Foundations of Physics Letters, Vol. 18, No. 7, December 2005 (© 2005)
DOI: 10.1007/s10702-005-1317-y

GENERALIZATIONS OF QUANTUM MECHANICS INDUCED BY CLASSICAL STATISTICAL FIELD THEORY

Andrei Khrennikov

International Center for Mathematical Modeling in Physics and Cognitive Sciences
University of Växjö, Sweden
Email:Andrei.Khrennikov@msi.vxu.se

Received 28 June 2004; revised 26 July 2005

We show that the Dirac-von Neumann formalism for quantum mechanics can be obtained as an approximation of classical statistical field theory. This approximation is based on the Taylor expansion (up to terms of the second order) of classical physical variables – maps $f : \Omega \to \mathbf{R}$, where Ω is the infinite-dimensional Hilbert space. The space of classical statistical states consists of Gaussian measures ρ on Ω having zero mean value and dispersion $\sigma^2(\rho) \approx h$. This viewpoint to the conventional quantum formalism gives the possibility to create generalized quantum formalisms based on expansions of classical physical variables in the Taylor series up to terms of nth order and considering statistical states ρ having dispersion $\sigma^2(\rho) = h^n$ (for $n = 2$ we obtain the conventional quantum formalism).

Key words: classical statistical field theory, Dirac-von Neumann formalism, Taylor expansion on the Hilbert space, small Gaussian fluctuations.

1. INTRODUCTION

In Ref. [1] we demonstrated that in the opposition to a rather common opinion it is possible to consider the Dirac-von Neumann formalism as a natural approximation of classical statistical mechanics. In this note we do not have any possibility to discuss different views on the problems of interpretations of quantum mechanics and the correspondence between classical and quantum theories, see, e.g., Dirac [2] or von Neumann

[3]; see also, e.g., De Muynck [4], Plotnitsky [5], Marchildon [6] and Khrennikov [7] for recent discussions.

The crucial point of our approach is that a prequantum classical mechanics is not the conventional classical mechanics on the phase-space $\Omega = \mathbf{R}^3 \times \mathbf{R}^3$, but on the infinite-dimensional phase-space:

$$\Omega = L_2(\mathbf{R}^3) \times L_2(\mathbf{R}^3). \quad (1)$$

Thus one can save classical views to causality and realism, but the price is the infinite dimension of the phase-space. Our result does not contradict to various "NO-GO" theorems. For example, there is no contradiction with Bell's theorem, because ranges of values of classical physical variables (functionals of classical fields) and quantum observables (self-adjoint operators) do not coincide: quantum mechanics is only an approximate representation of classical statistical mechanics on the infinite-dimensional phase-space (see, e.g., Khrennikov [8] and Hess and Philipp [9] for recent debates on Bell's inequality).

This viewpoint to the quantum formalism gives the possibility to create generalized quantum formalisms based on expansions of classical physical variables in the Taylor series up to terms of nth order (for $n = 2$ we obtain the conventional quantum formalism). In this note we construct such generalized quantum formalisms. They give better approximations of the prequantum classical statistical field theory (PCSFT) than the Dirac-von Neumann formalism.

It would be very hard to obtain a direct experimental confirmation of PCSFT. We should approach such a scale of measurements in that not only h, but also h^n for any $n \geq 1$, would be considered as "macroscopic parameters." But we might approach approximations of PCSFT of different orders (The Dirac-von Neumann formalism is the first order approximation of PCSFT).

This paper is a natural concluding of investigations on quantum-like representations of classical contextual probabilistic models, see Ref. [10]. In these works it was shown that the Hilbert space probabilistic formalism can be considered as a "projection" of the classical Kolmogorov model (with a contextual interpretation of conditional probabilities). The main open question was about physical realizations of contexts. In this note we represent physical contexts of Ref. [10] by Gaussian ensembles of classical fields (having very small dispersion: $\sigma^2(\rho) \approx h, h \to 0$).

Our approach is quite close to Nelson's stochastic quantum mechanics and SED; see, e.g., Ref. [11]-[13]. We also reproduce quantum statistics by taking into account Gaussian fluctuations of classical fields.

This paper can be considered as a contribution to the old debate on incompleteness of quantum mechanics that was started more than 70 years ego by Einstein, Podolsky and Rosen, see their famous paper [14] on the "EPR-paradox". Our investigation supports the EPR-thesis on incompleteness of quantum mechanics. In our model both the position and momentum operators, $\hat{q}$ and $\hat{p}$, represent the "elements of physical reality": not reality of particles, but reality of fields. In PCSFT the $\hat{q}$ and $\hat{p}$ are images of functionals of classical fields, $f_q(\phi)$ and $f_p(\phi)$.

Our approach is very close to attempts of E. Schrödinger [15,16] and A. Einstein [17,18] to create purely (classical) field model inducing quantum mechanics.

We start with a short presentation of results of Ref. [1]. To simplify considerations, we consider the case of the real (separable) Hilbert space H. In the definition of the quantum statistical model N_{quant} the complex Hilbert space H_c should be changed to the real Hilbert space H. In particular, in this paper we consider as the space of quantum observables the space of self-adjoint (bounded) operators $\mathcal{L}_{\text{s}} \equiv \mathcal{L}_{\text{s}}(H)$ and as the space of quantum states the space of density operators (i.e., self-adjoint positive trace-one operators) in the real Hilbert space $\mathcal{D} \equiv \mathcal{D}(H)$.

2. STATISTICAL MODELS

The crucial point of our considerations is that classical and quantum models give us two different levels of description of physical reality. We can say that prequantum and quantum models provide, respectively, *ontic* and *epistemic* descriptions. The first describes nature as it is (as it is "when nobody looks"). The second is an observational model. It gives an image of nature through a special collection of observables, cf. Ref. [15], [16], [19]-[22]. QM is an example of an epistemic model of nature. In fact, this was the point of view of N. Bohr and W. Heisenberg and many other adherents of the Copenhagen interpretation; cf. Ref. [23]. The only problem for us is that the majority of scientists supporting the Copenhagen interpretation deny the possibility to create a prequantum ontic model which would reproduce (in some way) quantum averages. We recall that (as was pointed out by one of the referees of this paper) it was not the whole Copenhagen school that "denied" the possibility to create prequantum models. Pauli, for example, just believed that such approaches would take away the efficiency of quantum formalism. The complete denial of these possibilities came later, mostly under the influence of the theorem of Bell.

In our approach the ontic description is given by a continuous field-model, cf. E. Schrödinger [16]: " We do give a complete description, continuous in space and time, without leaving any gaps, confirming the classical ideal of a description of something. But we do not claim that this something is the observed and observable facts."

We now discuss mathematical representations of *ontic and epistemic models.* Traditionally ontic models are represented as *"classical statistical models"*: (a) physical states ω are represented by points of some set Ω (state space); (b) physical variables are represented by functions $f : \Omega \to \mathbf{R}$ belonging to some functional space $V(\Omega)$; (c) statistical states are represented by probability measures on Ω belonging to some class $S(\Omega)$; d) the average of a physical variable (which is represented by a function $f \in V(\Omega)$) with respect to a statistical state (which is represented by a probability measure $\rho \in S(\Omega)$) is given by

$$< f >_\rho \equiv \int_\Omega f(\omega) d\rho(\omega). \tag{2}$$

A *classical statistical model* is a pair $M = (S(\Omega), V(\Omega))$.

In the Dirac-von Neumann formalism[(2),(3)] in the complex Hilbert space H_c the *quantum statistical model* N_{quant} is described in the following way: a) physical observables are represented by operators $A : H_c \to H_c$ belonging to the class of self-adjoint continuous operators $\mathcal{L}_s \equiv \mathcal{L}_s(H_c)$; b) statistical states are represented by density operators (the class of such operators is denoted by $\mathcal{D} \equiv \mathcal{D}(H_c)$; c) the average of a physical observable (which is represented by the operator $A \in \mathcal{L}_s(H_c)$) with respect to a statistical state (which is represented by the density operator $D \in \mathcal{D}(H_c)$) is given by von Neumann's formula:

$$< A >_D \equiv \text{Tr DA}. \tag{3}$$

The *quantum statistical model* is the pair $N_{\text{quant}} = (\mathcal{D}(H_c), \mathcal{L}_s(H_c))$.

3. QUANTUM APPROXIMATION OF CLASSICAL MECHANICS FOR FIELDS

Let us consider a classical statistical model in that the space of states $\Omega = H$. One may say "states of individual systems." But one should not forget that systems under consideration have the infinite number of degrees of freedom. In our classical model we choose the class of statistical states consisting of Gaussian measures with zero mean value and dispersion

$$\sigma^2(\rho) = \int_H \|x\|^2 d\rho(x) = h, \tag{4}$$

where $h > 0$ is a small real parameter. Denote such a class by the symbol $S_G^h(H)$. For $\rho \in S_G^h(H)$, we have Tr cov $\rho = \text{h}$. We recall that the covariation operator $B =$ cov ρ of a Gaussian measure ρ on the σ-field of Borel subsets of the Hilbert space H is defined by the equality

$$(By_1, y_2) = \int (y_1, x)(y_2, x) d\rho(x), y_1, y_2 \in H, \tag{5}$$

and it has the following properties: (a). $B \geq 0$, i.e., $(By, y) \geq 0, y \in H$; (b). B is a self-adjoint operator, $B \in \mathcal{L}_s(H)$; (c). B is a trace-class operator and

$$\mathrm{Tr}\ B = \int_H ||x||^2 d\rho(x) \tag{6}$$

The right-hand side of (6) defines *dispersion* of the probability ρ. Thus for a Gaussian probability we have

$$\sigma^2(\rho) = \mathrm{Tr}\ B. \tag{7}$$

We pay attention that the list of properties of the covariation operator of a Gaussian measure differs from the list of properties of a von Neumann density operator only by one condition: Tr D = 1, for a density operator D.

We denote the Gaussian measure with the zero mean value and the covariation operator $B = \mathrm{cov}\ \rho$ by the symbol ρ_B. We have for the Gaussian measure ρ_B :

$$< f >_{\rho_B} = \int_H f(x) d\rho_B(x). \tag{8}$$

We now make the change of variables (scaling) in this integral:

$$y = x/\sqrt{h}. \tag{9}$$

We remark that any linear transformation (in particular, scaling) preserves the class of Gaussian measures. Thus

$$\int_H f(x) d\rho_B(x) = \int_H f(\sqrt{h}y) d\rho_D(y). \tag{10}$$

To find the covariation operator D of the scaling ρ_D of the Gaussian measure ρ_B, we compute its Fourier transform

$$\tilde{\rho}_D(\xi) = \int_H e^{i(\xi, y)} d\rho_D(y) = \int_H e^{i(\xi, \frac{x}{\sqrt{h}})} d\rho_B(x) = e^{-\frac{1}{2h}(B\xi, \xi)}. \tag{11}$$

Thus

$$D = B/h = \mathrm{cov}\rho/h. \tag{12}$$

We shall use this formula later.

Let us consider a functional space $\mathcal{V}(H)$ that consists of analytic functions of exponential growth preserving the state of vacuum:

$$f(0) = 0 \text{ and there exist } C, \alpha \geq 0 : |f(x)| \leq Ce^{\alpha \|x\|}. \tag{13}$$

We remark that any function $f \in \mathcal{V}(H)$ is integrable with respect to any Gaussian measure on H; see, e.g., Ref. [24]. Let us consider the classical statistical model

$$M_a^h = (S_G^h(H), \mathcal{V}(H)). \tag{14}$$

Asymptotic expansion of classical averages: We recall that in mathematical considerations h is not a constant, but a small parameter. In our calculations we shall use the symbol $o(h)$ which is defined by

$$\lim_{h \to 0} \frac{o(h)}{h} = 0 \tag{15}$$

(so, e.g., $h^2 = o(h), h \to 0$). Let us find the average of a variable $f \in \mathcal{V}(H)$ with respect to a statistical state $\rho_B \in S_G^h(H)$:

$$< f >_{\rho_B} = \int_H f(x) d\rho_B(x) = \int_H f(\sqrt{h} y) d\rho_D(y)$$

$$= \sum_{n=2}^{\infty} \frac{h^{n/2}}{n!} \int_H f^{(n)}(0)(y, ..., y) d\rho_D(y), \tag{16}$$

where the covariation operator D is given by (12). We remark that

$$\int_H (f'(0), y) d\rho(y) = 0, \tag{17}$$

because the mean value of ρ is equal to zero. It is also important that any classical variable $f \in \mathcal{V}(H)$ preserves the state of vacuum, $f(0) = 0$: The field of the zero magnitude could not produce any effect.

Since $\rho_B \in S_G^h(H)$, we have

$$\text{Tr D} = 1. \tag{18}$$

The change of variables (9) can be considered as scaling of the magnitude of statistical (Gaussian) fluctuations. Fluctuations which were considered as very small,

$$\sigma(\rho) = \sqrt{h} \tag{19}$$

(where h is a small parameter), are considered in the new scale as standard normal fluctuations. We have (see Ref. [1] for details):

$$< f >_{\rho} = \frac{h}{2} \int_H (f''(0)y, y) \, d\rho_D(y) + o(h), \ h \to 0, \tag{20}$$

or

$$< f >_\rho = \frac{h}{2} \operatorname{Tr} D\, f''(0) + o(h),\ h \to 0. \tag{21}$$

We see that the classical average (computed in the model $M_a^h = (S_G^h(H), \mathcal{V}(H))$ by using measure-theoretic approach) is approximately equal to the quantum average (computed in the model $N_{\rm quant} = (\mathcal{D}(H), \mathcal{L}_{\rm s}(H))$ by the von Neumann trace-formula).

Classical $\to$ quantum correspondence: The equality (21) can be used as the motivation for defining the following classical $\to$ quantum map T from the classical statistical model $M_a^h = (S_G^h, \mathcal{V})$ to the quantum statistical model $N_{\rm quant} = (\mathcal{D}, \mathcal{L}_{\rm s})$:

$$T : S_G^h(H) \to \mathcal{D}(H), D = T(\rho) = \frac{\operatorname{cov} \rho}{h} \tag{22}$$

(the Gaussian measure ρ is represented by the density matrix D which is equal to the covariation operator of this measure normalized by the Planck constant h);

$$T : \mathcal{V}(H) \to \mathcal{L}_{\rm s}(H), A_{\rm quant} = T(f) = \frac{h}{2} f''(0). \tag{23}$$

Our previous considerations can be presented as

Theorem 1. *The map T defined by (22),(23) is one to one on the space of statistical states $S_G^h(H)$; the map $T : \mathcal{V}(H) \to \mathcal{L}_{\rm s}(H)$ is linear and the classical and quantum averages are asymptotically, $h \to 0$, equal, see (21).*

The correspondence between physical variables $f \in \mathcal{V}(H)$ and physical observables $A \in \mathcal{L}_{\rm s}(H)$ is not one to one, cf. E. Schrödinger(16): "The gaps eliminated from the wave picture have withdrawn to the connection between the wave picture and observable facts. The latter are not in one to one correspondence with the former." However, if we restrict the map T to the space of quadratic functionals of classical fields, then it becomes one to one correspondence.

Conclusion: *Quantum mechanics is an approximate statistical description of nature based on extracting Gaussian fluctuations of the magnitude $\sigma(\rho) = \sqrt{h}$ and neglecting by fluctuations of the magnitude $o(\sqrt{h})$.*

4. GENERALIZED QUANTUM MODELS

We have created the classical statistical model which induced the quantum statistical model. The quantum description is the result of neglecting by terms of the magnitude $o(h), h \to 0$, in the expansion of classical averages $< f >_\rho$ with respect to the small parameter $s = h^{1/2}, h \to 0$. (Here h is the Planck constant which is interpreted in purely statistical way as the magnitude of fluctuations of classical fields on the prequantum level, cf., e.g., Nelson [11] or de la Pena and Cetto [12].)

Since classical statistical states are given by Gaussian measures ρ with zero mean value, $a_\rho = 0$, terms of the magnitude $s = h^{1/2}$ are absent in the expansion of the average $< f >_\rho$ into a power series with respect to the parameter s.

This viewpoint to conventional quantum mechanics implies the evident possibility to generalize this formalism by considering higher order expansions of averages $< f >_\rho$ with respect to the small parameter $s = h^{1/2}$. We remark that for a Gaussian measure $\rho, a_\rho = 0$ implies that all its momenta of odd orders $a_\rho^{(k)}, k = 2n + 1, n = 0, 1, \dots,$ are also equal to zero[1]. Therefore the expansion of $< f >_\rho$ with respect to $s = h^{1/2}$ does not contain terms with s^{2n+1}. Hence this is the expansion with respect to $h^n (= s^{2n}), n = 1, 2, \dots$. We are able to create $o(h^n)$-generalization of quantum mechanics through neglecting by terms of the magnitude $o(h^n), h \to 0 (n = 1, 2, \dots)$ in the power expansion of the classical average. Of course, for $n = 1$ we obtain the conventional quantum mechanics. Let us consider the classical statistical model

$$M_a^h = (S_G^h(\Omega), \mathcal{V}(\Omega)), \tag{24}$$

where $\Omega = H$ is the real Hilbert space. By taking into account that $a_\rho^{2n+1} = 0, n = 0, 1, \dots,$ for $\rho \in S_G^h(\Omega)$, we have

$$< f >_\rho = \frac{h}{2} \mathrm{Tr}\, D f''(0) + \sum_{k=2}^{\infty} \frac{h^k}{(2k)!} \int_H f^{(2k)}(0)(y, \dots, y) d\rho_D(y), \tag{25}$$

where as always $D = \frac{\mathrm{cov} \rho}{h}$.

We now consider a new epistemic ("observational") statistical model which is a natural generalization of the conventional quantum mechanics. We start with some preliminary mathematical considerations. Let A and B be two n-linear symmetric forms. We define their trace by

$$\mathrm{Tr}\, BA = \sum_{j_1, \dots, j_n = 1}^{\infty} B(e_{j_1}, \dots, e_{j_h}) A(e_{j_1}, \dots, e_{j_n}), \tag{26}$$

[1] We recall that momentums of a measure ρ are defined by $a_\rho^{(k)}(z_1, \dots, z_k) = \int_H (z_1, x) \dots (z_k, x) d\rho(x)$.

if this series converges and its sum does not depend on the choice of an orthonormal basis $\{e_j\}$ in H. We remark that

$$< f >_\rho = \frac{h}{2}\text{Tr } D f''(0) + \sum_{k=2}^{n} \frac{h^k}{2k!}\text{Tr } a_{\rho_D}^{(2k)} f^{(2k)}(0) + o(h^n), h \to 0; \quad (27)$$

here we used the following result about Gaussian integrals:

Lemma 1. *Let A_k be a continuous k-linear form on H and let ρ_D be a Gaussian measure (with zero mean value and the covariation operator D). Then*

$$\int_H A_k(x, \ldots, x) d\rho_D(x) = \text{Tr } a_{\rho_D}^{(k)} A_k. \quad (28)$$

Proof. Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis in H. We apply the well known Lebesque theorem on majorant convergence. We set

$$f_N(x) = \sum_{j_1,\ldots,j_k=1}^{n} A_k(e_{j_1}, \ldots, e_{j_k})(e_{j_1}, x) \ldots (e_{j_k}, x). \quad (29)$$

We have

$$|f_N(x)| = |A_k(\sum_{j_1=1}^{N}(x, e_{j_1})e_{j_1} \ldots, \sum_{j_k=1}^{N}(x, e_{j_k})e_{j_k})| \leq ||A_k|| \, ||x||^k. \quad (30)$$

We therefore obtain

$$\int_H A_k(x, \ldots, x) d\rho_D(x) = \lim_{N\to\infty} \int_H f_N(x) d\rho_D(x)$$

$$= \sum_{j_1=1,\ldots,j_k=1}^{\infty} A_k(e_{j_1}, \ldots, e_{j_k}) \int_H (e_{j_1}, x) \ldots (e_{j_k}, x) d\rho_D(x) = \text{Tr } a_{\rho_D}^{(k)} A_k. \quad (31)$$

The proof is finished.

In particular, we obtained the inequality

$$|\text{Tr } a_{\rho_D}^{k} A_k| \leq ||A|| \int_H ||x||^k d\rho_D(x). \quad (32)$$

We now remark that for a Gaussian measure (with zero mean value) integrals (28) are equal to zero for $k = 2l + 1$. Thus

Tr $a_{\rho_D}^{(2l+1)} A_{2l+1} = 0$. It is easy to see that $2k$-linear forms (momenta of even order) $a_{\rho_D}^{2k}$ can be expressed through the covariance operator D :

$$a_{\rho_D}^{(2k)} = e(k, D) = \frac{d^{2k}}{dy^{2k}} e^{-\frac{1}{2}(Dy,y)}|_{y=0}. \tag{33}$$

In particular, $e(2, D)(z_1, z_2) = (Dz_1, z_2)$ and $e(4, D)(z_1, z_2, z_3, z_4)$

$$= (Dz_1, z_3)(Dz_2, z_4) + (Dz_2, z_3)(Dz_1, z_4) + (Dz_1, z_2)(Dz_3, z_4). \tag{34}$$

Thus (27) can be rewritten as

$$< f >_{\rho_B} = \frac{h}{2} \text{Tr } D f''(0) + \sum_{k=2}^{n} \frac{h^k}{2k!} \text{Tr } e(2k, D) f^{(2k)}(0) + o(h^n),\ h \to 0. \tag{35}$$

This formula is the basis of *a new quantum theory.* In this theory statistical states can be still represented by von Neumann density operators $D \in \mathcal{D}(H)$, but observables are represented by multiples $A = (A_2, A_4, \ldots, A_{2n})$, where A_{2j} are symmetric $2n$-linear forms on a Hilbert space H. In particular, the quadratic form A_2 can be represented by a self-adjoint operator. To escape mathematical difficulties, we can assume that forms A_{2j} are continuous. Denote the space of all such multiples A by $L_{2n}(H)$. We obtain the following generalization of the conventional quantum model:

$$N_{2n} = (\mathcal{D}(H), L_{2n}(H)). \tag{36}$$

Here the average of an observable $A \in L_{2n}(H)$ with respect to a state $D \in \mathcal{D}(H)$ is given by

$$< a >_D = \sum_{n=1}^{n} \text{Tr } e(2k, D) A_{2k}. \tag{37}$$

If one defines Tr $DA = \sum_{k=1}^{n}$ Tr $e(2k, D)A_{2k}$, then the formula (37) can be written as in the conventional quantum mechanics (von Neumann's formula of nth order):

$$< A >_D = \text{Tr} DA. \tag{38}$$

This model is the result of the following "quantization" procedure of the classical statistical model $M_a^h = (S_G^h(\Omega), \mathcal{V}(\Omega)$:

$$\rho \to D = \text{cov}\rho/h; \tag{39}$$

$$f \to A = (\frac{h}{2} f''(0), \frac{h^2}{4!} f^{(4)}(0), ..., \frac{h^n}{(2n)!} f^{(2n)}(0)) \tag{40}$$

(thus here $A_{2k} = h^k/(2k)! f^{(2k)}(0)$). The transformation T_{2n} given by (39), (40) maps the classical statistical model $M_a^h = (S_G^h(\Omega), \mathcal{V}(\Omega))$ onto generalized quantum model $N_{2n} = (H, \mathcal{D}(H), L_{2n}(H))$.

Theorem 1a. *For the classical statistical model $M_a^h = (S_G^h(\Omega), \mathcal{V}(\Omega))$, the classical $\to$ quantum map T_{2n}, defined by (39) and (40), is one-to-one for statistical states; it has a huge degeneration for variables. Classical and quantum averages are equal* $\bmod\, o(h^n), h \to 0$, *see (35).*

We pay attention to the simple mathematical fact that the degree of degeneration of the map $T_{2n} : \mathcal{V}(\Omega) \to L_{2n}(H)$ is decreasing for $n \to \infty$. Denote the space of polynomials of the degree $2n$ containing only terms of even degrees by the symbol $P_{2n}(H)$. Thus $f \in P_{2n}(H)$ iff $f(x) = Q_2(x,x) + Q_4(x,x,x,x) + \ldots + Q_{2n}(x,\ldots,x)$, where $Q_{2j} : H^{2j} \to \mathbf{R}$ is a symmetric $2j$-linear (continuous) form. The restriction of the map T_{2n} on the subspace $P_{2n}(H)$ of the space $\mathcal{V}(\Omega)$ is one-to-one. One can also consider a generalized quantum model

$$N_\infty = (H, \mathcal{D}(H), L_\infty(H)), \tag{41}$$

where $L_\infty(H)$ consists of infinite sequences of $2n$-linear (continuous) forms on H :

$$A = (A_2, \ldots, A_{2n}, \ldots). \tag{42}$$

As a simple consequence of the expansion (35) of the classical average for a variable $f \in \mathcal{V}(\Omega)$, we obtain:

Theorem 2. *Averages given by the classical statistical model M and the generalized quantum model N_∞ coincide.*

5. COMPLEX HILBERT SPACE

We now briefly present a classical statistical field model for the conventional Dirac-von Neumann quantum model on the complex Hilbert space H_c. Here the most interesting problem is the establishing of correspondence between classical and quantum dynamical equations[1]; however, we are not able to present such a study in this letter.

We choose the phase space $\Omega = Q \times P$, where $Q = P = H$ and H is the infinite-dimensional real (separable) Hilbert space. We consider Ω as the real Hilbert space with the scalar product $(\omega_1, \omega_2) = (q_1, q_2) + (p_1, p_2)$. We denote by J the symplectic operator on $\Omega : J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Let us consider the class $\mathcal{L}_{\rm symp}(\Omega)$ of bounded $(\mathbf{R})$-linear operators $A : \Omega \to \Omega$ which commute with the symplectic operator $J : AJ = JA$. This is a subalgebra of the algebra of bounded linear operators $\mathcal{L}(\Omega)$. We also consider the subspace $\mathcal{L}_{\rm symp,s}(\Omega)$ of the space $\mathcal{L}_{\rm symp}(\Omega)$ consisting of self-adjoint operators.

Let us consider the functional space $\mathcal{V}_{\rm symp}(\Omega)$ consisting of real analytic functions, $f : \Omega \to \mathbf{R}$, such that: (a) the state of vacuum is preserved : $f(0) = 0$; (b) f is symplectically invariant: $f(J\phi) = f(\phi)$ for any $\phi \in \Omega$; (c) f is a (real) analytic function; (d) f has the exponential growth.

The following trivial mathematical result plays the fundamental role in establishing classical $\to$ quantum correspondence: *Let $f : \Omega \to \mathbf{R}$ be a smooth symplectically invariant function. Then* $f''(0) \in \mathcal{L}_{\rm symp,s}(\Omega)$. In particular, a quadratic form is symplectically invariant iff it is determined by an operator belonging to $\mathcal{L}_{\rm symp,s}(\Omega)$.

In our model we choose the space statistical states $S^h_{G,\rm symp}(\Omega)$ consisting of measures ρ on Ω such that: (a) ρ has zero mean value; (b) it is a Gaussian measure; (c) it is symplectically invariant (this is equivalent to the symplectic invariance of the Fourier transform $\tilde{\rho}$ of the measure ρ) ; (d) its dispersion has the magnitude h.

The following trivial mathematical result plays the fundamental role in establishing classical $\to$ quantum correspondence: *Let a measure ρ be symplectically invariant. Then its covariation operator* $B = \mathrm{cov}\,\rho \in \mathcal{L}_{\rm symp,s}(\Omega)$.

We introduce on the phase-space Ω the complex structure: $H_c = Q \oplus iP$ and the corresponding complex scalar product $< \cdot, \cdot >$; here the symplectic operator $J = -i$. We remark that the class of operators $\mathcal{L}_{\rm symp}(\Omega)$ is mapped onto the class of $\mathbf{C}$-linear (bounded) operators $\mathcal{L}(H_c)$.

We consider the classical statistical model

$$M^h_a = (S^h_{G,\rm symp}(\Omega), \mathcal{V}_{\rm symp}(\Omega)) \tag{43}$$

and the classical $\to$ quantum map T; see Sec. 3. We have:

Theorem 3. *The map T given by (22),(23) is well defined. It maps the space of statistical states $S^h_{G,\rm symp}(\Omega))$ onto the space of von Neumann density operators $\mathcal{D}(H_c)$ (it is one-to-one on this space) and the space of classical variables $\mathcal{V}(\Omega)$ onto the space of quantum observables $\mathcal{L}_{\rm s}(H_c)$. The classical and quantum averages are asymptotically,* $h \to 0$, *equal; see (21).*

I would like to thank A. Aspect, L. Ballentine, E. Nelsson, S. Albeverio, D. Greenberger, S. Gudder, G. ‘t Hooft, G. Adenier, I. Begtsson, V. Belavkin, G. Emch, Th. Nieuwenhuizen, A. Leggett, A. Peres, A. S. Holevo, H. Atmanspacher, A. Plotnitsky, I. Volovich, K. Hess, W.

Philipp, L. Marchildon, L. De La Pena and A. M. Cetto, D. Cole, S. Stenholm, B. Helsing, A. Kracklauer, B. Hiley, E. Loubents, L. Levitin, A. Sergienko, G. Jaeger, D. Mermin, P. Kwait, H. Rauch, M. Ohya, L. Accardi, P. Lahti, and A. Grib for fruitful discussions on the foundations of QM. I am especially thankful to Gerard ‘t Hooft for a series of discussions on the transition from classical to quantum description as the process of lost of information (“factorization on the space of ontic states”).

REFERENCES

1. A. Yu. Khrennikov, “Prequantum classical statistical model with infinite dimensional phase-space,” quant-ph/0505228, v1-v3; “Prequantum classical statistical model with infinite dimensional phase-space-2,” quant-ph/0505230.
2. P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford University Press, 1930).
3. J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, N.J., 1955).
4. W. M. De Muynck, *Foundations of Quantum Mechanics, an Empiricists Approach* (Kluwer Academic, Dordrecht, 2002); “Interpretations of quantum mechanics, and interpretations of violations of Bell’s inequality,” in *Foundations of Probability and Physics*, A. Yu. Khrennikov, ed., *Q. Prob. White Noise Anal.* **13**, 95-114 (2004).
5. A. Plotnitsky, *Found. Phys.* **33**, 1649 (2003).
6. L. Marchildon, *Found. Phys.* **34**, 1453 (2004); *Quantum Mechanics: From Basic Principles to Numerical Methods and Applications* (Springer, New York, 2002).
7. A. Yu. Khrennikov, *Found. Phys.* **35**, 1 (2005).
8. A. Yu. Khrennikov, *Found. Phys.* **32**, 1159 (2002); *Phys. Lett.*, A **278**, 307 (2001); *Nuovo Cimento* **119**, 131 (2004); *Found. Phys. Lett.* **17**, 691 (2004); *Interpretations of Probability* (VSP International Science Publishers, Utrecht, 1999).
9. K. Hess and W. Philipp, *Proc. Nat. Acad. Sc.* **98**, 14224 (2001); **98**, 14227(2001); **101**, 1799 (2004); *Europhys. Lett.* **57**, 775 (2002); “Bell’s theorem: critique of proofs with and without inequalities,” in *Foundations of Probability and Physics*-3, A. Yu. Khrennikov, ed. (AIP Conference Proceedings, 2005), pp. 150-157.
10. A. Yu. Khrennikov, *Ann. Phys.* **12**, 575 (2003); “On the classical limit for the hyperbolic quantum mechanics,” quant-ph/0401035; *J. Phys.A: Math. Gen.* **34**, 9965 (2001); *Nuovo Cimento B* **117**, 267 (2002); *J. Math. Phys.* **43** 789 (2002); ibid **45**, 902 (2004); *Doklady Math.* **71**, 363 (2005).

11. E. Nelson , *Quantum Fluctuations* (Princeton University Press, Princeton, 1985).
12. L. de la Pena and A. M. Cetto, *The Quantum Dice: An Introduction to Stochastic Electrodynamics* (Kluwer Academic, Dordrecht, 1996). T. H. Boyer, "A brief survey of stochastic electrodynamics," in *Foundations of Radiation Theory and Quantum Electrodynamics*, A. O. Barut, ed. (Plenum, New York, 1980). T. H. Boyer, *Sci. Am.*, August 1985, pp. 70-78.
13. L. de la Pena, *Found. Phys.* **12**, 1017 (1982); *J. Math. Phys.* **10**, 1620 (1969). L. de la Pena and A. M. Cetto, *Phys. Rev. D* **3**, 795 (1971).
14. A. Einstein, B. Podolsky, and N. Rosen, *Phys. Rev.* **47**, 777 (1935).
15. E. Schrödinger, *Philosophy and the Birth of Quantum Mechanics*, M. Bitbol and O. Darrigol, eds. (Editions Frontieres, Gif-sur-Yvette, 1992), especially the paper of S. D'Agostino, "Continuity and completeness in physical theory: Schrödinger's return to the wave interpretation of quantum mechanics in the 1950's," pp. 339-360.
16. E. Schrödinger, *E. Schrödinger, Gesammelte Abhandlungen* (Vieweg, Vienna, 1984); especially the paper "What is an elementary particle?," pp. 456-463.
17. A. Einstein, *The Collected Papers of Albert Einstein* (Princeton University Press, Princeton, 1993).
18. A. Einstein and L. Infeld, *The Evolution of Physics. From Early Concepts to Relativity and Quanta* (Free Press, London, 1967).
19. H. Atmanspacher and H. Primas, "Epistemic and ontic quantum realities," in *Foundations of Probability and Physics*-3, A. Yu. Khrennikov, ed. (AIP Conference Proceedings, 2005), pp. 10-25.
20. G. 't Hooft, "Quantum mechanics and determinism," hep-th/0105105.
21. G. 't Hooft,"Determinism beneath quantum mechanics," quant-ph/0212095.
22. A. Plotnitsky, *The Knowable and Unknowable (Modern Science, Nonclassical Thought, and the "Two Cultures")* (University of Michigan Press, 2002).
23. A. Plotnitsky, "Quantum atomicity and quantum information: Bohr, Heisenberg, and quantum mechanics as an information theory," in *Quantum Theory: Reconsideration of Foundations*, A. Yu. Khrennikov, ed.(Växjö University Press, 2002), pp. 309-343.
24. A. V. Skorohod, *Integration in Hilbert Space* (Springer, Berlin, 1974). A. L. Daletski and S. V. Fomin, *Measures and Differential Equations in Infinite-Dimensional Spaces* (Kluwer Academic, Dordrecht, 1991). O. G. Smolyanov, *Analysis on Topological Vector Spaces* (Moscow State University Press, Moscow, 1981). S. Albeverio, and M. Röckner, *Prob. Theory and Related Fields* **89**, 347 (1991).