L_2(\mathbf{R}^3)\times L_2(\mathbf{R}^3)} \int_{\mathbf{R}^3} [p^2(x) + q^2(x)] dx d\rho(q,p) = \alpha, \quad \alpha \to 0. \qquad (1)$$

This dispersion is a small parameter of the model. Quantum mechanics is obtained as the $\lim_{\alpha\to 0}$ of PCSFT.

Let us consider the "classical vacuum field." In PCSFT it is represented by the function $\psi_{\text{vacuum}} \equiv 0$. Since a Gaussian ensemble of classical fields has the zero mean value, these fields can be considered as random fluctuations of the "classical vacuum field." Since dispersion is very small, these are very small fluctuations. There is some similarity with SED and stochastic QM, cf. [9]–[14]. The main difference is that we consider fluctuations not on "physical space" $\mathbf{R}^3$, but on infinite dimensional space of classical fields. In fact, in PCSFT the latter space plays the fundamental role. Conventional "physical space"

$\mathbf{R}^3$ plays just a subsidiary role and it appears through a special representation of the *infinite dimensional space* of fields. Therefore it is more natural to consider the latter space as physical space. Thus one of unexpected consequences of our approach is the *infinite dimension of physical space.* We shall discuss this fundamental consequence and in particular comparing with string theory in section 8.

In [8] we studied asymptotic expansions of Gaussian integrals of analytic functionals and obtained an asymptotic equality coupling the Gaussian integral and the trace of the composition of scaling of the covariation operator of a Gaussian measure and the second derivative of a functional. In this way we coupled the classical average (given by an infinite-dimensional Gaussian integral) and the quantum average (given by the von Neumann trace formula). In [8] there was obtained generalizations of QM that were based on expansions of classical field-functionals into Taylor series up to terms of the degree $n = 2, 4, 6, ..$ (for $n = 2$ we obtain the ordinary QM).

In the present paper we change crucially the interpretation of the small parameter of our model. In [8] this parameter was identified with the Planck constant h (in making such a choice I was very much stimulated by discussions with people working in SED and stochastic quantum mechanics, cf. [9]–[14]). In this paper we consider α as a new parameter giving the dispersion of prequantum fluctuations. We construct a one parameter family of classical statistical models $M^\alpha,\ \alpha \geq 0$. QM is obtained as the limit of classical statistical models when $\alpha \to 0$:

$$\lim_{\alpha \to 0} M^\alpha = N_{\text{quant}}, \tag{2}$$

where N_{quant} is the Dirac-von Neumann quantum model [15], [16].

We remark that the problem of classical limit of quantum mechanics was discussed since the first days of quantum mechanics. There are a few approaches showing that (in some sense) the limit of quantum mechanics gives classical statistical mechanics on the phase space $\Omega_{2n} = \mathbf{R}^{2n}$. This problem is known as *the principle of correspondence* (between quantum and classical mechanics), see, e.g., [17,18]. In that framework the Planck constant h is considered as a small parameter: $N_{\text{quant}} \equiv N^h_{\text{quant}}$. Classical mechanics on the phase-space $\Omega_{2n} = \mathbf{R}^{2n}$is obtained as the $\lim_{h \to 0}$ of quantum mechanics and formally

$$\lim_{h \to 0} N^h_{\text{quant}} = M_{\text{conv.class}}, \tag{3}$$

where $M_{\text{conv.class.}}$ is the conventional classical model with the phase-space Ω_{2n}. In contrast to this conventional correspondence principle, we consider *the quantum limit of classical statistical mechanics on the infinite-dimensional phase space.*

The main problem is that our model does not provide the magnitude of α. We may just speculate that there might be some relations with scales of quantum gravity and string theory.

In this article we pay attention to the interpretation of so called *pure states* in PCSFT, especially so called *stationary states.* We show, see Theorem 2, that pure states of QM can be interpreted simply as labels for Gaussian measures concentrated on one dimensional complex subspaces of phase space that are invariant with respect to the Schrödinger dynamics. Thus PCSFT implies the following viewpoint to quantum stationarity. First of all this is not deterministic classical stationarity. Nevertheless, this is purely classical, but stochastic stationarity, cf. [19]. "A quantum system in a stationary state ψ" in PCSFT is nothing else than a Gaussian ensemble of classical fields (fluctuations of the vacuum field of a very small magnitude) which is not changed in the process of Schrödinger's evolution. We interpret in this way the problem of *stability of hydrogen atom,* see section 7. Here "an electron on a stationary orbit" is a stationary Gaussian ensemble of classical fields. The structure of these Gaussian fluctuations provides the picture of a *bound state.*

We analyze PCSFT by comparing it with known "NO-GO" theorems, in particular, the Bell theorem [20]. We also note that PCSFT might be considered as a realization of Einstein's dream on a purely field model of physical reality, cf. [21].

To simplify the introduction to PCSFT, in papers [8] we considered quantum models over the real Hilbert space and only in section 5 of the second paper in [8] there were given main lines of generalization to the complex Hilbert space. In this paper we start directly with the complex case. Here the crucial role is played by the symplectic structure on the infinite-dimensional phase space Ω. In particular, in our model all classical physical variables should be invariant with respect to the symplectic operator $J,\ J^2 = -I.$

We show that the Schrödinger dynamics is nothing else than the Hamilton dynamics on Ω. Therefore quantum stationary states can be considered as invariant measures (concentrated on J-invariant planes of phase space Ω) of special infinite-dimensional Hamiltonian systems.

In contrast to [8], in this paper we study asymptotics of classical averages (given by Gaussian functional integrals [22]) on the mathematical level of rigorousness. We find a correct functional class in that such expansions are valid and obtain an estimate of the rest term in the fundamental asymptotic formula coupling classical and quantum averages.

2. ASYMPTOTIC DEQUANTIZATION

We define *"classical statistical models"* in the following way, see [8] for more detail (and even philosophic considerations): a) physical states ψ are represented by points of some set Ω (state space); b) physical variables are represented by functions $f : \Omega \to \mathbf{R}$ belonging to some functional space $V(\Omega)$; c) statistical states are represented by probability measures on Ω belonging to some class $S(\Omega)$; d) the average of a physical variable (which is represented by a function $f \in V(\Omega)$) with respect to a statistical state (which is represented by a probability measure $\rho \in S(\Omega)$) is given by

$$\langle f \rangle_\rho \equiv \int_\Omega f(\psi) d\rho(\psi). \tag{4}$$

A *classical statistical model* is a pair $M = (S, V)$. We recall that classical statistical mechanics on the phase space $\Omega_{2n} = \mathbf{R}^n \times \mathbf{R}^n$ gives an example of a classical statistical model. But we shall not be interested in this example in our further considerations. We shall develop a classical statistical model with *an infinite-dimensional phase-space.*

The conventional quantum statistical model with the complex Hilbert state space Ω_c is described in the following way (see Dirac-von Neumann [15], [16] for the conventional complex model): a) physical observables are represented by operators $A : \Omega_c \to \Omega_c$ belonging to the class of continuous self-adjoint operators $\mathcal{L}_s \equiv \mathcal{L}_s(\Omega_c)$; b) statistical states are represented by von Neumann density operators [16] (the class of such operators is denoted by $\mathcal{D} \equiv \mathcal{D}(\Omega_c)$); c) the average of a physical observable (which is represented by the operator $A \in \mathcal{L}_s(\Omega_c)$) with respect to a statistical state (which is represented by the density operator $D \in \mathcal{D}(\Omega_c)$) is given by von Neumann's formula [16]:

$$\langle A \rangle_D \equiv \text{Tr DA}. \tag{5}$$

The *quantum statistical model* is the pair $N_{\text{quant}} = (\mathcal{D}, \mathcal{L}_s)$.

We are looking for a classical statistical model $M = (S, V)$ which will give "dequantization" of the quantum model $N_{\text{quant}} = (\mathcal{D}, \mathcal{L}_s)$. Here the meaning of "dequantization" should be specified. In fact, all "NO-GO" theorems (e.g., von Neumann, Kochen-Specker, Bell, etc.) can be interpreted as theorems about impossibility of various dequantization procedures. Therefore we should define the procedure of dequantization in such a way that there will be no contradiction with known "NO-GO" theorems, but our dequantization procedure still will be natural from the physical viewpoint. We define (asymptotic) dequantization as a family $M^\alpha = (S^\alpha, V)$ of classical statistical models depending on small parameter $\alpha \geq 0$. There should exist maps $T : S^\alpha \to \mathcal{D}$ and $T : V \to \mathcal{L}_s$ such that: a) both maps are *surjections* (so all quantum states and observables can be represented as images of classical statistical states and variables, respectively); b) the map $T : V \to \mathcal{L}_s$ is $\mathbf{R}$-linear (we recall that we consider real-valued classical physical variables); c) the map $T : S \to \mathcal{D}$ is injection (there is one-to one correspondence between classical and quantum statistical states); d) classical and quantum averages are coupled through the following asymptotic equality:

$$\langle f\rangle_\rho = \alpha\langle T(f)\rangle_{T(\rho)} + o(\alpha), \quad \alpha\to 0 \tag{6}$$

(here $\langle T(f)\rangle_{T(\rho)}$ is the quantum average); so

$$\int_\Omega f(\psi)d\rho(\psi) = \alpha \text{ Tr DA} + \text{o}(\alpha), \quad \text{A} = \text{T(f)}, \text{D} = \text{T}(\rho). \tag{7}$$

This equality can be interpreted in the following way.

Let $f(\psi)$ be a classical physical variable (describing properties of microsystems - classical fields having very small magnitude α). We define its *amplification* by setting:

$$f_\alpha(\psi) = \frac{1}{\alpha} f(\psi). \tag{8}$$

Thus any micro effect is amplified in $(1/\alpha)$-times. If we interpret α as the intensity of vacuum fluctuations, then $f_\alpha(\psi)$ is the relative intensity of $f(\psi)$ with respect to vacuum fluctuations.

By dividing both sides of the equation (8) by α we obtain: $\langle f_\alpha\rangle_\rho = \langle T(f)\rangle_{T(\rho)} + o(1), \quad \alpha\to 0$, or

$$\int_\Omega f_\alpha(\psi)d\rho(\psi) = \text{Tr DA} + \text{o}(1), \quad \text{A} = \text{T(f)}, \text{D} = \text{T}(\rho). \tag{9}$$

The quantum term gives the main contribution into the relative intensity with respect to vacuum fluctuations. QM is the mathematical formalism describing the statistical approximation of the amplification of vacuum fluctuations.

We see that for physical variables/quantum observables and classical and quantum statistical states the dequantization maps have different features. The map $T: V\to\mathcal{L}_s$ is not injective. Different classical physical variables f_1 and f_2 can be mapped into one quantum observable A. This is not surprising. Such a viewpoint on the relation between classical variables and quantum observables was already presented by J. Bell, see [20]. In principle, experimenter could not distinguish classical ("ontic") variables by his measurement devices. In contrast, the map $T: S^\alpha\to\mathcal{D}$ is injection. Here we suppose that quantum statistical states represent uniquely ("ontic") classical statistical states.

We recall that in the von Neumann "NO-GO" theorem, see [16], there was assumed that the correspondence T between classical variables and quantum observables is *one-to-one.* Thus our dequantization violates this von Neumann condition. Therefore the von Neumann theorem could not be applied to PCSFT. On the other hand, the map

$T : V \to \mathcal{L}_s$ (given by (20)) is $\mathbf{R}$-linear as it was postulated by J. von Neumann in [16] (thus, e.g., $T(f_1 + f_2) = T(f_1) + T(f_2)$ even in the case of noncommuting operators $T(f_1)$ and $T(f_2)$). We recall that this assumption was criticized by many authors, in particular, by J. Bell [20].

The crucial difference with dequantizations considered in known "NO-GO" theorems is that in our case *classical and quantum averages are equal only asymptotically.* We also remark that a classical variable f and the corresponding quantum observable $A = T(f)$ can have *different ranges of values.* In particular, the latter possibility blocks application of Bell's theorem to PCSFT. We recall that in Bell's theorem classical variables should have the same range of values, namely ± 1, as the spin-observables. This condition was generalized by Clauser-Horn-Shimony-Holt. They obtained the so called CHSH-inequality (generalizing Bell's inequality) under the condition that classical variables are bounded by one. In [23] we demonstrated (as it might be expected) that if classical variables can take values larger than one, then CHSH-inequality can be violated.

We point out that the essence of Bell's considerations was the problem of quantum nonlocality. From the very beginning Bell wanted to show that any prequantum model with hidden variables should be nonlocal. In the complete accordance with the original Bell's interpretation experimental violation of Bell's inequality is typically interpreted as the evidence of quantum nonlocality. We emphasize that such an interpretation is oversimplified. We recall that the original Bell's theorem is a purely probabilistic statement and physical space is not involved in considerations at all, see [23] for detailed analysis of this problem. There can be presented purely probabilistic arguments blocking Bell's considerations, see, e.g., [24], [6]. Nevertheless, if one chooses the original Bell interpretation, i.e., nonlocality of any model with hidden variables, there can be possed the problem of locality PCSFT. As we have already mentioned, the direct application of Bell's theorem is blocked, because its conditions are violated in PCSFT-dequantization. Therefore the PCSFT-model with hidden variables might be local, in spite Bell's theorem. However, it seems that the problem of locality is not well possed for PCSFT. The conventional (Einsteinian) locality is locality in physical space given by its mathematical model $\mathbf{R}^3$. Surprisingly this space could not be considered as basic physical space in PCSFT, since it plays just a subsidiary role in our approach. *Natural physical space of PCSFT has infinite dimension,* see section 8 for details.

Finally, we remark that the application of Kochen-Specker theorem to PCSFT is blocked, e.g., because in this theorem (in the same way as in the Von Neumann theorem) there was postulated one-to-to one correspondence between classical variables and quantum observables.

3. PREQUANTUM CLASSICAL STATISTICAL MODEL

We choose the phase space $\Omega = Q \times P$, where $Q = P = H$ and H is the infinite-dimensional real (separable) Hilbert space. We consider Ω as the real Hilbert space with the scalar product $(\psi_1, \psi_2) = (q_1, q_2) + (p_1, p_2)$. We denote by J the symplectic operator on $\Omega : J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Let us consider the class $\mathcal{L}_{\rm symp}(\Omega)$ of bounded $\mathbf{R}$-linear operators $A : \Omega \to \Omega$ which commute with the symplectic operator:

$$AJ = JA. \tag{10}$$

This is a subalgebra of the algebra of bounded linear operators $\mathcal{L}(\Omega)$. We also consider the space of $\mathcal{L}_{\rm symp,s}(\Omega)$ consisting of self-adjoint operators.

By using the operator J we can introduce on the phase space Ω the complex structure. Here J is realized as $-i$. We denote Ω endowed with this complex structure by $\Omega_c : \Omega_c \equiv Q \oplus iP$. We shall use it later. At the moment consider Ω as a real linear space and consider its complexification $\Omega^{\mathbf{C}} = \Omega \oplus i\Omega$.

Let us consider the functional space $\mathcal{V}_{\rm symp}(\Omega)$ consisting of functions $f : \Omega \to \mathbf{R}$ such that:

(a) the state of vacuum is preserved: $f(0) = 0$;

(b) f is J-invariant: $f(J\psi) = f(\psi)$;

(c) f can be extended to the analytic function $f : \Omega^{\mathbf{C}} \to \mathbf{C}$ having the exponential growth

$$|f(\psi)| \leq c_f e^{r_f \|\psi\|} \tag{11}$$

for some $c_f, r_f \geq 0$ and for all $\psi \in \Omega^{\mathbf{C}}$. We remark that the possibility to extend a function f analytically onto $\Omega^{\mathbf{C}}$ and the exponential estimate on $\Omega^{\mathbf{C}}$ plays the important role in the asymptotic expansion of Gaussian integrals. To get a mathematically rigorous formulation, conditions in [8] should be reformulated in the similar way.

The following trivial mathematical result plays the fundamental role in establishing classical $\to$ quantum correspondence: *Let f be a smooth J-invariant function. Then* $f''(0) \in \mathcal{L}_{\rm symp,s}(\Omega)$. In particular, a quadratic form is J-invariant iff it is determined by an operator belonging to $\mathcal{L}_{\rm symp,s}(\Omega)$.

We consider the space statistical states $S^{\alpha}_{G,\rm symp}(\Omega)$ consisting of measures ρ on Ω such that: a) ρ has zero mean value; b) it is a Gaussian measure; c) it is J-invariant; d) its dispersion has the magnitude α. Thus these are J-invariant Gaussian measures such that

$$\int_\Omega \psi d\rho(\psi) = 0 \text{ and } \sigma^2(\rho) = \int_\Omega \|\psi\|^2 d\rho(\psi) = \alpha, \ \alpha \to 0. \tag{12}$$

Such measures describe small Gaussian fluctuations of the vacuum field.

The following trivial mathematical result plays the fundamental role in establishing classical $\rightarrow$ quantum correspondence: *Let a measure ρ be J-invariant. Then its covariation operator $B = \mathrm{cov}\ \rho \in$* $\mathcal{L}_{\mathrm{symp,s}}(\Omega)$. Here $(By_1, y_2) = \int (y_1, \psi)(y_2, \psi) d\rho(\psi)$.

We now consider the complex realization Ω_c of the phase space and the corresponding complex scalar product $\langle \cdot, \cdot \rangle$. We remark that the class of operators $\mathcal{L}_{\mathrm{symp}}(\Omega)$ is mapped onto the class of $\mathbf{C}$-linear operators $\mathcal{L}(\Omega_c)$. We also remark that, for any $A \in \mathcal{L}_{\mathrm{symp,s}}(\Omega)$, real and complex quadratic forms coincide:

$$(A\psi, \psi) = \langle A\psi, \psi \rangle. \tag{13}$$

We also define for any measure its complex covariation operator $B^c = \mathrm{cov}^{\mathrm{c}}\rho$ by

$$\langle B^c y_1, y_2 \rangle = \int \langle y_1, \psi \rangle \ \langle \psi, y_2 \rangle d\rho(\psi). \tag{14}$$

We remark that for a J-invariant measure ρ its complex and real covariation operators are related as $B^c = 2B$. As a consequence, we obtain that any J-invariant Gaussian measure is uniquely determined by its complex covariation operator.

Remark. (The origin of complex numbers) In our approach the complex structure of QM has a natural physical explanation. The prequantum classical field $\psi(x)$ ("background field") is a vector field, so $\psi(x)$ has two real components $q(x)$ and $p(x)$. And these components are coupled in such a way that physical variables of the ψ-field, $f = f(q,p)$, are J-invariant. Second derivatives of such functionals are J-invariant $\mathbf{R}$-linear symmetric operators, $f''(0) \in \mathcal{L}_{\mathrm{symp,s}}(\Omega)$. As pointed out, this space of operators can be represented as the space of $\mathbf{C}$-linear operators $\mathcal{L}_s(\Omega_c)$. But QM takes into account only second derivatives of functionals of the vector prequantum field.

As in the real case [8], we can prove that for any operator $A \in \mathcal{L}_{\mathrm{symp,s}}(\Omega)$:

$$\int_\Omega \langle A\psi, \psi \rangle d\rho(\psi) = \mathrm{Tr}\ \mathrm{cov}^{\mathrm{c}}\rho\ \mathrm{A}. \tag{15}$$

We point out that the trace is considered with respect to the complex inner product. We consider now the one parameter family of classical statistical models:

$$M^\alpha = (S^\alpha_{G,\mathrm{symp}}(\Omega), \mathcal{V}_{\mathrm{symp}}(\Omega)),\ \alpha \geq 0. \tag{16}$$

Lemma 1. *Let $f \in \mathcal{V}_{\mathrm{symp}}(\Omega)$ and let $\rho \in S^\alpha_{G,\mathrm{symp}}(\Omega)$. Then the following asymptotic equality holds:*

$$\langle f \rangle_\rho = \frac{\alpha}{2}\ \mathrm{Tr}\ \mathrm{D^c}\ \mathrm{f}''(0) + \mathrm{o}(\alpha),\ \alpha \to 0, \tag{17}$$

where the operator $D^c = (\mathrm{cov}^c\ \rho)/\alpha$. *Here*

$$o(\alpha) = \alpha^2 R(\alpha, f, \rho), \tag{18}$$

where $|R(\alpha, f, \rho)| \leq c_f \int_\Omega e^{r_f \|\psi\|} d\rho_{D^c}(\psi)$.

Here ρ_{D^c} is the Gaussian measure with zero mean value and the complex covariation operator D^c. The proof of this lemma can be found in the appendix. We point out that Lemma 1 is a purely mathematical result giving the expansion of Gaussian integrals over the infinite dimensional Hilbert phase space with respect to the small parameter (dispersion of vacuum fluctuations).

We see that the classical average (computed in the model $M^\alpha = (S^\alpha_{G,\mathrm{symp}}(\Omega), \mathcal{V}_{\mathrm{symp}}(\Omega))$ by using the measure-theoretic approach) is coupled through (17) to the quantum average (computed in the model $N_{\mathrm{quant}} = (\mathcal{D}(\Omega_c), \mathcal{L}_{\mathrm{s}}(\Omega_c))$ by the von Neumann trace-formula).

The equality (17) can be used as the motivation for defining the following classical $\to$ quantum map T from the classical statistical model $M^\alpha = (S^\alpha_{G,\mathrm{symp}}, \mathcal{V}_{\mathrm{symp}})$ onto the quantum statistical model $N_{\mathrm{quant}} = (\mathcal{D}, \mathcal{L}_{\mathrm{s}})$:

$$T : S^\alpha_{G,\mathrm{symp}}(\Omega) \to \mathcal{D}(\Omega_c), \quad D^c = T(\rho) = (\mathrm{cov}^c\ \rho)/\alpha \tag{19}$$

(the Gaussian measure ρ is represented by the density matrix D^c which is equal to the complex covariation operator of this measure normalized by α);

$$T : \mathcal{V}_{\mathrm{symp}}(\Omega) \to \mathcal{L}_{\mathrm{s}}(\Omega_c), \quad A_{\mathrm{quant}} = T(f) = \frac{1}{2} f''(0). \tag{20}$$

Our previous considerations can be presented as

Theorem 1. *The one parametric family of classical statistical models* $M^\alpha = (S^\alpha_{G,\mathrm{symp}}(\Omega), \mathcal{V}_{\mathrm{symp}}(\Omega))$ *provides dequantization of the quantum model* $N_{\mathrm{quant}} = (\mathcal{D}(\Omega_c), \mathcal{L}_{\mathrm{s}}(\Omega_c))$ *through the pair of maps (19) and (20). The classical and quantum averages are coupled by the asymptotic equality (17).*

4. PURE STATES

Let $\Psi = u + iv \in \Omega_c$, so $u \in Q, v \in P$ and let $||\Psi|| = 1$. By using the conventional terminology of quantum mechanics we say that such a normalized vector of the complex Hilbert space Ψ represents a *pure quantum state.* By Born's interpretation of the wave function a pure state Ψ determines the statistical state with the density matrix:

$$D_\Psi = \Psi \otimes \Psi. \tag{21}$$

This Born's interpretation of the Ψ – which is, on one hand, the pure state (normalized vector $\Psi \in \Omega_c$) and, on the other hand, the statistical state D_Ψ – was the root of appearance in QM such a notion as individual (or irreducible) randomness. Such a randomness could not be reduced to classical ensemble randomness, see von Neumann [16].

In our approach the density matrix D_Ψ has nothing to do with the individual state (classical field). The density matrix D_Ψ is the image of the classical statistical state – the J-invariant Gaussian measure $\rho_\Psi \equiv \rho_{B_\Psi}$ on the phase space that has the zero mean value and the complex covariation operator $B_\Psi = \alpha D_\psi$.

PCSFT-interpretation of pure states. *There are no "pure quantum states." States that are interpreted in the conventional quantum formalism as pure states, in fact, represent J-invariant Gaussian measures having two dimensional supports. Such states can be imagined as fluctuations of fields concentrated on two dimensional real planes of the infinite dimensional state phase-space.*

5. SCHRÖDINGER DYNAMICS

States of systems with the infinite number of degrees of freedom - classical fields – are represented by points $\psi = (q,p) \in \Omega$; evolution of a state is described by the Hamiltonian equations. We consider a quadratic Hamilton function: $\mathcal{H}(q,p) = \frac{1}{2}(\mathbf{H}\psi, \psi)$, where $\mathbf{H} : \Omega \to \Omega$ is an arbitrary symmetric (bounded) operator; the Hamiltonian equations have the form: $\dot{q} = \mathbf{H}_{21}q + \mathbf{H}_{22}p$, $\dot{p} = -(\mathbf{H}_{11}q + \mathbf{H}_{12}p)$, or

$$\dot{\psi} = \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = J\mathbf{H}\psi. \tag{22}$$

(Thus quadratic Hamilton functions induce linear Hamilton equations.) From (22) we get $\psi(t) = U_t\psi$, where $U_t = e^{J\mathbf{H}t}$. The map $U_t\psi$ is a linear Hamiltonian flow on the phase space Ω. Let us consider an operator $\mathbf{H} \in \mathcal{L}_{\text{symp,s}}(\Omega)$: $\mathbf{H} = \begin{pmatrix} R & T \\ -T & R \end{pmatrix}$. This operator defines the quadratic Hamilton function $\mathcal{H}(q,p) = \frac{1}{2}[(Rp,p) + 2(Tp,q) + (Rq,q)]$, where the operator R is symmetric and the operator T is skew symmetric. Corresponding Hamiltonian equations have the form

$$\dot{q} = Rp - Tq, \ \dot{p} = -(Rq + Tp). \tag{23}$$

We point out that for a J-invariant Hamilton function, the Hamiltonian flow $U_t \in \mathcal{L}_{\text{symp}}(\Omega)$. By considering the complex structure on the infinite-dimensional phase space Ω we write the Hamiltonian equations (22) in the form of the Schödinger equation on Ω_c :

$$id\psi/dt = \mathbf{H}\psi. \tag{24}$$

Its solution has the following complex representation: $\psi(t) = U_t\psi, \quad U_t = e^{-i\mathbf{H}t}$. We consider the Planck system of units in that $h = 1$. This is *the complex representation of flows corresponding to quadratic J-invariant Hamilton functions.*

By choosing $H = L_2(\mathbf{R}^n)$ we see that the interpretation of the solution of this equation coincides with the original interpretation of Schrödinger – this is a classical field $\psi(t, x) = (q(t, x), p(t, x)$.

Example 1. Let us consider an important class of Hamilton functions

$$\mathcal{H}(q, p) = \frac{1}{2}[(Rp, p) + (Rq, q)], \tag{25}$$

where R is a symmetric operator. The corresponding Hamiltonian equations have the form:

$$\dot{q} = Rp, \quad \dot{p} = -Rq. \tag{26}$$

We now choose $H = L_2(\mathbf{R}^3)$, so $q(x)$ and $p(x)$ are components of the vector-field $\psi(x) = (q(x), p(x))$. We can call fields $q(x)$ and $p(x)$ *mutually inducing.* The field $p(x)$ induces dynamics of the field $q(x)$ and vice versa, cf. with electric and magnetic components, $q(x) = E(x)$ and $p(x) = B(x)$, of the electromagnetic field, cf. Einstein and Infeld [21], p. 148: "Every change of an electric field produces a magnetic field; every change of this magnetic field produces an electric field; every change of ..., and so on." We can write the form (25) as $\mathcal{H}(q, p) = \frac{1}{2}\int_{\mathbf{R}^6} R(x, y)[q(x)q(y) + p(x)p(y)]dxdy$ or $\mathcal{H}(\psi) = \frac{1}{2}\int_{\mathbf{R}^6} R(x, y)\psi(x)\bar{\psi}(y)dxdy$, where $R(x, y) = R(y, x)$ is in general a distribution on $\mathbf{R}^6$. We call such a kernel $R(x, y)$ a *self-interaction potential* for the background field $\psi(x) = (q(x), p(x))$.We point out that $R(x, y)$ induces a self-interaction of each component of the $\psi(x)$, but there is no cross-interaction between components $q(x)$ and $p(x)$ of the vector-field $\psi(x)$.

6. STATIONARY PURE STATES AS INVARIANT GAUSSIAN MEASURES

All Gaussian measures considered in this section are supposed to be J-invariant. As we have seen in section 4, so called pure states $\Psi, ||\Psi|| = 1$, are just labels for Gaussian measures concentrated on one dimensional (complex) subspaces Ω_Ψ of the infinite-dimensional phase-space Ω. In this section we study the case of so called *stationary (pure) states* in more detail. The α-scaling does not play any role in present considerations. Therefore we shall not take it into account. We consider a pure state $\Psi, ||\Psi|| = 1$, as the label for the Gaussian measure ν_Ψ having the zero mean value and the complex covariation operator $\mathrm{cov}^c \nu_\Psi = \Psi \otimes \Psi$.

Theorem 2. *Let ν be a Gaussian measure (with zero mean value) concentrated on the one-dimensional (complex) subspace corresponding to a normalized vector Ψ. Then ν is invariant with respect to the unitary dynamics $U_t = e^{-it\mathbf{H}}$, where $\mathbf{H} : \Omega \to \Omega$ is a bounded self-adjoint operator, iff Ψ is an eigenvector of $\mathbf{H}$.*

Proof. A). Let $\mathbf{H}\Psi = \lambda\Psi$. The Gaussian measure $U_t^*\nu$ has the covariation operator $B_t^c = U_t(\Psi \otimes \Psi)U_t^* = U_t\Psi \otimes U_t\Psi = e^{-it\lambda}\Psi \otimes e^{-it\lambda}\Psi = \Psi \otimes \Psi$. Since all measures under consideration are Gaussian, this implies that $U_t^*\nu = \nu$. Thus ν is an invariant measure.

B). Let $U_t^*\nu = \nu$ and $\nu = \nu_\Psi$ for some $\Psi, ||\Psi|| = 1$. We have that $U_t\Psi \otimes U_t\Psi = \Psi \otimes \Psi$. Thus, for any $\psi_1, \psi_2 \in \Omega$, we have

$$\langle \psi_1, U_t\Psi \rangle \langle U_t\Psi, \psi_2 \rangle = \langle \psi_1, \Psi \rangle \langle \Psi, \psi_2 \rangle. \tag{27}$$

Let us set $\psi_2 = \Psi$. We obtain: $\langle \psi_1, \overline{c(t)} U_t\Psi \rangle = \langle \psi_1, \Psi \rangle$, where $c(t) = \langle U_t\Psi, \Psi \rangle$. Thus $\overline{c(t)} U_t\Psi = \Psi$.We point out that $c(0) = ||\Psi||^2 = 1$. Thus $\overline{c'(0)}\Psi - i\mathbf{H}\Psi = 0$, or $\mathbf{H}\Psi = -i\overline{c'(0)}\Psi$. Thus Ψ is an eigenvector of $\mathbf{H}$ with the eigenvalue $-i\overline{c'(0)}$. We remark that $c'(0) = -i\langle \mathbf{H}\Psi, \Psi \rangle$; so $\overline{c'(0)} = i\langle \mathbf{H}\Psi, \Psi \rangle$. Hence, $\lambda = -i\overline{c'(0)} = \langle \mathbf{H}, \Psi, \Psi \rangle$.

Conclusion. *Stationary states of the quantum Hamiltonian (represented by a bounded self-adjoint operator $\mathbf{H}$) are just labels for Gaussian one-dimensional measures (with the zero mean value) that are invariant with respect to the Schrödinger dynamics $U_t = e^{-it\mathbf{H}}$.*

We now describe all possible Gaussian measures which are U_t-invariant.

Theorem 3. *Let $\mathbf{H}$ be a bounded self-adjoint operator with purely discrete nondegenerate spectrum: $\mathbf{H}\Psi_k = \lambda_k \Psi_k$, so $\{\Psi_k\}$ is an orthonormal basis consisting of eigenvectors of $\mathbf{H}$. Then any U_t-invariant Gaussian measure ν (with the zero mean value) has the covariance operator of the form:*

$$B^c = \sum_{k=1}^{\infty} c_k \Psi_k \otimes \Psi_k, c_k \geq 0, \tag{28}$$

and vice versa.

Proof. (A) Let $\mathrm{cov}^c\nu = B^c$ has the form (28). Then

$$\mathrm{cov}^c U_t^*\nu = U_t B U_t^* = \sum_{k=1}^{\infty} c_k e^{-i\lambda_k t}\Psi_k \otimes e^{-i\lambda_k t}\Psi_k = \mathrm{cov}^c\nu = B^c. \quad (29)$$

Since measures are Gaussian, this implies that $U_t^*\nu = \nu$ for any t.

(B) Let $U_t^*\nu = \nu$ for any t. We remark that any covariation operator B^c can be represented in the form:

$$B^c = \sum_{k=1}^{\infty}\langle B^c\Psi_k, \Psi_k\rangle\Psi_k \otimes \Psi_k + \sum_{k\neq j}\langle B^c\Psi_k, \Psi_j\rangle\Psi_k \otimes \Psi_j. \quad (30)$$

We shall show that $\langle B^c\Psi_k, \Psi_j\rangle = 0$ for $k \neq j$. Denote the operator corresponding to $\sum_{k\neq j}$ by Z. We have

$$\langle U_t Z U_t \psi_1, \psi_2\rangle = \sum_{k\neq j}\langle B^c\Psi_k, \Psi_j\rangle e^{it(\lambda_j-\lambda_k)}\langle\Psi_k, \psi_2\rangle\langle\psi_1, \Psi_j\rangle = \langle Z\psi_1, \psi_2\rangle. \quad (31)$$

Set $\psi_1 = \Psi_j, \psi_2 = \Psi_k$; then

$$\langle U_t Z U_t^* \Psi_j, \Psi_k\rangle = \langle B^c\Psi_k, \Psi_j\rangle e^{it(\lambda_j-\lambda_k)} = \langle B^c\Psi_k, \Psi_j\rangle. \quad (32)$$

Thus $\langle B^c\Psi_k, \Psi_j\rangle = 0, k \neq j$.

7. STABILITY OF HYDROGEN ATOM

As we have seen, in PCSFT so called stationary (pure) states of quantum mechanics are just labels for Gaussian measures (which are J-invariant and have zero mean value) that are U_t-invariant. We now apply our standard α-scaling argument and we see that a stationary state Ψ is a label for the Gaussian measure ρ_Ψ with $\mathrm{cov}^c\rho_\Psi = \alpha\Psi\otimes\Psi$. This measure is concentrated on one-dimensional (complex) subspace Ω_Ψ of phase space Ω. Therefore each realization of an element of the Gaussian ensemble of classical fields corresponding to the statistical state ρ_Ψ gives us the field of the shape $\Psi(x)$, but magnitudes of these fields vary from one realization to another. But by the well known Chebyshov inequality probability that $\mathcal{E}(\Psi) = \int_{\mathbf{R}^3}|\Psi(x)|^2dx$ is large is negligibly small.

Thus we have Gaussian fluctuations of very small magnitudes of the same shape $\Psi(x)$. In PCSFT a stationary quantum state can not be identified with a stationary classical field, but only with an ensemble of fields having the same shape $\Psi(x)$. Let us now compare descriptions of dynamics of electron in hydrogen atom given by quantum mechanics and our prequantum field theory.

In quantum mechanics stationary bound states of hydrogen atom are of the form

$$\Psi_{nlm}(r, \theta, \phi) = c_{n,l} R^l L_{n+l}^{2l+1}(R) e^{-R/2} Y_l^m(\theta, \phi), \tag{33}$$

where $R = 2r/na_0$, and $a_0 = \frac{h^2}{\mu e^2}$ is a characteristic length for the atom (Bohr radius). We are mainly interested in the presence of the component $e^{-R/2}$.

In PCSFT this stationary bound state is nothing else, but the label for the Gaussian measure $\rho_{\Psi_{nlm}}$ which is concentrated on the subspace $\Omega_{\Psi_{nlm}}$. Thus PCSFT says that "electron in atom" is nothing else than Gaussian fluctuations of a certain classical field, namely the field $\Psi_{nlm}(r, \theta, \phi)$:

$$\psi_{nlm}(r, \theta, \phi; \psi) = \gamma(\psi) \Psi_{nlm}(r, \theta, \phi), \tag{34}$$

where $\gamma(\psi)$ is the C-valued Gaussian random variable: $E\gamma = 0, E|\gamma|^2 = \alpha$.

The intensiveness of the field $\Psi_{nlm}(r, \theta, \phi, \psi)$ varies, but the shape is the same. Therefore this random field does not produce any significant effect for large R (since $e^{-R/2}$ eliminates such effects).

Thus in PCSFT the hydrogen atom stable, since the prequantum random fields $\psi_{nlm}(r, \theta, \phi; \psi)$ have a special shape (decreasing exponentially $R \to \infty$).

8. INFINITE DIMENSION OF SPACE

This is a good place to discuss the role of physical space represented by $\mathbf{R}^3$ in our model. In PCSFT the *physical space* is Hilbert space. If we choose the realization

$$H = L_2(\mathbf{R}^3), \tag{35}$$

then we obtain the realization of H as the space of classical fields on $\mathbf{R}^3$. So the *conventional space* $\mathbf{R}^3$ *appears only through this special representation of the Hilbert configuration space.* Dynamics in $\mathbf{R}^3$ in just a shadow of dynamics in the space of fields. However, we can choose other representations of Hilbert configuration space. In this way we shall obtain classical fields defined on other "physical spaces."

We remark that at first sight the situation with development of PCSFT is somewhat reminiscent of the one confronted by Schrödinger in his introduction of his wave equation, which "maps" waves in the configuration space (the idea that in part derives from Hamilton's mechanical-optical analogy that led Hamilton to his version of classical mechanics, cf. section 5 of this paper). However, as is well known, in the specific case considered by Schrödinger, the configuration space and the physical space were both $\mathbf{R}^3$, which coincidence was in part responsible for Scrödinger's hope that his equation describes an actual physical (wave) process in space-time. This hope did not materialize, given that in general the configuration space of physical systems is not $\mathbf{R}^3$. The difficulties of Schrödinger's program quickly led to Born's interpretation of the wave function in terms of probability or, as Scrödinger himself came to call it "expectation catalogue", which view is central in the orthodox or Copenhagen interpretation of quantum mechanics. Even though he had, just as did Einstein, major reservations concerning quantum mechanics as the ultimate theory of quantum phenomena, Schrödinger never went so far as to see any space other than $\mathbf{R}^3$ as real.

The same can be said about Einstein's attempts to go beyond quantum mechanics. His attempts (see, e.g., [21]) to create purely field model of physical reality did not induce rejection of the conventional model of physical space. Nevertheless, some of his comments might be interpreted as signs as coming rejection of the conventional model of physical space, see [25]: "Space-time does not claim existence on its own, but only as a structural quality of the field." "The requirement of general covariance takes away from space and time the last remnant of physical objectivity." And the following Einstein's remark is especially important for PCSFT's view to physical space: "There is no such thing as an empty space, i.e., a space without field. Space-time does not claim existence on its own, but only as a structural quality of the field." L. De Broglie in his theory of double solution (the first hidden variable model) emphasized the fundamental role of physical space $\mathbf{R}^3$, see, e.g., [26]. Such a viewpoint also was common for adherents of Bohmian mechanics (in any case for D. Bohm and J. Bell). We can conclude that all former models with field-like hidden variables were based on the conventional model of physical space, namely $\mathbf{R}^3$.

On the other hand, string theory does introduce spaces of higher dimensions, although not of infinite dimensions. This approach was one of inspirations for our radical viewpoint to physical space. One could speculate that on scales of quantum gravity and string theory space became infinite dimensional, just as those theories the space has the (finite) dimension higher than three. (In our approach quantum theory is not the ultimate theory. It has its boundaries of applications. Therefore there are no reasons to expect that "quantum gravity" should exist at all. Thus it would be better to speak not about scales of quantum gravity and string theory, but simply about the Planck scale for length and time.) Starting with classical statistical mechanics on the infinite dimensional physical space (PCSFT), we first obtain quantum mechanics and then classical statistical mechanics on the finite-dimensional phase space:

$$\lim_{h\to 0}\lim_{\alpha\to 0} M^{\alpha} = \lim_{h\to 0} N^{h}_{\text{quant}} = M_{\text{conv.class}}. \tag{36}$$

An intriguing question is what locality itself means in the physical space H, where H is the infinite dimensional Hilbert space. The first evident remark is that in PCSFT there are present interactions between field-systems which are not reduced to "physical potentials" $V(x,y)$ defined on the $\mathbf{R}^3 \times \mathbf{R}^3$. The Hamilton function for a pair of field-systems ψ and ϕ can contain nonquadratic terms corresponding to interactions between fields which are not given by "physical potentials" $V(x,y)$, e.g., $W(\psi,\phi) = \|\psi^n \phi^m\|^2$. Thus field systems ψ and ϕ can interact even if $V \equiv 0$. Hence our model is definitely nonlocal in the sense of the conventional locality in $\mathbf{R}^3$. Therefore it is clear that PCSFT-locality should be defined without direct relation to the conventional physical space $\mathbf{R}^3$. The discussion on PCSFT-locality is really going outside the main framework of the present paper. We postpone it to further publications.

9. APPENDIX. PROOF OF LEMMA 1

Proof of Lemma 1. In the Gaussian integral $\int_\Omega f(\psi)d\rho(\psi)$ we make the scaling:

$$\psi \to \psi/\sqrt{\alpha}. \tag{37}$$

We denote the image of the measure ρ under this change of variables by ρ_{D^c}, since the latter measure (which is also Gaussian) has the complex covariation operator D^c. We have:

$$\langle f\rangle_\rho = \int_\Omega f(\sqrt{\alpha}\psi)d\rho_{D^c}(\psi) = \frac{\alpha}{2}\int_\Omega (f''(0)\psi,\psi)\; d\rho_{D^c}(\psi) + \alpha^2 R(\alpha, f, \rho D^c), \tag{38}$$

where

$$R(\alpha,f,\rho) = \int_\Omega g(\alpha,f;\psi)d\rho_{D^c}(\psi), g(\alpha,f;\psi) = \sum_{n=4}^{\infty}\frac{\alpha^{n/2-2}}{n!}f^{(n)}(0)(\psi,...,\psi). \tag{39}$$

We point out that

$$\int_\Omega (f'(0),\psi)d\rho_{D^c}(\psi) = 0, \qquad \int_\Omega f'''(0)(\psi,\psi,\psi)d\rho_{D^c}(\psi) = 0, \tag{40}$$

because the mean value of ρ (and, hence, of ρ_{D^c}) is equal to zero. Since $\rho \in S^{\alpha}_{G,\text{symp}}(\Omega)$, we have $\text{Tr D}^{\text{c}} = 1$. We now estimate the rest term $R(\alpha, f, \rho)$. We recall the following inequality for functions of the exponential growth:

$$\|f^{(n)}(0)\| \leq c\, r^n, \; n = 0, 1, 2, \ldots . \tag{41}$$

This inequality is well known for analytic functions of the exponential growth $f : \mathbf{C}^n \to \mathbf{C}$. It was generalized to infinite-dimensional case in [22].

By using this inequality we have for $\alpha \leq 1$:

$$|g(\alpha, f; \psi)| = \sum_{n=4}^{\infty} \frac{\|f^{(n)}(0)\| \|\psi\|^n}{n!} \leq c_f \sum_{n=4}^{\infty} \frac{r_f^n \|\psi\|^n}{n!} = C_f e^{r_f \|\psi\|}. \tag{42}$$

Thus: $|R(\alpha, f, \rho)| \leq c_f \int_\Omega e^{r_f \|\psi\|} d\rho_{D^c}(\psi)$. We obtain:

$$\langle f \rangle_\rho = \frac{\alpha}{2} \int_\Omega (f''(0)\psi, \psi) \; d\rho_{D^c}(\psi) + o(\alpha), \; \alpha \to 0. \tag{43}$$

By using the equalities (13) and (15) we finally come the asymptotic equality (17).

10. APPENDIX. MAXWELL EQUATION AS A VERSION OF SCHRÖDINGER EQUATION

Our formalism can easily be generalized to the case of vector fields. We consider the configuration space: $H \equiv L_2(\mathbf{R}^n, \mathbf{R}^m) = \{\phi : \mathbf{R}^n \to \mathbf{R}^m, \phi(x) = (\phi_1(x), \ldots, \phi_m(x)), ||\phi||^2 = \int_{\mathbf{R}^m} \sum_j \phi_j^2(x) dx < \infty\}$. In this case $q(x) = (q_1(x), \ldots, q_m(x)), \; p(x) = (p_1(x), \ldots, p_m(x))$. For example, we can consider quadratic functionals: $\mathcal{H}(q_1, ..., q_m, p_1, ..., p_m) = \int_{\mathbf{R}^6} \sum_{ij} R_{ij}(x, y)(q_i(x) q_j(y)$
$+ p_i(x) p_j(y)) d^3x d^3y$. In our approach any smooth functional of prequantum fields is represented by its second derivative – its quantum image:

$$T(f) == \frac{1}{2} \begin{pmatrix} \frac{\partial^2 f}{\partial q_i \partial q_j}(0) & \frac{\partial^2 f}{\partial q_i \partial p_j}(0) \\ \frac{\partial^2 f}{\partial p_i \partial q_j}(0) & \frac{\partial^2 f}{\partial p_i \partial p_j}(0) \end{pmatrix}, \tag{44}$$

where all derivatives are functional derivatives.

Moreover, we need not consider the whole space $H = L_2(\mathbf{R}^n, \mathbf{R}^m)$ as the state space. Depending on the model we can choose some subspace H_0 of the Hilbert space H and consider the phase-space $\Omega = H_0 \times H_0$. Typically such a linear subspace is defined through a linear constraint for prequantum classical fields: $Cq = 0, Cp = 0$. In this case we choose $H_0 = \text{Ker C}$.

We start with the derivation of an "abstract Maxwell equation" in the Hamiltonian formalism on the infinite-dimensional phase space. Let us consider a J-commuting operator $\mathbf{H} : \Omega \to \Omega, \mathbf{H} = \text{diag}\,(R, R)$, where $R : H \to H$, and the corresponding Hamilton function $\mathcal{H}(\psi) = \frac{1}{2}(\mathbf{H}\psi, \psi) = \frac{1}{2}[(Rp, p) + (Rq, q)]$. The corresponding Hamilton equations have the form:

$$\frac{dq}{dt} = Rp, \quad \frac{dp}{dt} = -Rq. \tag{45}$$

Let us now consider the subspace H_0 of the space $L_2(\mathbf{R}^3, \mathbf{R}^3)$ consisting of all functions satisfying the condition:

$$(\nabla, \phi) = 0. \tag{46}$$

Let us choose the phase space $\Omega = H_0 \times H_0$ and let us consider the operator

$$R\phi = \nabla \times \phi. \tag{47}$$

We remark that the Hilbert spaceH_0 is an invariant subspace of the operator R, since $(\nabla, \nabla \times \phi) = 0$ for any ϕ.

Of course, in the rigorous mathematical framework we should start with a space $\mathcal{S}(\mathbf{R}^3, \mathbf{R}^3)$ of Schwartz's test functions, then choose in it the subspace $\mathcal{S}_0(\mathbf{R}^3, \mathbf{R}^3)$ consisting of functions satisfying (46), and complete this space with respect to the L_2-norm. The resulting space is that one we have denoted by H_0.

We note that $R : H_0 \to H_0$ is an unbounded operator. We define it on the domain of definition $\mathcal{S}_0(\mathbf{R}^3, \mathbf{R}^3)$. It is easy to see that R is symmetric on $\mathcal{S}_0(\mathbf{R}^3, \mathbf{R}^3)$:

$$\int_{\mathbf{R}^3} (\nabla \times \phi(x), \psi(x))dx = \int_{\mathbf{R}^3} (\phi(x), \nabla \times \psi(x))dx. \tag{48}$$

We consider the corresponding self-adjoint operator $R : H_0 \to H_0$. We now set in the Hamilton equations (45): $q(x) \equiv E(x) = (E_1(x), E_2(x), E_3(x))$, $p(x) \equiv B(x) = (B_1(x), B_2(x), B_3(x))$. Then this system coincides with the system of Maxwell equations in the empty space (for the system of units in which $c = 1$) :

$$\frac{\partial E}{\partial t}(t, x) = \nabla \times B(t, x), \quad \frac{\partial B}{\partial t}(t, x) = -\nabla \times E(t, x); \tag{49}$$

we have automatically $(\nabla, B(t, x)) = 0$ and $(\nabla, E(t, x)) = 0$; these conditions were incorporated into the definition of the state space. The corresponding Hamilton function has the form

$$\mathcal{H}(E, B) = \frac{1}{2}\int_{\mathbf{R}^3} [(\nabla \times E(x), E(x)) + (\nabla \times B(x), B(x))]d^3x. \tag{50}$$

We finish this section with the remark that, as any dynamics with J-invariant quadratic Hamilton function, the Maxwell dynamics can be written in the form of Schrödinger equation

$$i\partial\psi(t)/\partial t = \mathbf{H}\psi(t), \tag{51}$$

where $\psi(t,x) = E(t,x) + iB(t,x)$ and $\mathbf{H} = \text{diag}\,\{R, R\}, R\phi = \nabla \times \phi.$

The results of this paper were presented at seminars at Universities of Bonn and Mannheim. I would like to thank all participants of seminars for discussions and comments and Sergio Albeverio and Ernst Binz for hospitality.

REFERENCES

1. L. Marchildon, *Found. Phys.* **34**, 1453 (2004); *Quantum Mechanics: from Basic Principles to Numerical Methods and Applications* (Springer, New York, 2002).
2. G. ‘t Hooft, “Quantum mechanics and determinism,” hep-th/0105105; “Determinism beneath quantum mechanics,” quant-ph/0212095.
3. W. M. De Muynck, *Foundations of Quantum Mechanics, an Empiricists Approach* (Kluwer, Dordrecht, 2002); “Interpretations of quantum mechanics, and interpretations of violations of Bell’s inequality,” in *Foundations of Probability and Physics*, A. Yu. Khrennikov, ed.; *Q. Prob. White Noise Anal.* **13**, 95-114 (2001).
4. A. Plotnitsky, *Found. Phys.* **33**, 1649 (2003); *The Knowable and Unknowable (Modern Science, Nonclassical Thought, and the “Two Cultures)* (University of Michigan Press, 2002); “Quantum atomicity and quantum information: Bohr, Heisenberg, and quantum mechanics as an information theory,” in *Quantum Theory: Reconsideration of Foundations*, A. Yu. Khrennikov,ed. (Växjö University Press, Växjö, 2002), pp. 309-343.
5. A. Yu. Khrennikov, ed., *Foundations of Probability and Physics*; *Q. Prob. White Noise Anal.* **13** (WSP, Singapore, 2001); *Quantum Theory: Reconsideration of Foundations* (Ser. Math. Modeling **2**) (Växjö University Press, Växjö, 2002); *Foundations of Probability and Physics*-2 (Ser. Math. Modeling **5**) (Växjö University Press, Växjö, 2003); *Quantum Theory: Reconsideration of Foundations*-2 (Ser. Math. Modeling **10**) (Växjö University Press, Växjö, 2004); *Foundations of Probability and Physics-3* **750** (AIP Conference Proceedings, 2005).
6. K. Hess and W. Philipp, *Proc. Nat. Acad. Sc.* **98**, 14224 (2001); *Europhys. Lett.* **57**, 775 (2002); “Bell’s theorem: critique of proofs with and without inequalities,” in *Foundations of Probability and Physics*-3, A. Yu. Khrennikov, ed., **750** (AIP Conference Proceedings, 2005), pp. 150-157.

7. A. Yu. Khrennikov, *Ann. Phys.* **12**, 575 (2003); *J. Phys. A: Math. Gen.* **34**, 9965 (2001); *Nuovo Cimento B* **117**, 267 (2002); *J. Math. Phys.* **43** 789 (2002); ibid **45**, 902 (2004); *Doklady Mathematics* **71**, 363 (2005).
8. A. Yu. Khrennikov, *J. Phys. A: Math. Gen.*, **38**, 9051-9073 (2005); *Found. Phys. Lett.*, **18**, 637-650 (2005).
9. E. Nelson, *Quantum fluctuation* (Princeton University Press, Princeton, 1985).
10. L. de la Pena and A. M. Cetto, *The Quantum Dice: An Introduction to Stochastic Electrodynamics* (Kluwer Academic, Dordrecht, 1996).
11. T. H. Boyer, "A brief survey of stochastic electrodynamics," in *Foundations of Radiation Theory and Quantum Electrodynamics*, A. O. Barut, ed. (Plenum, New York, 1980).
12. T. H. Boyer, *Scientific American*, pp. 70-78, Aug. 1985.
13. L. de La Pena, *Found. Phys.* **12**, 1017 (1982); *J. Math. Phys.* **10**, 1620 (1969).
14. L. de La Pena and A. M. Cetto, *Phys. Rev. D* **3**, 795 (1971).
15. P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford University Press, 1930).
16. J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, NJ, 1955).
17. E. Binz, R. Honegger, and A. Rieckers, *Ann. Henri Poincaré* **5**, 327 (2004).
18. A. Yu. Khrennikov, *Supernalysis* (Nauka, Fizmatlit, Moscow, 1997), in Russian (English translation, Kluwer Academic, Dordreht, 1999).
19. L. Arnold, *Random Dynamical Systems* (Springer, New York, 1998).
20. J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, 1987).
21. A. Einstein and L. Infeld, *The Evolution of Physics. From Early Concepts to Relativity and Quanta* (Free Press, London, 1967).
22. A. Yu. Khrennikov, "Equations with infinite-dimensional pseudo-differential operators," Dissertation for the degree of candidate (phys-math. sc.) (Moscow, State University Press, Moscow, 1983).
23. A. Khrennikov and I. Volovich, *Soft Computing — A Fusion of Foundations, Methodologies and Applications* **10**, 521 (2005).
24. A. Yu. Khrennikov, *Found. Phys.* **32**, 1159 (2002); *Phys. Essays* **16**, 26 (2003).
25. A. Einstein, *The Collected Papers of Albert Einstein* (Princeton University Press, Princeton, 1993).
26. L. De Broglie, *The Current Interpretation of Wave Mechanics, Critical Study* (Elsevier, Amsterdam, 1964).