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Quantum Entanglement: An Analysis via the Orthogonality Relation

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Abstract
In the literature there has been evidence that a kind of relational structure called a quantum Kripke frame captures the essential characteristics of the orthogonality relation between pure states of quantum systems, and thus is a good qualitative mathematical model of quantum systems. This paper adds another piece of evidence by providing a tensor-product construction of two finite-dimensional quantum Kripke frames. We prove that this construction is exactly the qualitative counterpart of the tensor-product construction of two finite-dimensional Hilbert spaces over the complex numbers, and thus show that composition of quantum systems, especially the phenomenon of quantum entanglement, can be modelled in the framework of quantum Kripke frames. The assumptions used in our construction hint that we need complex numbers in quantum theory. Moreover, for this construction, we give a new and interesting characterization of linear maps of trace 0 in terms of the orthogonality relation.

Keywords Orthogonality relation · Mathematical foundations of quantum theory · Quantum entanglement · Quantum logic

1 Introduction

Developed since the early twentieth century, quantum theory is a successful theoretical framework in describing microscopic objects on the one hand and a great source of conceptual dispute on the other. Mathematical foundations of quantum theory [1, 2], considered as initiated by Birkhoff and von Neumann [3], is one of the fields devoted to investigating the foundations of quantum theory and improving the conceptual understanding of it. According to my understanding, one of the

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main questions of this field is: *what are the characteristics in quantum physics of some basic physical concepts, which force us to use Hilbert spaces over $\mathbb{C}$ as mathematical models of physical systems*? To answer this question, the paradigm mainly consists of four steps: first, choose and start from some basic concepts in physics; second, use simple mathematical structures to model these concepts; third, find axioms, as simple and natural as possible, to characterize the features of these concepts in quantum theory; fourth, prove representation theorems of the mathematical structures satisfying the axioms via Hilbert spaces over $\mathbb{C}$. Besides these four steps, the mathematical structures emerging from this research are also applied to model quantum phenomena and tested whether they are good models of (certain aspects of) quantum systems.

In the literature, there have been many successful approaches in this field. The oldest and the most famous one was initiated by Birkhoff and von Neumann and carried on by Mackey [4], Jauch [5], Piron [6], Aerts [7] and others. It uses lattices to model the structure formed by testable properties of quantum systems. For other approaches, Mielnik [8] and Zabey [9] use probabilistic transition systems to model the structure of pure states formed with the transition probabilities; Dalla Chiara, Giuntini and others [10] use algebras to model the structure formed by effects of quantum systems; Baltag and Smets [11] use relational structures to model the structure of pure states formed under tests of properties; Isham, Döring, Butterfield and others [12–17] use topoi to model quantum measurements; Holik and others [18] use Cox's probability theory to model mixed states. If we may, we also consider the work of Abramsky, Coecke and others [19–21] as using monoidal categories to model composition of quantum systems and the work in the book [22] as using operational probabilistic theory to model quantum information.

This paper belongs to mathematical foundations of quantum theory, and we follow another approach which is in the work of Foulis and Randall [23], Dishkant [24], Goldblatt [25] and others. In this approach, we use sets each equipped with a binary relation to model the structure of pure states formed with the orthogonality relation. Considering the physical meaning and fundamental theoretical role of the orthogonality relation, we think that this approach is natural and close to the way how most physicists think about quantum systems. To be more precise, we will focus on a kind of mathematical structures called quantum Kripke frames. In [26] it is argued from two perspectives that they capture the essential characteristics of the (non-)orthogonality relation: from the mathematical perspective, a representation theorem of quantum Kripke frames via Hilbert spaces over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ is presented (Theorem 2.13); from the conceptual perspective, natural hypotheses are proposed to reveal the empirical rationale behind the definition of quantum Kripke frames. Moreover, in [27], which is an extension of [28], a duality is established between a category of quantum Kripke frames and one of Piron lattices, which shows a close connection between this approach and the traditional lattice-theoretic approach.

In this paper, we try to describe quantum entanglement using quantum Kripke frames. In quantum theory, quantum entanglement is described by the tensor product of Hilbert spaces over $\mathbb{C}$. Hence, mathematically the description of quantum entanglement is called the tensor product problem: *if we use two or more mathematical structures of a particular kind to model two or more quantum systems, respectively,* *how to construct the mathematical structure of the same kind that models the system formed by composing these quantum systems*? Many approaches in mathematical foundations of quantum theory have addressed this problem. In the traditional lattice-theoretic approach, this problem has been addressed at an early stage. By introducing probability measures on lattices, Randall and Foulis [29] show that, if we use orthomodular lattices to model two quantum systems, it is possible that there is no orthomodular lattice that models the composition of these two systems. In [30] Foulis and Bennett study orthoalgebras, which are more general than orthomodular lattices, and give existence conditions for the tensor product of two orthoalgebras. In [31] the authors give a characterization of $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ from $\mathcal{L}(\mathcal{H}_1)$ and $\mathcal{L}(\mathcal{H}_2)$[1], and discuss the physical justification of this characterization. In both [32] and [33] the authors give characterizations when a lattice of a particular kind is the tensor product of two lattices of the same kind, respectively. In [32] a construction is also given but is proved to be failed to construct $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ from $\mathcal{L}(\mathcal{H}_1)$ and $\mathcal{L}(\mathcal{H}_2)$; in [33] the characterization involves probability measures and the relation among $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, $\mathcal{L}(\mathcal{H}_1)$ and $\mathcal{L}(\mathcal{H}_2)$ is shown to be a special case of this characterization. In other approaches, Baltag and Smets [34] give an informational-logical characterization of entangled states in a bipartite quantum system in terms of local observations and remote evolutions. Finally, Abramsky, Coecke and others focus primarily on composition of quantum systems, and their work turns out to be of great value in both theory and application.

In this paper, we study the tensor product problem of quantum Kripke frames. Unlike [29] and [33], we do not introduce probability measures and stick to the purely qualitative framework. Moreover, we restrict our goal only to mimic one way of constructing the tensor product of two finite-dimensional Hilbert spaces over $\mathbb{C}$: the vectors in the tensor product are linear maps from one Hilbert space to the other, and two linear maps, f and g, are orthogonal if the trace of $f^\dagger \circ g$ is 0. As it turns out, we achieve our goal under some assumptions in an arguably natural way: Relying heavily on results in projective geometry in the literature, we find that, among the arrows in the category of quantum Kripke frames in [27] called continuous homomorphisms, there are counterparts of linear maps between finite-dimensional Hilbert spaces, which model the states of the composite system. Then we prove a theorem characterizing linear maps of trace 0 in terms of the orthogonality relation in a Hilbert space, and use it to define the (non-)orthogonality relation between continuous homomorphisms. Our construction is such that, if two quantum Kripke frames are abstracted from two finite-dimensional Hilbert spaces over $\mathbb{C}$, respectively, then our construction results in the quantum Kripke frame abstracted from the tensor product of them.

In a word, our work models quantum entanglement in terms of the (non-)orthogonality relation which is physically intuitive. Moreover, the result in this paper yields hope of solving the tensor product problem for approaches in mathematical foundations of quantum theory which are purely qualitative and primarily focus on physical concepts about a quantum system as a whole, like the relational one we use and the

[1] Here $\mathcal{L}(\mathcal{H})$ denotes the lattice formed by the closed linear subspaces of a Hilbert space $\mathcal{H}$.

lattice-theoretic one. Finally, the assumptions we use reflect some features of complex numbers which are not shared by real numbers and quaternions. This hints at why we use complex numbers in quantum theory and, in addition, that the reason may only be found when we describe composition of quantum systems and may not be found if we only consider a quantum system as a whole. This echoes some recent theoretical and experimental research on ruling out quantum theory over real numbers [35–37].

The rest of this paper is organized as follows: In Sect. 2, we review elements of the Hilbert space formalism of quantum theory, of quantum Kripke frames and of some relevant results in projective geometry. Section 3 introduces continuous homomorphisms which model quantum entanglement as correlation in terms of the (non-) orthogonality relation, and we find among them counterparts of linear maps from a Hilbert space to another which can model the states of a composite system. In Sect. 4, as a preparatory step for defining the (non-)orthogonality relation between the states of a composite system, we prove a characterization of linear maps of trace 0 in terms of the orthogonality relation. Section 5 presents the main result of this paper based on the previous two sections, and Sect. 6 contains the conclusion and some discussion. Appendix A deals with a technical subtlety; Appendix B lists the definitions and the results in projective geometry in [38] and [39] used in this paper.

2 Preliminaries

2.1 States in Quantum Theory

According to quantum theory, a quantum system is described by a Hilbert space[2] $\mathcal{H}$ over $\mathbb{C}$ in such a way that the (pure) states of the system correspond to one-dimensional subspaces of $\mathcal{H}$.

In mathematics it is well known that the one-dimensional subspaces of a vector space form a projective geometry. Therefore and in particular, the states of a quantum system form a special kind of projective geometry called an *irreducible Hilbertian geometry*. The definition, adapted from [38, 40, 41], is as follows:

Definition 2.1 A *projective geometry* is a tuple $\mathfrak{G} = (G, \star)$, where G is a non-empty set and $\star$ is a function from $G \times G$ to the power set of G such that all of the following hold: for all $a, b, c, d, r \in G$,

(P1) $a \star a = \{a\}$;
(P2) $a \in b \star a$;
(P3) if $a \in b \star r$, $r \in c \star d$ and $a \neq c$, then $(a \star c) \cap (b \star d) \neq \emptyset$.

[2] In this paper, we assume that every vector space V, including a Hilbert space, is of dimension at least 1, i.e. $V \neq \{\mathbf{0}\}$.

An *orthogeometry* is a tuple $\mathfrak{G} = (G, \star, \perp)$, where $(G, \star)$ is a projective geometry and $\perp \subseteq G \times G$ is such that all of the following hold: for all $a, b, c, q \in G$,

(O1) $a \perp b$ implies that $b \perp a$;
(O2) if $a \perp q$, $b \perp q$ and $c \in a \star b$, then $c \perp q$;
(O3) if $b \neq c$, then there is a $q \in b \star c$ such that $q \perp a$;
(O4) there is a $b \in G$ such that $a \not\perp b$.

For any $E, F \subseteq G$, $E^{\perp} \stackrel{\text{def}}{=} \{a \in G \mid a \perp b, \text{ for every } b \in E\}$ is called *the orthocomplement of* E *in* $\mathfrak{G}$; and $E \boxplus F$, called *the linear sum* of E and F, is defined as follows:

$$E \boxplus F \stackrel{\text{def}}{=} \begin{cases} \bigcup\{a \star b \mid a \in E, b \in F\}, & \text{if } E \neq \emptyset \text{ and } F \neq \emptyset \\ E \cup F, & \text{otherwise} \end{cases}$$

An orthogeometry is a *Hilbertian geometry*, if it satisfies the following:

(Hil) for each $E \subseteq G$, if $E = E^{\perp\perp}$, then $E \boxplus E^{\perp} = G$.

A projective geometry is *irreducible*, if it satisfies the following:

(Irr) for any $a, b \in G$ satisfying $a \neq b$, $a \star b$ contains at least three elements.

The following proposition formally expresses the idea that the states of a quantum system form a projective geometry.

Proposition 2.2 *For each Hilbert space $\mathcal{H}$ over $\mathbb{C}$, the tuple $\mathfrak{G}_{\mathcal{H}} = (\Sigma_{\mathcal{H}}, \star_{\mathcal{H}}, \perp_{\mathcal{H}})$ defined as follows is an irreducible Hilbertian geometry*:

- $\Sigma_{\mathcal{H}} \stackrel{\text{def}}{=} \big\{\langle \mathbf{v} \rangle \mid \mathbf{v} \in \mathcal{H} \setminus \{\mathbf{0}\}\big\}$, where $\langle \mathbf{v} \rangle$ is the one-dimensional subspace generated by $\mathbf{v}$;
- for any $s, t \in \Sigma_{\mathcal{H}}$, $s \star_{\mathcal{H}} t \stackrel{\text{def}}{=} \big\{\langle \mathbf{s} + \mathbf{t} \rangle \mid \mathbf{s} \in s, \mathbf{t} \in t \text{ and } \mathbf{s} + \mathbf{t} \neq \mathbf{0}\big\}$
- for any $s, t \in \Sigma_{\mathcal{H}}$, define that

$$s \perp_{\mathcal{H}} t, \text{ if and only if } \langle \mathbf{s}, \mathbf{t} \rangle = 0 \text{ is true for any } \mathbf{s} \in s \text{ and } \mathbf{t} \in t,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of $\mathcal{H}$. The relation $\perp_{\mathcal{H}}$ is called the *orthogonality relation*.

Proof A direct verification is not hard. For an indirect proof, combine Theorem 2.16 below and Proposition 2.7 in [26]. □

We will need some definitions and theorems about projective geometry.

Definition 2.3 (*Definition 2.3.1 in* [38]) A *subspace* of a projective geometry $\mathfrak{G} = (G, \star)$ is a set $E \subseteq G$ satisfying that, for any $a, b \in E$, $a \star b \subseteq E$.

Theorem 2.4 (*Proposition* 2.3.3 *in* [38]) *In a projective geometry* $\mathfrak{G} = (G, \star)$, *for each* $E \subseteq G$, *there is a smallest* (*in the sense of set inclusion*) *subspace of* $\mathfrak{G}$ *that contains* E.

This subspace is called the linear closure of E *and denoted by* $\mathcal{C}(E)$.

With the notion of linear closure, we build a dimension theory of projective geometry. The key definitions and theorems are as follows:

Definition 2.5 (*Definitions 4.1.1 and 4.1.7 in* [38]) Let $\mathfrak{G} = (G, \star)$ be a projective geometry and $E, F \subseteq G$.

- E is *independent*, if, for each $a \in E$, $a \notin \mathcal{C}(E \backslash \{a\})$.
- E is a *basis* of F, if E is independent and $F = \mathcal{C}(E)$.

Theorem 2.6 (*Theorems* 4.1.9 *and* 4.2.2 *in* [38]) *Let* $\mathfrak{G} = (G, \star)$ *be a projective geometry*.

1. *Every independent set contained in a subspace* E *of* $\mathfrak{G}$ *can be extended to a basis of* E.
2. *For each subspace* E *of* $\mathfrak{G}$, *each basis of* E *is of the same cardinality*, *which is called the rank of* E.

Remark 2.7 In [38] Theorems 4.1.9 and 4.2.2 are about matroids. According to Proposition 2.3.3 and Proposition 3.1.13, a projective geometry is a matroid.

2.2 Bipartite Entanglement in Quantum Theory

According to quantum theory, if two quantum systems are described by two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ over $\mathbb{C}$, respectively, then the quantum system consisting of these two systems is described by the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$.

There are many ways of constructing $\mathcal{H}_A \otimes \mathcal{H}_B$. We briefly review the one that is used here. We need the notion of the conjugate space of a Hilbert space.

Definition 2.8 (*Page 131 in* [42]) For a Hilbert space $\mathcal{H}$ over $\mathbb{C}$, its *conjugate space*, denoted by $\overline{\mathcal{H}}$, is a Hilbert space over $\mathbb{C}$ such that

1. $\mathcal{H}$ and $\overline{\mathcal{H}}$ have the same set of vectors;
2. for any $\mathbf{u}, \mathbf{v} \in \mathcal{H}$, $\mathbf{u} \bar{+} \mathbf{v} = \mathbf{u} + \mathbf{v}$;
3. for any $\mathbf{u} \in \mathcal{H}$ and $c \in \mathbb{C}$, $c \bar{\cdot} \mathbf{u} = c^* \cdot \mathbf{u}$;

4. for any $\mathbf{u}, \mathbf{v} \in \mathcal{H}$, $[\mathbf{u}, \mathbf{v}] = \langle \mathbf{v}, \mathbf{u} \rangle$,

where $+$, $\cdot$ and $\langle \cdot, \cdot \rangle$ are the addition, the scalar multiplication and the inner product of $\mathcal{H}$, $\overline{+}$, $\bar{\cdot}$ and $[\cdot, \cdot]$ are the addition, the scalar multiplication and the inner product of $\overline{\mathcal{H}}$ and $(\cdot)^*$ is the complex conjugate.[3]

Theorem 2.9 *For any two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ over $\mathbb{C}$, $\mathscr{HS}\mathscr{O}(\overline{\mathcal{H}_A}, \mathcal{H}_B)$ equipped with the addition $+$, the scalar multiplication $\cdot$ and the inner product $\langle \cdot, \cdot \rangle$ is the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$, where*

1. *$\mathscr{HS}\mathscr{O}(\overline{\mathcal{H}_A}, \mathcal{H}_B)$ is the set of all Hilbert-Schmidt operators from $\overline{\mathcal{H}_A}$ to $\mathcal{H}_B$; a Hilbert-Schmidt operator from $\overline{\mathcal{H}_A}$ to $\mathcal{H}_B$ is a bounded linear map $f : \overline{\mathcal{H}_A} \to \mathcal{H}_B$ such that $Tr(f^\dagger \circ f) = \sum_{\mathbf{v}_A \in B_A} \langle f(\mathbf{v}_A), f(\mathbf{v}_A) \rangle_B < \infty$, where Tr is the trace function and B_A is an orthonormal basis of $\mathcal{H}_A$ whose choice can be proved to be irrelevant;*
2. *for any $f, g \in \mathscr{HS}\mathscr{O}(\overline{\mathcal{H}_A}, \mathcal{H}_B)$, $(f + g)(\mathbf{v_A}) = f(\mathbf{v}_A) + g(\mathbf{v}_A)$, for each $\mathbf{v}_A \in \mathcal{H}_A$;*
3. *for any $f \in \mathscr{HS}\mathscr{O}(\overline{\mathcal{H}_A}, \mathcal{H}_B)$ and $c \in \mathbb{C}$, $(cf)(\mathbf{v_A}) = cf(\mathbf{v}_A)$, for each $\mathbf{v}_A \in \mathcal{H}_A$;*
4. *for any $f, g \in \mathscr{HS}\mathscr{O}(\overline{\mathcal{H}_A}, \mathcal{H}_B)$, $\langle f, g \rangle = Tr(f^\dagger \circ g) = \sum_{\mathbf{v}_A \in B_A} \langle f(\mathbf{v}_A), g(\mathbf{v}_A) \rangle_B$, where B_A is an orthonormal basis of $\mathcal{H}_A$ whose choice can be proved to be irrelevant.*

Proof Please refer to Sect. 2.6, especially pp.125-142, in [42]. □

Remark 2.10 When at least one of $\mathcal{H}_A$ and $\mathcal{H}_B$ is finite-dimensional, Hilbert-Schmidt operators coincide with linear maps. In other words, if we introduce the notation $Hom(\overline{\mathcal{H}_A}, \mathcal{H}_B)$ to denote the set of all linear maps from $\overline{\mathcal{H}_A}$ to $\mathcal{H}_B$, then in this special case $\mathscr{HS}\mathscr{O}(\overline{\mathcal{H}_A}, \mathcal{H}_B) = Hom(\overline{\mathcal{H}_A}, \mathcal{H}_B)$.

According to quantum theory, the physical intuition behind this construction is very clear. The main idea is: *a state of the bipartite system can be not only a juxtaposition of two states one from each subsystem but also a way of correlation between the states of the two subsystems*. (No correlation is a correlation, so the former is a special case of the latter.) To be concrete, consider the situation when the bipartite system is in a state described by $\langle f \rangle \in \Sigma_{\mathscr{HS}\mathscr{O}(\overline{\mathcal{H}_A}, \mathcal{H}_B)}$. On the one hand, after performing a measurement only on the subsystem described by $\mathcal{H}_A$, if from the outcome we learn that the state of the subsystem is now the one described by $\langle \mathbf{v}_A \rangle \in \Sigma_{\mathcal{H}_A}$, then the state of the other subsystem is now the one described by $\langle f(\mathbf{v}_A) \rangle \in \Sigma_{\mathcal{H}_B}$ and the state of the bipartite system is no longer the one described by $\langle f \rangle$. At this moment, no matter what measurement only on the subsystem described by $\mathcal{H}_B$ do we perform, if from the outcome we can learn the state of the subsystem, it will be non-orthogonal to the one described by $\langle f(\mathbf{v}_A) \rangle$. On the other hand, after performing a measurement only on the subsystem described by $\mathcal{H}_B$, if from the outcome we learn that the state of the subsystem is now the one described by $\langle \mathbf{v}_B \rangle \in \Sigma_{\mathcal{H}_B}$, then the

[3] For simplicity, we henceforth omit all '$\cdot$' for the scalar multiplication in a vector space.

state of the other subsystem is now the one described by $\langle f^{\dagger}(\mathbf{v}_B)\rangle \in \Sigma_{\mathcal{H}_A}$ and the state of the bipartite system is no longer the one described by $\langle f \rangle$. At this moment, no matter what measurement only on the subsystem described by $\mathcal{H}_A$ do we perform, if from the outcome we can learn the state of the subsystem, it will be non-orthogonal to the one described by $\langle f^{\dagger}(\mathbf{v}_B)\rangle$. Since there is no causal relation between measurements on the two subsystems, both orders of performance are possible and the description using f and its adjoint $f^{\dagger}$ is generic enough to model such a correlation.

Finally, the notion of the conjugate space is needed just because we define a Hilbert-Schmidt operator to be a linear map. In fact, $\mathscr{HSO}(\overline{\mathcal{H}_A}, \mathcal{H}_B)$ is the set of *anti-linear maps* from $\mathcal{H}_A$ to $\mathcal{H}_B$ satisfying the additional condition about trace, where an anti-linear map is defined as follows:

Definition 2.11 An *anti-linear map* from $\mathcal{H}_A$ to $\mathcal{H}_B$ is a function $f : \mathcal{H}_A \to \mathcal{H}_B$ such that

1. for any $\mathbf{u}_A, \mathbf{v}_A \in \mathcal{H}_A$, $f(\mathbf{u}_A + \mathbf{v}_A) = f(\mathbf{u}_A) + f(\mathbf{v}_A)$;
2. for any $\mathbf{u}_A \in \mathcal{H}_A$ and $c \in \mathbb{C}$, $f(c\mathbf{u}_A) = c^* f(\mathbf{u}_A)$.

2.3 Quantum Kripke Frame

In Subsect. 2.1, given a Hilbert space $\mathcal{H}$ over $\mathbb{C}$, we define a binary relation $\perp_{\mathcal{H}}$ on $\Sigma_{\mathcal{H}}$ such that, for any $s, t \in \Sigma_{\mathcal{H}}$, $s \perp_{\mathcal{H}} t$, if and only if, for any $\mathbf{s} \in s$ and $\mathbf{t} \in t$, $\langle \mathbf{s}, \mathbf{t} \rangle = 0$. This relation is called the *orthogonality relation*. According to quantum theory, this relation has a clear and important physical significance as the (perfect) discriminability relation between states [43]. To be precise:

> for two states s and t, $s \perp_{\mathcal{H}} t$, if and only if there is an observable of the system and two possible values i and j of the observable such that, if we know that the system is in either the state s or the state t, then after a measurement of this observable we will know which is the case: the state is s if and only if the outcome is i, and the state is t if and only if the outcome is j.

In this paper, following [26, 27, 41], it is more convenient to focus on the complement of this relation in $\Sigma_{\mathcal{H}} \times \Sigma_{\mathcal{H}}$, called the *non-orthogonality relation* and denoted by $\to_{\mathcal{H}}$. To be precise, for any $s, t \in \Sigma_{\mathcal{H}}$, $s \to_{\mathcal{H}} t$, if and only if there are $\mathbf{s} \in s$ and $\mathbf{t} \in t$ such that $\langle \mathbf{s}, \mathbf{t} \rangle \neq 0$.

According to Proposition 2.7 in [26], the mathematical structure $\mathfrak{F}_{\mathcal{H}} = (\Sigma_{\mathcal{H}}, \to_{\mathcal{H}})$ is a quantum Kripke frame defined via the following definitions from [26, 41]:

Definition 2.12 A *Kripke frame* $\mathfrak{F}$ is an ordered pair $(\Sigma, \to)$ in which Σ is a non-empty set and $\to$ is a binary relation on Σ.

Definition 2.13 Let $\mathfrak{F} = (\Sigma, \to)$ be a Kripke frame.

- $s \nrightarrow t$ means that $s \to t$ does not hold.

- For each $P \subseteq \Sigma$, the *orthocomplement* of P, denoted by $\sim P$, is the set $\{s \in \Sigma \mid s \nrightarrow t, \text{ for each } t \in P\}$.
- $P \subseteq \Sigma$ is *closed*, if $P = \sim\sim P$.
- $s, t \in \Sigma$ are *indistinguishable with respect to* $P \subseteq \Sigma$, denoted by $s \approx_P t$, if $s \to w \Leftrightarrow t \to w$ holds for each $w \in P$.
- For each $P \subseteq \Sigma$ and $s \in \Sigma$, $s' \in \Sigma$ is called a *representative of s in P*, if $s' \in P$ and $s \approx_P s'$.
- $P \subseteq \Sigma$ is *orthogonal*, if, for any $s, t \in P$, $s \neq t$ implies $s \nrightarrow t$.
- $P \subseteq \Sigma$ is *maximal orthogonal*, if P is orthogonal and $\sim\sim P = \Sigma$.

Definition 2.14 A *quantum Kripke frame* is a Kripke frame $\mathfrak{F} = (\Sigma, \to)$ satisfying all of the following five conditions:

1. **Reflexivity**: for each $s \in \Sigma$, $s \to s$.
2. **Symmetry**: for any $s, t \in \Sigma$, if $s \to t$, then $t \to s$.
3. **Separation**: for any $s, t \in \Sigma$, if $s \neq t$, then there is a $w \in \Sigma$ such that $w \to s$ but $w \nrightarrow t$.
4. **Superposition**: for any $s, t \in \Sigma$, there is a $w \in \Sigma$ such that $w \to s$ and $w \to t$.
5. **Representation**: for any $P \subseteq \Sigma$ and $s \in \Sigma$, if $P = \sim\sim P$ and $s \notin \sim P$, then there is a representative of s in P.

For **Reflexivity** and **Symmetry**, technically they follow directly from positive definiteness and conjugate symmetry of the inner product; intuitively they are natural properties of the indiscriminability relation. For **Separation**, technically w is the projection of s onto the orthocomplement of t; intuitively it is an idealization saying that there are enough measurements to non-perfectly discriminate any two distinct states. For **Superposition**, technically w is the superposition of s and t; intuitively this is the feature that distinguishes quantum physics from classical physics. For **Representation**, technically the representative of s in P is the projection of s on P whose existence is a consequence of the orthogonal decomposition theorem; intuitively it guarantees that, given any state, any measurement and any result of the measurement, there is always a point in the Kripke frame which properly describes the state after the given measurement on the given state yielding the given result, if this is possible to happen. A detailed discussion of the physical intuitions behind the definition of quantum Kripke frames can be found in Sec. 4 in [26].

The following lemma collects some basic and useful results about quantum Kripke frames.

Lemma 2.15 (Lemma 2.8 in [26]) *Let* $\mathfrak{F} = (\Sigma, \to)$ *be a quantum Kripke frame.*

1. $\sim\emptyset = \Sigma$ *and* $\sim\Sigma = \emptyset$, *and thus* $\sim\sim\emptyset = \emptyset$ *and* $\sim\sim\Sigma = \Sigma$.
2. *For any* $P, Q \subseteq \Sigma$, $P \subseteq Q$ *implies that* $\sim Q \subseteq \sim P$.
3. *For each* $P \subseteq \Sigma$, $P \subseteq \sim\sim P$.

4. *For each* $P \subseteq \Sigma$, $\sim P$ *is closed.*
5. $\bigcap_{i \in I} P_i$ *is closed, if* P_i *is closed for each* $i \in I$.
6. *For each* $s \in \Sigma$, $\{s\}$ *is closed.*
7. *For any* $P \subseteq \Sigma$ *and* $s, t, t' \in \Sigma$, *if* P *is closed and both* t *and* t' *are representatives of* s *in* P, *then* $t = t'$.

A very important fact about quantum Kripke frames is that they correspond to irreducible Hilbertian geometries, as is manifested by the following theorem:

Theorem 2.16 (Theorem 4.22 in [41])

1. *For every quantum Kripke frame* $\mathfrak{F} = (\Sigma, \to)$, $\mathbf{G}(\mathfrak{F}) = (\Sigma, \sim\sim\{\cdot,\cdot\}, \nrightarrow)$ *is an irreducible Hilbertian geometry.*
2. *For every irreducible Hilbertian geometry* $\mathfrak{G} = (G, \star, \perp)$, $\mathbf{F}(\mathfrak{G}) = (G, \not\perp)$ *is a quantum Kripke frame.*
3. $\mathbf{G}$ *is a class function from the class of quantum Kripke frames to that of irreducible Hilbertian geometries.*

   $\mathbf{F}$ *is a class function from the class of irreducible Hilbertian geometries to that of quantum Kripke frames.*

   For any quantum Kripke frame $\mathfrak{F}$ *and irreducible Hilbertian geometry* $\mathfrak{G}$,

$$(\mathbf{F}\circ\mathbf{G})(\mathfrak{F}) = \mathfrak{F} \qquad\qquad (\mathbf{G}\circ\mathbf{F})(\mathfrak{G}) = \mathfrak{G}$$

This theorem enables us to use powerful results in projective geometry to study quantum Kripke frames. Here we illustrate this by two examples which will be used later: one is a dimension theory of quantum Kripke frames, and the other is a representation theorem of quantum Kripke frames via generalized Hilbert spaces.

For the dimension theory, we need the following results and definition.

Proposition 2.17 *Let* $\mathfrak{F} = (\Sigma, \to)$ *be a quantum Kripke frame*, $n \in \mathbb{N}$ *and* $s_1, \ldots, s_n \in \Sigma$. $\sim\sim\{s_1, \ldots, s_n\} = \mathcal{C}(\{s_1, \ldots, s_n\})$, *where the right-hand side is the linear closure of* $\{s_1, \ldots, s_n\}$ *in* $\mathbf{G}(\mathfrak{F})$.

Proof We leave it to the Appendix. □

Corollary 2.18 *Let* $\mathfrak{F} = (\Sigma, \to)$ *be a quantum Kripke frame*, $n \in \mathbb{N}$ *and* $s_1, \ldots, s_n \in \Sigma$. *If* $\{s_1, \ldots, s_n\}$ *is orthogonal, then* $\{s_1, \ldots, s_n\}$ *is independent in* $\mathbf{G}(\mathfrak{F})$.

Proof If $n = 0$, by convention $\{s_1, \ldots, s_n\} = \emptyset$ and thus it is independent.

If $n \neq 0$, take an arbitrary $s \in \{s_1, \ldots, s_n\}$. Without loss of generality we assume that $s = s_1$. Since $\{s_1, \ldots, s_n\}$ is orthogonal, $s_1 \in \sim\{s_2, \ldots, s_n\}$. By Reflexivity and Proposition 2.17

$$s_1 \notin \sim\sim\{s_2, \dots, s_n\} = \mathcal{C}(\{s_2, \dots, s_n\}) = \mathcal{C}(\{s_1, \dots, s_n\} \setminus \{s_1\})$$

Therefore, $\{s_1, \dots, s_n\}$ is independent in $\mathbf{G}(\mathfrak{F})$. □

Definition 2.19 Let $\mathfrak{F} = (\Sigma, \to)$ be a quantum Kripke frame and $n \in \mathbb{N}$. $\mathfrak{F}$ is *n-dimensional*, if there is a maximal orthogonal set $P \subseteq \Sigma$ of cardinality n.

Remark 2.20 By Proposition 2.17 and its corollary a quantum Kripke frame $\mathfrak{F} = (\Sigma, \to)$ is n-dimensional, if and only if Σ is of rank n in $\mathbf{G}(\mathfrak{F})$. Therefore, this definition does not depend on the choice of P, and thus is legitimate.

For the representation theorem, we start from reviewing the definition of generalized Hilbert spaces.

Definition 2.21 (*Definition 52 in* [40]) A *division ring* is a ring in which every non-zero element has a multiplicative inverse; if further the multiplication is commutative, it is called a *field*.

An *involution* on a division ring $\mathcal{K} = (K, +, \cdot, 0, 1)$[4] is a function $\mu : K \to K$ satisfying all of the following:

1. μ is bijective;
2. $\mu(x + y) = \mu(x) + \mu(y)$ and $\mu(xy) = \mu(y)\mu(x)$, for any $x, y \in K$;
3. $(\mu \circ \mu)(x) = x$, for every $x \in K$;

An *Hermitian form* on a vector space V over a division ring $\mathcal{K} = (K, +, \cdot, 0, 1)$ is a function $\Phi : V \times V \to K$ satisfying all of the following:

1. $\Phi(\mathbf{u} + \mathbf{v}, \mathbf{w}) = \Phi(\mathbf{u}, \mathbf{w}) + \Phi(\mathbf{v}, \mathbf{w})$, for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$;
2. $\Phi(x\mathbf{v}, \mathbf{w}) = x\Phi(\mathbf{v}, \mathbf{w})$, for any $\mathbf{v}, \mathbf{w} \in V$ and $x \in K$[5];
3. there is an involution μ on $\mathcal{K}$ such that $\Phi(\mathbf{v}, \mathbf{w}) = \mu(\Phi(\mathbf{w}, \mathbf{v}))$ holds for any $\mathbf{v}, \mathbf{w} \in V$.

μ is called *the accompanying involution* of Φ.

A *generalized Hilbert space* is a vector space V over some division ring $\mathcal{K}$ equipped with an Hermitian form Φ satisfying the following condition:

$(*)$ for every $E \subseteq V$, if $E = (E^{\perp})^{\perp}$, then $V = \{\mathbf{u} + \mathbf{v} \mid \mathbf{u} \in E \text{ and } \mathbf{v} \in E^{\perp}\}$;

where $E^{\perp} \overset{\text{def}}{=} \{\mathbf{u} \in V \mid \Phi(\mathbf{u}, \mathbf{v}) = 0 \text{ holds for each } \mathbf{v} \in E\}$.

[4] In the following, for simplicity, we omit all '·' for the multiplication in a ring.

[5] An Hermitian form is defined to be conjugate in the second argument, while an inner product is usually defined to be conjugate in the first argument. This difference is just a matter of notation.

It can be verified that every generalized Hilbert space naturally gives rise to a quantum Kripke frame:

Proposition 2.22 *Let V be a generalized Hilbert space where Φ is the Hermitian form on it. $\mathfrak{F}_V = (\Sigma_V, \rightarrow_V)$ defined below is a quantum Kripke frame*:

1. Σ_V *is the set of one-dimensional subspaces of* V;
2. *for any* $s, t \in \Sigma_V$, $s \rightarrow_V t$, *if and only if there are* $\mathbf{s} \in s$ *and* $\mathbf{t} \in t$ *such that* $\Phi(\mathbf{s}, \mathbf{t}) \neq 0$.

In fact, Theorem 2.11 in [26] is a representation theorem of quantum Kripke frames via generalized Hilbert spaces, and it facilitates studying quantum Kripke frames using the analytic method. Moreover, as a generalized Hilbert space is a mild algebraic generalization of a Hilbert space over $\mathbb{C}$, this theorem means that the conditions defining a quantum Kripke frame capture the essential properties of the non-orthogonality relation, and that quantum Kripke frames are good qualitative models of quantum systems. For a more detailed discussion, please refer to [26].

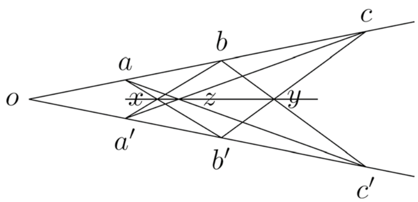
In this paper we use a more specific version of this representation theorem. For this, we need the notion of Pappian quantum Kripke frames.

Definition 2.23 A quantum Kripke frame $\mathfrak{F} = (\Sigma, \rightarrow)$ is *Pappian*, if it has an orthogonal set of cardinality at least 3 and, for any $a, b, c, a', b', c', x, y, z \in \Sigma$ satisfying all of the following:

1. a, b, c, a', b', c' are all distinct;
2. $c \in {\sim\sim}\{a, b\}$;
3. $c' \in {\sim\sim}\{a', b'\}$;
4. ${\sim\sim}\{a, b\} \cap {\sim\sim}\{a', b'\} = \{o\}$, for some $o \in \Sigma \setminus \{a, b, c, a', b', c'\}$;
5. $x \in {\sim\sim}\{a, b'\} \cap {\sim\sim}\{a', b\}$;
6. $y \in {\sim\sim}\{b, c'\} \cap {\sim\sim}\{b', c\}$;
7. $z \in {\sim\sim}\{c, a'\} \cap {\sim\sim}\{c', a\}$;

it holds that $s_1 \in {\sim\sim}\{s_2, s_3\}$, for some $s_1, s_2, s_3 \in \Sigma$ with $\{s_1, s_2, s_3\} = \{x, y, z\}$.

This definition involves complicated configurations. The following picture of the analogue in an affine plane may help to make sense of it:



Remark 2.24 It can be verified that a quantum Kripke frame $\mathfrak{F}$ is Pappian, if and only if the theorem of Pappus holds in the irreducible Hilbertian geometry $\mathbf{G}(\mathfrak{F})$ in the sense of the definition on page 62 in [39].

The following is a representation theorem of Pappian quantum Kripke frames via generalized Hilbert spaces over fields.

Theorem 2.25 *For a Kripke frame* $\mathfrak{F} = (\Sigma, \rightarrow)$, *the following are equivalent*:

(i) $\mathfrak{F}$ *is a Pappian quantum Kripke frame*;
(ii) *there is a generalized Hilbert space V of dimension at least* 3 *over some field* $\mathcal{K}$ *such that* $\mathfrak{F} \cong \mathfrak{F}_V = (\Sigma_V, \rightarrow_V)$.

Moreover, if they exist, both V and $\mathcal{K}$ are unique up to isomorphism, and the Hermitian form is unique up to a constant multiple.
Proof The proof is very similar to that of Theorem 2.11 in [26] except for two differences. One of them is in the direction from (ii) to (i). We use Theorem 2.2.2 in [39] to prove that $\mathfrak{F}$ is Pappian. The other is in the direction from (i) to (ii). When $\mathfrak{F}$ is Pappian, the theorem of Pappus holds in $\mathbf{G}(\mathfrak{F})$, so the theorem of Desargues holds in $\mathbf{G}(\mathfrak{F})$ by Theorem 2.2.3 in [39], which is the main result in [44]. Then the division ring is a field, according to Theorem 2.2.2 in [39]. □

3 States of a Bipartite Quantum System

In the coming three sections, we try to answer the following question:

> Given two quantum Kripke frames $\mathfrak{F}_A = (\Sigma_A, \rightarrow_A)$ and $\mathfrak{F}_B = (\Sigma_B, \rightarrow_B)$ describing two quantum systems, respectively, what is the quantum Kripke frame $\mathfrak{F} = (\Sigma, \rightarrow)$ that describes the bipartite quantum system consisting of the two given quantum systems?

In this section, we are going to construct the right candidate for Σ from $\mathfrak{F}_A = (\Sigma_A, \rightarrow_A)$ and $\mathfrak{F}_B = (\Sigma_B, \rightarrow_B)$, which models the set of states of the bipartite quantum system. In the next two sections, we are going to define a non-orthogonality relation $\rightarrow$ on Σ, which models the indiscriminability relation between the states of the bipartite quantum system, and thus finish the construction.

3.1 Adjunction and Continuous Homomorphism

To construct the right candidate for Σ from $\mathfrak{F}_A = (\Sigma_A, \rightarrow_A)$ and $\mathfrak{F}_B = (\Sigma_B, \rightarrow_B)$, we will adopt the idea explained in Subsect. 2.2: a state of a bipartite quantum system is a way of correlation between the states of the two subsystems. We first use relations to model correlations, and this leads to the definition of adjunctions between $\mathfrak{F}_A$ and $\mathfrak{F}_B$. We will study the properties of adjunctions to find the right candidate for Σ.

3.1.1 Adjunction

Definition 3.1 An *adjunction* between two Kripke frames $\mathfrak{F}_A = (\Sigma_A, \to_A)$ and $\mathfrak{F}_B = (\Sigma_B, \to_B)$ satisfying $\Sigma_A \cap \Sigma_B = \emptyset$ is a set $R \subseteq (\Sigma_A \times \Sigma_B) \cup (\Sigma_B \times \Sigma_A)$ such that: for any $s_A \in \Sigma_A$ and $s_B \in \Sigma_B$,

1. for any $t_A \in \Sigma_A$ and $t_B \in \Sigma_B$, if $s_A R s_B$ and $t_B R t_A$, then $s_A \to_A t_A \Leftrightarrow s_B \to_B t_B$;
2. there is no $w_B \in \Sigma_B$ such that $s_A R w_B$, if and only if, for each $t_A \in \Sigma_A$, if there is a $t_B \in \Sigma_B$ such that $t_B R t_A$, then $s_A \not\to_A t_A$;
3. there is no $w_A \in \Sigma_A$ such that $s_B R w_A$, if and only if, for each $t_B \in \Sigma_B$, if there is a $t_A \in \Sigma_A$ such that $t_A R t_B$, then $s_B \not\to_B t_B$.

In this definition, an adjunction is defined to be a relation satisfying special conditions which models the correlation between elements in Σ_A and those in Σ_B. These special conditions are all in terms of the primitive relations $\to_A$ and $\to_B$. Moreover, they are not very complicated: intuitively, they reflect the idea of correlation explained in Subsect. 2.2; mathematically, in a qualitative way they reflect the defining equation $\langle f(\mathbf{w}_A), \mathbf{w}_B \rangle = \langle \mathbf{w}_A, f^\dagger(\mathbf{w}_B) \rangle$ relating a linear map and its adjoint, if it exists. However, the special conditions in fact impose mathematical properties on an adjunction which will be very useful and thus should be made more explicit. In the following, we will obtain some characterization of adjunctions. In the end, we will arrive at the counterparts of linear maps at the level of one-dimensional subspaces to construct Σ.

The following proposition shows that an adjunction is in fact the union of two partial functions in opposite directions.

Proposition 3.2 *Let* $\mathfrak{F}_A = (\Sigma_A, \to_A)$ *and* $\mathfrak{F}_B = (\Sigma_B, \to_B)$ *be two quantum Kripke frames such that* $\Sigma_A \cap \Sigma_B = \emptyset$ *and* $R \subseteq (\Sigma_A \times \Sigma_B) \cup (\Sigma_B \times \Sigma_A)$. *The following are equivalent*:

(i) *R is an adjunction between $\mathfrak{F}_A$ and $\mathfrak{F}_B$;*
(ii) *there are two partial functions $F : \Sigma_A \dashrightarrow \Sigma_B$ and $G : \Sigma_B \dashrightarrow \Sigma_A$ such that $R = F \cup G$ and all of the following are true for F and G:*

1. $\mathrm{Ker}(F) = {\sim_A} G[\Sigma_B]$[6];
2. $\mathrm{Ker}(G) = {\sim_B} F[\Sigma_A]$;
3. *for any $s_A, t_A \in \Sigma_A$ and $s_B, t_B \in \Sigma_B$, if $(s_A, s_B) \in F$ and $(t_B, t_A) \in G$, then $s_A \to_A t_A \Leftrightarrow s_B \to_B t_B$.*

Here $\mathrm{Ker}(F) \stackrel{\text{def}}{=} \{s_A \in \Sigma_A \mid F(s_A) \text{ is undefined}\}$ is called the *kernel of F*.

Proof **From (i) to (ii):** Let $F = R \cap (\Sigma_A \times \Sigma_B)$ and $G = R \cap (\Sigma_B \times \Sigma_A)$. Obviously $R = F \cup G$. The crux is to prove that both F and G are partial functions. After

6 Here $G[\Sigma_B]$ denotes the image of Σ_B under the partial function $G : \Sigma_B \dashrightarrow \Sigma_A$.

proving this, the three items follow immediately from the three items in the definition. Moreover, since the definition of adjunction is symmetric in the subscripts A and B, it suffices to prove that F is a partial function, for the proof of G is similar.

Assume that $s_A \in \Sigma_A$ and $s_B, s'_B \in \Sigma_B$ are such that $s_A R s_B$ and $s_A R s'_B$. Suppose (towards a contradiction) that $s_B \neq s'_B$. By Separation there is a $t_B \in \Sigma_B$ such that $t_B \rightarrow_B s_B$ but $t_B \nrightarrow_B s'_B$. Since $s_A R s_B$ and $t_B \rightarrow_B s_B$, by Symmetry and Item 3 in the definition there is a $t_A \in \Sigma_A$ such that $t_B R t_A$. On the one hand, since $s_A R s_B$ and $t_B R t_A$, by Item 1 in the definition $s_A \rightarrow_A t_A \Leftrightarrow s_B \rightarrow_B t_B$. Since $t_B \rightarrow_B s_B$, $s_A \rightarrow_A t_A$. On the other hand, since $s_A R s'_B$ and $t_B R t_A$, by Item 1 in the definition $s_A \rightarrow_A t_A \Leftrightarrow s'_B \rightarrow_B t_B$. Since $t_B \nrightarrow_B s'_B$, $s_A \nrightarrow_A t_A$. Hence we have got a contradiction. Therefore, $s_B = s'_B$, so F is a partial function.

From (ii) to (i): This follows easily from the definition of adjunctions. □

In the following, the term 'adjunction' will be used in the sense of the characterization, i.e. Item (ii), in this proposition, instead of the formal definition.

Remark 3.3 (F, G) is an adjunction between $\mathfrak{F}_A$ and $\mathfrak{F}_B$, if and only if it is an adjunction between $\mathbf{G}(\mathfrak{F}_A)$ and $\mathbf{G}(\mathfrak{F}_B)$ (Definition 14.4.1 in [38]).

3.1.2 Continuous Homomorphism

In this subsubsection, we introduce continuous homomorphisms. We will see in the next subsubsection that they are exactly the partial functions each serving as a component in an adjunction.

Definition 3.4 (*Definition 9 in* [27]) A *continuous homomorphism* from a Kripke frame $\mathfrak{F}_A = (\Sigma_A, \rightarrow_A)$ to a Kripke frame $\mathfrak{F}_B = (\Sigma_B, \rightarrow_B)$ is a partial function $F : \Sigma_A \dashrightarrow \Sigma_B$ such that, for each $t_B \in \Sigma_B$, if there is an $s_A \in \Sigma_A$ such that $F(s_A)$ is defined and $F(s_A) \rightarrow_B t_B$, then there is a $t_A \in \Sigma_A$ satisfying the following[7]:

$$(\mathrm{Ad})_F \quad \text{for any } w_A \in \Sigma_A, t_A \rightarrow_A w_A, \text{ if and only if } F(w_A) \text{ is defined and } t_B \rightarrow_B F(w_A)$$

Remark 3.5 For every $P_B \subseteq \Sigma_B$, recall that *the inverse image of* P_B *under* F, denoted by $F^{-1}[P_B]$, is $\{s_A \in \Sigma_A \mid (s_A, s_B) \in F$, for some $s_B \in P_B\}$. With this terminology, it can be shown that a partial function $F : \Sigma_A \dashrightarrow \Sigma_B$ is a continuous homomorphism, if and only if, for every $t_B \in \Sigma_B$, $\mathrm{Ker}(F) \cup F^{-1}[\sim_B\{t_B\}] = \Sigma_A$ or $\mathrm{Ker}(F) \cup F^{-1}[\sim_B\{t_B\}] = \sim_A\{t_A\}$ for some $t_A \in \Sigma_A$.

Moreover, it can be shown that, for every $t_A \in \Sigma_A$, there is no closed set which lies strictly between $\sim_A\{t_A\}$ and Σ_A in the set inclusion ordering. Hence a partial function

7 Please remember that the following condition $(\mathrm{Ad})_F$ is not a property of a Kripke frame or a partial function; instead it is a property of an (ordered) pair of two elements in two Kripke frames, in this case, (t_A, t_B).

$F : \Sigma_A \dashrightarrow \Sigma_B$ is a continuous homomorphism, if and only if, for each $t_B \in \Sigma_B$, $\mathrm{Ker}(F) \cup F^{-1}[\sim_B\{t_B\}]$ is closed and includes $\sim_A\{t_A\}$ for some $t_A \in \Sigma_A$.

The following is a lemma which will be used immediately.

Lemma 3.6 *Let $\mathfrak{F}_A = (\Sigma_A, \rightarrow_A)$ and $\mathfrak{F}_B = (\Sigma_B, \rightarrow_B)$ be two quantum Kripke frames, F a continuous homomorphism from $\mathfrak{F}_A$ to $\mathfrak{F}_B$ and $t_B \in \Sigma_B$. If it exists, $t_A \in \Sigma_A$ with (t_A, t_B) satisfying $(Ad)_F$ is unique.*

Proof Suppose (towards a contradiction) that $t'_A \in \Sigma_A \setminus \{t_A\}$ is also such that (t'_A, t_B) satisfies $(\mathrm{Ad})_F$. Since $t_A \neq t'_A$, by Separation there is a $w_A \in \Sigma_A$ such that $t_A \not\rightarrow_A w_A$ and $t'_A \rightarrow_A w_A$. Since (t'_A, t_B) satisfies $(\mathrm{Ad})_F$, $F(w_A)$ is defined and $t_B \rightarrow_B F(w_A)$. Since (t_A, t_B) satisfies $(\mathrm{Ad})_F$, $t_A \rightarrow_A w_A$, contradicting that $t_A \not\rightarrow_A w_A$. □

3.1.3 Correspondence

In this subsubsection, we prove a correspondence between adjunctions and continuous homomorphisms. It means that continuous homomorphisms are exactly the partial functions each serving as a component in an adjunction. Later on Σ will be constructed based on them.

Throughout this subsubsection, we fix two quantum Kripke frames $\mathfrak{F}_A = (\Sigma_A, \rightarrow_A)$ and $\mathfrak{F}_B = (\Sigma_B, \rightarrow_B)$.

First we show how to get an adjunction given a continuous homomorphism.

Proposition 3.7 *Let $F : \Sigma_A \dashrightarrow \Sigma_B$ be a continuous homomorphism from $\mathfrak{F}_A$ to $\mathfrak{F}_B$. $F^\dagger \overset{\text{def}}{=} \{(s_B, s_A) \in \Sigma_B \times \Sigma_A \mid (s_A, s_B)$ satisfies $(Ad)_F\}$ is a partial function from Σ_B to Σ_A, called the adjoint of F, and $(F, F^\dagger)$ is an adjunction between $\mathfrak{F}_A$ and $\mathfrak{F}_B$.*

Proof By Lemma 3.6 $F^\dagger$ is a partial function from Σ_B to Σ_A.

To show that $(F, F^\dagger)$ is an adjunction, we prove three facts.

Fact 1: $\mathrm{Ker}(F) = \sim_A F^\dagger[\Sigma_B]$.

Let $w_A \in \Sigma_A$ be arbitrary.

First assume that $w_A \notin \mathrm{Ker}(F)$. By Reflexivity $F(w_A) \rightarrow_B F(w_A)$. Since F is a continuous homomorphism, there is a $w_A^+ \in \Sigma_A$ such that $(w_A^+, F(w_A))$ satisfies $(\mathrm{Ad})_F$. Since $F(w_A)$ is defined and $F(w_A) \rightarrow_B F(w_A)$, $w_A^+ \rightarrow_A w_A$. Hence $w_A \rightarrow_A w_A^+ = F^\dagger(F(w_A))$, and thus $w_A \notin \sim_A F^\dagger[\Sigma_B]$.

Second assume that $w_A \notin \sim_A F^\dagger[\Sigma_B]$. Then there are $s_A \in \Sigma_A$ and $s_B \in \Sigma_B$ such that $F^\dagger(s_B) = s_A$ and $w_A \rightarrow_A s_A$. By the definition of $F^\dagger$ $(\mathrm{Ad})_F$ holds for (s_A, s_B). Since $s_A \rightarrow_A w_A$, $F(w_A)$ is defined. Hence $w_A \notin \mathrm{Ker}(F)$.

Fact 2: $\mathrm{Ker}(F^\dagger) = \sim_B F[\Sigma_A]$.

Let $w_B \in \Sigma_B$ be arbitrary.

First assume that $w_B \notin \mathrm{Ker}(F^\dagger)$. Then there is an $w_A \in \Sigma_A$ such that $(w_B, w_A) \in F^\dagger$, i.e. $(\mathrm{Ad})_F$ holds for (w_A, w_B). By Reflexivity $w_A \rightarrow_A w_A$, so $F(w_A)$ is defined and $w_B \rightarrow_B F(w_A)$. Therefore, $w_B \notin \sim_B F[\Sigma_A]$.

Second assume that $w_B \notin \sim_B F[\Sigma_A]$. Then there are $s_A \in \Sigma_A$ and $s_B \in \Sigma_B$ such that $F(s_A) = s_B$ and $w_B \to_B s_B$. Since F is a continuous homomorphism, there is a $w_A \in \Sigma_A$ such that $(\mathrm{Ad})_F$ holds for (w_A, w_B). By the definition of $F^\dagger$ $(w_B, w_A) \in F^\dagger$, so $w_B \notin \mathrm{Ker}(F^\dagger)$.

Fact 3: $s_A \to_A F^\dagger(s_B) \Leftrightarrow F(s_A) \to_B s_B$, if $s_A \notin \mathrm{Ker}(F)$ and $s_B \notin \mathrm{Ker}(F^\dagger)$.

Let $s_A \notin \mathrm{Ker}(F)$ and $s_B \notin \mathrm{Ker}(F^\dagger)$ be arbitrary. By the definition of $F^\dagger$ $(\mathrm{Ad})_F$ holds for $(F^\dagger(s_B), s_B)$. Hence $F^\dagger(s_B) \to_A s_A$, if and only if $F(s_A)$ is defined and $s_B \to_B F(s_A)$. Since $s_A \notin \mathrm{Ker}(F)$, the required equivalence follows. □

Remark 3.8 We denote by $\mathbf{A}_{\mathfrak{F}_B}^{\mathfrak{F}_A}$ the function from the set of continuous homomorphisms from $\mathfrak{F}_A$ to $\mathfrak{F}_B$ to the set of adjunctions between $\mathfrak{F}_A$ and $\mathfrak{F}_B$, which maps every continuous homomorphism F to the adjunction $(F, F^\dagger)$ defined as in this proposition. When the context is clear, we abbreviate $\mathbf{A}_{\mathfrak{F}_B}^{\mathfrak{F}_A}$ to $\mathbf{A}$.

Besides, note that, for any $s_A \in \Sigma_A$ and $s_B \in \Sigma_B$, $(\mathrm{Ad})_F$ holds for (s_A, s_B), if and only if $(s_B, s_A) \in F^\dagger$, i.e. $F^\dagger(s_B)$ is defined and equals to s_A.

Second we show how to get a continuous homomorphism given an adjunction.

Proposition 3.9 *Let (F, G) be an adjunction between $\mathfrak{F}_A$ and $\mathfrak{F}_B$ with $F : \Sigma_A \dashrightarrow \Sigma_B$ and $G : \Sigma_B \dashrightarrow \Sigma_A$. Then both F and G are continuous homomorphisms.*

Proof We show that F is a continuous homomorphism from $\mathfrak{F}_A$ to $\mathfrak{F}_B$. Let $s_A \in \Sigma_A$ and $s_B, t_B \in \Sigma_B$ be arbitrary such that $(s_A, s_B) \in F$ and $s_B \to_B t_B$. Then $t_B \notin \sim_B F[\Sigma_A] = \mathrm{Ker}(G)$. We claim that $(G(t_B), t_B)$ satisfies $(\mathrm{Ad})_F$.

Let $w_A \in \Sigma_A$ be arbitrary. First assume that $G(t_B) \to_A w_A$. Then $w_A \notin \sim_A G[\Sigma_B] = \mathrm{Ker}(F)$, so $F(w_A)$ is defined. Since $G(t_B) \to_A w_A$, by the definition of adjunction and Symmetry $t_B \to_B F(w_A)$. Second assume that $F(w_A)$ is defined and $t_B \to_B F(w_A)$. It follows directly from the definition of adjunction and Symmetry that $G(t_B) \to_A w_A$.

Since t_B is arbitrary, F is a continuous homomorphism.

Finally, by definition (G, F) is an adjunction between $\mathfrak{F}_B$ and $\mathfrak{F}_A$. Hence by what was just proved G is a continuous homomorphism from $\mathfrak{F}_B$ to $\mathfrak{F}_A$. □

Remark 3.10 We denote by $\mathbf{C}_{\mathfrak{F}_B}^{\mathfrak{F}_A}$ the function from the set of adjunctions between $\mathfrak{F}_A$ and $\mathfrak{F}_B$ to the set of continuous homomorphisms from $\mathfrak{F}_A$ to $\mathfrak{F}_B$, which maps every adjunction (F, G) to the continuous homomorphism F from $\mathfrak{F}_A$ to $\mathfrak{F}_B$ defined as in this proposition. When the context is clear, we abbreviate $\mathbf{C}_{\mathfrak{F}_B}^{\mathfrak{F}_A}$ to $\mathbf{C}$.

Finally, we prove the correspondence.

Proposition 3.11

1. *For every continuous homomorphism F from $\mathfrak{F}_A$ to $\mathfrak{F}_B$, $(\mathbf{C} \circ \mathbf{A})(F) = F$.*

2. *For every adjunction* (F, G) *between* $\mathfrak{F}_A$ *and* $\mathfrak{F}_B$ *with* $F : \Sigma_A \dashrightarrow \Sigma_B$ *and* $G : \Sigma_B \dashrightarrow \Sigma_A$, $(\mathbf{A}\circ\mathbf{C})((F, G)) = (F, G)$.

Proof **For *1***: By the previous two propositions, $(\mathbf{C}\circ\mathbf{A})(F) = \mathbf{C}((F, F^\dagger)) = F$.

For *2*: We start with proving that $G = F^\dagger$.

First note that $\mathrm{Ker}(F^\dagger) = \mathrm{Ker}(G)$. Since both $(F, F^\dagger)$ and (F, G) are adjunctions, by definition $\mathrm{Ker}(F^\dagger) = \sim_B F[\Sigma_A] = \mathrm{Ker}(G)$.

Second show that $G(s_B) = F^\dagger(s_B)$, for every $s_B \notin \mathrm{Ker}(F^\dagger) = \mathrm{Ker}(G)$. By the proof of Proposition 3.9 $(G(s_B), s_B)$ satisfies $(\mathrm{Ad})_F$. By definition $(F^\dagger(s_B), s_B)$ satisfies $(\mathrm{Ad})_F$. By Lemma 3.6 $G(s_B) = F^\dagger(s_B)$.

As a result, $G = F^\dagger$, and thus $(\mathbf{A}\circ\mathbf{C})((F, G)) = \mathbf{A}(F) = (F, F^\dagger) = (F, G)$. □

In this correspondence, on the one hand, adjunctions have an intuitive definition based on the idea of correlation; on the other hand, continuous homomorphisms have a definition which is simple and easy to use in proofs. In the following investigation, we will mostly focus on continuous homomorphisms.

Remark 3.12 Combining this correspondence with Proposition 14.4.4 in [38], F is a continuous homomorphism from $\mathfrak{F}_A$ to $\mathfrak{F}_B$, if and only if it is a continuous homomorphism from $\mathbf{G}(\mathfrak{F}_A)$ to $\mathbf{G}(\mathfrak{F}_\mathbf{B})$ as is defined in [38].

3.2 Continuous Homomorphism and Continuous Quasi-Linear Map

Since the analytic method is powerful and handy, we will facilitate its application. In this subsection, we study the analytic counterpart of continuous homomorphisms. We start from reviewing the notion of quasi-linear maps.

Definition 3.13 (*Definition 6.6.10 and Lemma 14.4.9 in* [38]) Let V_A and V_B be two vector spaces over two division rings $\mathcal{K}_A = (K_A, +_A, \cdot_A, 0_A, 1_A)$ and $\mathcal{K}_B = (K_B, +_B, \cdot_B, 0_B, 1_B)$[8], respectively. A *quasi-linear map* from V_A to V_B is a function $f : V_A \to V_B$ such that:

1. for any $\mathbf{u}_A, \mathbf{v}_A \in V_A$, $f(\mathbf{u}_A + \mathbf{v}_A) = f(\mathbf{u}_A) + f(\mathbf{v}_A)$;
2. there is a division ring isomorphism $\sigma : \mathcal{K}_A \to \mathcal{K}_B$, called *the division ring isomorphism associated to f*, such that, for any $\mathbf{v}_A \in V_A$ and $x \in K_A$, $f(x\mathbf{v}_A) = \sigma(x)f(\mathbf{v}_A)$.

When both V_A and V_B are generalized Hilbert spaces whose Hermitian forms are Φ_A and Φ_B, respectively, a quasi-linear map f from V_A to V_B is *continuous*, if there is a quasi-linear map $f^\dagger : V_B \to V_A$ such that, for any $\mathbf{v}_A \in V_A$ and $\mathbf{v}_B \in V_B$, $\Phi_B(f(\mathbf{v}_A), \mathbf{v}_B) = \sigma(\Phi_A(\mathbf{v}_A, f^\dagger(\mathbf{v}_B)))$.

[8] In the following, for simplicity, we omit all subscripts of function symbols and constant symbols of division rings. We also omit the dot for multiplication in a division ring.

Remark 3.14 By Lemma 14.4.9 in [38], for each continuous quasi-linear map *f*, the quasi-linear map $f^\dagger$ with the property prescribed by the definition is unique, and it is called the *adjoint* of *f*.

According to the following proposition, each continuous quasi-linear map induces a continuous homomorphism.

Proposition 3.15 *Let V_A and V_B be two generalized Hilbert spaces over two division rings $\mathcal{K}_A$ and $\mathcal{K}_B$, respectively, and f a continuous quasi-linear map from V_A to V_B. Define a partial function $\mathcal{P}(f) : \Sigma_{V_A} \dashrightarrow \Sigma_{V_B}$ as follows: for each $s_A \in \Sigma_{V_A}$,*

$$\mathcal{P}(f)(s_A) = \begin{cases} f[s_A], & \text{if } f[s_A] \neq \{\mathbf{0}_B\} \\ \text{undefined}, & \text{otherwise} \end{cases}$$

Then $\mathcal{P}(f)$ is a continuous homomorphism from $\mathfrak{F}_{V_A}$ to $\mathfrak{F}_{V_B}$, called the continuous homomorphism induced by f.

Proof Assume that $s_A \in \Sigma_{V_A}$ and $t_B \in \Sigma_{V_B}$ are such that $\mathcal{P}(f)(s_A)$ is defined and $\mathcal{P}(f)(s_A) \rightarrow_B t_B$. By definition $\mathcal{P}(f)(s_A)$ is $f[s_A]$, and it is not $\{\mathbf{0}_B\}$. Let $\mathbf{t}_B$ be a non-zero vector such that $t_B = \langle \mathbf{t}_B \rangle$. Since *f* is a continuous quasi-linear map, by definition there is a quasi-linear map $f^\dagger : V_B \rightarrow V_A$ such that, for any $\mathbf{v}_A \in V_A$ and $\mathbf{v}_B \in V_B$, $\Phi_B(f(\mathbf{v}_A), \mathbf{v}_B) = \sigma(\Phi_A(\mathbf{v}_A, f^\dagger(\mathbf{v}_B)))$.

We claim that $f^\dagger(\mathbf{t}_B) \neq \mathbf{0}_A$: Suppose (towards a contradiction) that $f^\dagger(\mathbf{t}_B) = \mathbf{0}_A$. Then, for each $\mathbf{s}_B \in \mathcal{P}(f)(s_A)$ and $\mathbf{v}_B \in t_B$, there are $\mathbf{s}_A \in s_A$ and $x \in K_B$ such that $\mathbf{s}_B = f(\mathbf{s}_A)$ and $\mathbf{v}_B = x\mathbf{t}_B$, and thus $\Phi_B(\mathbf{s}_B, \mathbf{v}_B) = \Phi_B(f(\mathbf{s}_A), x\mathbf{t}_B) = \mu_B(x)\Phi_B(f(\mathbf{s}_A), \mathbf{t}_B) = \mu_B(x)\sigma(\Phi_A(\mathbf{s}_A, f^\dagger(\mathbf{t}_B))) = \mu_B(x)\sigma(\Phi_A(\mathbf{s}_A, \mathbf{0}_A)) = 0$, where μ_B is the accompanying involution of Φ_B. It follows that $\mathcal{P}(f)(s_A) \not\rightarrow_B t_B$, contradicting the assumption.

Now that $f^\dagger(\mathbf{t}_B) \neq \mathbf{0}_A$, $\langle f^\dagger(\mathbf{t}_B) \rangle = f^\dagger[t_B]$ is an element of Σ_{V_A}. We try to show that $(f^\dagger[t_B], t_B)$ satisfies $(\mathrm{Ad})_{\mathcal{P}(f)}$. Let $w_A \in \Sigma_{V_A}$ be arbitrary.

First, assume that $f^\dagger[t_B] \rightarrow_A w_A$. By definition there are $\mathbf{v}_A \in f^\dagger[t_B]$ and $\mathbf{w}_A \in w_A$ such that $\Phi_A(\mathbf{w}_A, \mathbf{v}_A) \neq 0$. Then there is an $x \in K_A \backslash \{0\}$ such that $\mathbf{v}_A = x f^\dagger(\mathbf{t}_B)$. Hence $\Phi_B(f(\mathbf{w}_A), \mathbf{t}_B) = \sigma(\Phi_A(\mathbf{w}_A, f^\dagger(\mathbf{t}_B))) = \sigma(\Phi_A(\mathbf{w}_A, x^{-1}\mathbf{v}_A)) = \sigma(\mu_A(x^{-1})\Phi_A(\mathbf{w}_A, \mathbf{v}_A)) \neq 0$ where μ_A is the accompanying involution of Φ_A. It follows that $f(\mathbf{w}_A) \neq \mathbf{0}_B$, and thus $f[w_A] \neq \{\mathbf{0}_B\}$, so $\mathcal{P}(f)(w_A)$ is defined. It also follows that $t_B \rightarrow_B \mathcal{P}(f)(w_A)$.

Second, assume that $\mathcal{P}(f)(w_A)$ is defined and $t_B \rightarrow_B \mathcal{P}(f)(w_A)$. By definition there are $\mathbf{v}_B \in t_B$ and $\mathbf{w}_B \in \mathcal{P}(f)(w_A)$ such that $\Phi_B(\mathbf{v}_B, \mathbf{w}_B) \neq 0$. Hence there are $\mathbf{w}_A \in w_A$ and $x \in K_B \backslash \{0\}$ such that $\mathbf{w}_B = f(\mathbf{w}_A)$ and $\mathbf{t}_B = x\mathbf{v}_B$. Hence $\sigma(\Phi_A(\mathbf{w}_A, f^\dagger(\mathbf{t}_B))) = \Phi_B(f(\mathbf{w}_A), \mathbf{t}_B) = \Phi_B(\mathbf{w}_B, x\mathbf{v}_B) = \mu_B(x)\Phi_B(\mathbf{w}_B, \mathbf{v}_B) \neq 0$. Hence $\Phi_A(\mathbf{w}_A, f^\dagger(\mathbf{t}_B)) \neq 0$. Therefore, $f^\dagger[t_B] \rightarrow_A w_A$. □

Since every linear map having an adjoint is a quasi-linear map, it seems from the above proposition that continuous homomorphisms are the counterparts of linear maps at the level of one-dimensional subspaces. However, not every continuous homomorphism can be proved to be induced by a quasi-linear map. To characterize the continuous homomorphisms that are induced by quasi-linear maps, we need the following definition.

Definition 3.16 Let $\mathfrak{F}_A = (\Sigma_A, \to_A)$ and $\mathfrak{F}_B = (\Sigma_B, \to_B)$ be two quantum Kripke frames and F a continuous homomorphism from $\mathfrak{F}_A$ to $\mathfrak{F}_B$.

1. F is *non-degenerate*, if $F[\Sigma_A]$ includes an orthogonal set of cardinality at least 3.
2. F is *arguesian*, if it is the composition of two non-degenerate continuous homomorphisms between quantum Kripke frames. (These non-degenerate continuous homomorphisms may involve quantum Kripke frames other than $\mathfrak{F}_A$ and $\mathfrak{F}_B$.)

Remark 3.17 A partial function $F : \Sigma_A \dashrightarrow \Sigma_B$ is an arguesian continuous homomorphism from $\mathfrak{F}_A$ to $\mathfrak{F}_B$, if and only if it is an arguesian continuous homomorphism from $\mathbf{G}(\mathfrak{F}_A)$ to $\mathbf{G}(\mathfrak{F}_B)$ in the sense of Definition 10.3.2 in [38].

Now we can characterize the continuous homomorphisms induced by quasi-linear maps.

Theorem 3.18 *Let V_A and V_B be two generalized Hilbert spaces of dimension at least* 3 *over two division rings $\mathcal{K}_A$ and $\mathcal{K}_B$, respectively. For each partial function $F : \Sigma_{V_A} \dashrightarrow \Sigma_{V_B}$, the following are equivalent*:

(i) *F is an arguesian continuous homomorphism from $\mathfrak{F}_{V_A}$ to $\mathfrak{F}_{V_B}$;*
(ii) *there is a continuous quasi-linear map $f : V_A \to V_B$ such that $F = \mathcal{P}(f)$.*

Moreover, if $F[\Sigma_{V_A}]$ is not a singleton and the f in (ii) exists, f is unique up to scalar multiplication.

Proof **From (i) to (ii):** Assume that F is an arguesian continuous homomorphism from $\mathfrak{F}_{V_A}$ to $\mathfrak{F}_{V_B}$. Then there is a generalized Hilbert space V_C and two non-degenerate continuous homomorphisms $G : \Sigma_{V_A} \dashrightarrow \Sigma_{V_C}$ and $G' : \Sigma_{V_C} \dashrightarrow \Sigma_{V_B}$ such that $F = G' \circ G$. Note that, for each $\mathbf{v} \neq \mathbf{0}$ in a generalized Hilbert space V, there is a linear functional, namely $\frac{\Phi(\cdot,\mathbf{v})}{\Phi(\mathbf{v},\mathbf{v})}$, such that its value on $\mathbf{v}$ is 1. Hence every generalized Hilbert space is a dualized vector space in the sense of Example 13.4.2 in [38]. By Proposition 13.5.3 in [38] there are two continuous quasi-linear maps $g : V_A \to V_C$ and $g' : V_C \to V_B$ such that $G = \mathcal{P}(g)$ and $G' = \mathcal{P}(g')$. It can be verified that $g' \circ g : V_A \to V_B$ is a continuous quasi-linear map and $\mathcal{P}(g' \circ g) = \mathcal{P}(g') \circ \mathcal{P}(g) = G' \circ G = F$.

From (ii) to (i): By Proposition 3.15 $\mathcal{P}(f)$ is a continuous homomorphism. We can also prove that $\mathcal{P}(f)$ is arguesian by the strategy used in the proof of Theorem 10.3.1 in [38] as follows:

Since V_B is of dimension at least 3, we can find a set of three linearly independent vectors in V_B. Using the Gram-Schmidt process we get three pairwise orthogonal vectors $\mathbf{v}_B^1$, $\mathbf{v}_B^2$, $\mathbf{v}_B^3$ in V_B[9]. Let $W = V_A \times \mathcal{K}_A \times \mathcal{K}_A \times \mathcal{K}_A$ and define a function

$$\begin{aligned}\Phi :\ & W \times W \to \mathcal{K}_A \\ & :: \big((\mathbf{u}_A^1, x_A^1, x_A^2, x_A^3), (\mathbf{u}_A^2, y_A^1, y_A^2, y_A^3)\big) \\ & \qquad \mapsto \Phi_A(\mathbf{u}_A^1, \mathbf{u}_A^2) + \mu_A(x_A^1) \cdot y_A^1 + \mu_A(x_A^2) \cdot y_A^2 + \mu_A(x_A^3) \cdot y_A^3\end{aligned}$$

where μ_A is the accompanying involution of Φ_A. It can be shown that W over $\mathcal{K}_A$ equipped with Φ is a generalized Hilbert space. Then define the functions

$$\begin{aligned}f_A &: V_A \to W :: \mathbf{u}_A \mapsto (\mathbf{u}_A, 0, 0, 0) \\ f_B &: W \to V_B :: (\mathbf{u}_A, x_A^1, x_A^2, x_A^3) \mapsto f(\mathbf{u}_A) + \sigma(x_A^1)\mathbf{v}_B^1 + \sigma(x_A^2)\mathbf{v}_B^2 + \sigma(x_A^3)\mathbf{v}_B^3\end{aligned}$$

Here σ is the division ring isomorphism associated to f. It can be verified that f_A is linear and f_B is quasi-linear, and $f = f_B \circ f_A$. By definition the dimension of the image of f_A is the same as V_A which is at least 3, and the image of f_B includes all of $\mathbf{v}_B^1$, $\mathbf{v}_B^2$ and $\mathbf{v}_B^3$; so both $\mathcal{P}(f_A)$ and $\mathcal{P}(f_B)$ are non-degenerate. By Proposition 3.15 both $\mathcal{P}(f_A)$ and $\mathcal{P}(f_B)$ are continuous homomorphisms and it can be verified that $\mathcal{P}(f) = \mathcal{P}(f_B) \circ \mathcal{P}(f_A)$. Therefore, $\mathcal{P}(f)$ is arguesian.

Uniqueness: It follows from Lemma 6.3.4 in [38]. □

Remark 3.19 It is crucial for the direction from (ii) to (i) to assume that both V_A and V_B are at least 3-dimensional. Here we need to show $\mathcal{P}(f) = G \circ H$, for two continuous homomorphisms G and H such that both are non-degenerate, i.e. the range of each of them contains three pairwise orthogonal elements. If V_A were at most 2-dimensional, H would have at most two orthogonal elements in its range and thus not be non-degenerate, even if it is an embedding. If V_B were at most 2-dimensional, G would have at most two orthogonal elements in its range and thus not be non-degenerate.

3.3 Projective Collineations in Projective Geometries and in Quautum Kripke Frames

The work in the previous subsection results in a correspondence between arguesian continuous homomorphisms and continuous quasi-linear maps.

Suppose that we are in a concrete setting with two finite-dimensional Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ over $\mathbb{C}$. By Theorem 2.9 and Remark 2.10 we can construct $\mathcal{H}_A \otimes \mathcal{H}_B$ from $Hom(\overline{\mathcal{H}_A}, \mathcal{H}_B)$, so $\Sigma_{Hom(\overline{\mathcal{H}_A}, \mathcal{H}_B)}$ can be defined as the set of all continuous homomorphisms of the form $\mathcal{P}(f)$, where f is a linear map whose image is not

[9] Here we use superscripts as indices, for subscriptes have been used to distinguish between objects related to V_A and those related to V_B.

$\{\mathbf{0}_B\}$. Please be aware that, if we want to define $\Sigma_{Hom(\overline{\mathcal{H}_A},\mathcal{H}_B)}$ using $\mathfrak{F}_{\mathcal{H}_A}$ and $\mathfrak{F}_{\mathcal{H}_B}$, we can *not* take all arguesian continuous homomorphims from $\mathfrak{F}_{\mathcal{H}_A}$ to $\mathfrak{F}_{\mathcal{H}_B}$; otherwise, by Theorem 3.18 the anti-linear maps would sneak in.

As a result, we see the need to characterize the arguesian continuous homomorphisms induced by linear maps. More precisely, we need to characterize the arguesian continuous homomorphisms each induced by a quasi-linear map with the same associated division ring isomorphism; then, without loss of generality, we can assume that this isomorphism is the identity.

Our way of doing this will employ some results in projective geometry about projective collineations. The following definition is adapted from that on Page 96 in [39].

Definition 3.20 Let $\mathfrak{G} = (G, \star)$ be a projective geometry. A *central collineation* on $\mathfrak{G}$ is a bijection $C : G \to G$ such that:

1. for any $a, b, c \in G$, $c \in a \star b$ if and only if $C(c) \in C(a) \star C(b)$;
2. there is a hyperplane such that each element of it is a fixed point of C, where a *hyperplane* is defined to be a subspace H of $\mathfrak{G}$ such that G is the only subspace E satisfying $H \subsetneq E$;
3. there is an $o \in G$ such that, for any $a, b \in G$, if $b \in a \star o$, then $C(b) \in a \star o$.

A *projective collineation* on $\mathfrak{G}$ is a composition of finitely many central collineations on $\mathfrak{G}$.

Theorem 3.21 *Let V be a vector space and $\mathfrak{G}_V = (\Sigma_V, \star_V)$. For each function $F : \Sigma_V \to \Sigma_V$, the following are equivalent*:

(i) *F is a projective collineation on $\mathfrak{G}_V$;*
(ii) *there is a bijective linear map $f : V \to V$ such that $F = \mathcal{P}(f)$.*

Proof It follows from Theorem 3.6.7 and the definition on page 126 in [39]. Here we use the characterizing condition in the theorem in [39] as the definition and the definition there as a characterizing condition. □

According to Theorem 2.16, we can introduce the notion of projective collineations into the framework of quantum Kripke frames.

Definition 3.22 Let $\mathfrak{F} = (\Sigma, \to)$ be a quantum Kripke frame.

1. For any two distinct elements s and t of Σ, the *line* determined by s and t is the set $\sim\sim\{s, t\}$.
2. A *subspace* of $\mathfrak{F}$ is a subset E of Σ such that, for any $s, t \in \Sigma$, $s \in E$ and $t \in E$ imply that $\sim\sim\{s, t\} \subseteq E$.

3. A *hyperplane* of $\mathfrak{F}$ is a subspace H such that Σ is the only subspace P satisfying $H \subsetneqq P$.
4. A *central collineation* on $\mathfrak{F}$ is a bijection F on Σ such that:

   (a) for any $s, t, u \in \Sigma$, $u \in {\sim\sim}\{s, t\}$ if and only if $F(u) \in {\sim\sim}\{F(s), F(t)\}$;
   (b) there is a hyperplane H of $\mathfrak{F}$ such that every element in H is a fixed point of F;
   (c) there is an $o \in \Sigma$ such that, for any $s, t \in \Sigma$, if $t \in {\sim\sim}\{s, o\}$, then $F(t) \in {\sim\sim}\{s, o\}$.

5. A collineation on $\mathfrak{F}$ is *projective*, if it is a composition of finitely many central collineations on $\mathfrak{F}$.

Combining Theorems 2.16 and 3.21 we have the following theorem:

Theorem 3.23 *Let V be a vector space and $\mathfrak{F}_V = (\Sigma_V, \rightarrow_V)$. For each function $F : \Sigma_V \to \Sigma_V$, the following are equivalent*:

(i) *F is a projective collineation on $\mathfrak{F}_V$;*
(ii) *there is a bijective linear map $f : V \to V$ such that $F = \mathcal{P}(f)$.*

3.4 Continuous Homomorphisms with the Same Associated Division Ring Isomorphism

Now we start to characterize the arguesian continuous homomorphisms each induced by a quasi-linear map with the same associated division ring isomorphism. First we deal with the degenerate case.

Lemma 3.24 *Let V_A and V_B be two vector spaces over two division rings $\mathcal{K}_A$ and $\mathcal{K}_B$, respectively, $g : V_A \to V_B$ a quasi-linear map whose image is at most one-dimensional, and $\sigma : \mathcal{K}_A \to \mathcal{K}_B$ a division ring isomorphism. There is a quasi-linear map $f : V_A \to V_B$ such that σ is the division ring isomorphism associated to f and $\mathcal{P}(f) = \mathcal{P}(g)$.*

Proof Let τ be the division ring isomorphism associated to g. Two cases need to be considered.

Case 1: $g[V_A] = \{\mathbf{0}_B\}$. In this case g is the zero map, and it is not hard to verify that g is a quasi-linear map with σ as the associated division ring isomorphism, and trivially $\mathcal{P}(g) = \mathcal{P}(g)$.

Case 2: $g[V_A]$ is one-dimensional. From linear algebra there is a Hamel basis[10] $\{\mathbf{v}_A^i \mid i \in I \cup \{e\}\}$ of V_A such that $e \notin I$ and $g(\mathbf{v}_A^i) \neq \mathbf{0}_B$ if and only if $i = e$, for every $i \in I \cup \{e\}$. For every vector $\mathbf{v}_A \in V_A$, if $\mathbf{v}_A = \sum_{i\in J} x_i \mathbf{v}_A^i$ for a finite set $J \subseteq I \cup \{e\}$, define $f(\mathbf{v}_A)$ to be $\sum_{i\in J} \sigma(x_i) g(\mathbf{v}_A^i)$. Then f is a function from V_A to V_B. We claim that f has the required properties.

First show that f is additive. For any two vectors $\mathbf{s}_A, \mathbf{t}_A \in V$, suppose that under the basis $\mathbf{s}_A = \sum_{i\in J_s} x_i \mathbf{v}_A^i$ and $\mathbf{t}_A = \sum_{i\in J_t} y_i \mathbf{v}_A^i$. Then

$$\begin{aligned}
&f(\mathbf{s}_A + \mathbf{t}_A)\\
&\quad = f(\sum_{i\in J_s} x_i \mathbf{v}_A^i + \sum_{i\in J_t} y_i \mathbf{v}_A^i)\\
&\quad = f(\sum_{i\in J_s\cap J_t} (x_i + y_i)\mathbf{v}_A^i + \sum_{i\in J_s\setminus(J_s\cap J_t)} x_i \mathbf{v}_A^i + \sum_{i\in J_t\setminus(J_s\cap J_t)} y_i \mathbf{v}_A^i)\\
&\quad = \sum_{i\in J_s\cap J_t} \sigma(x_i + y_i) g(\mathbf{v}_A^i) + \sum_{i\in J_s\setminus(J_s\cap J_t)} \sigma(x_i) g(\mathbf{v}_A^i) + \sum_{i\in J_t\setminus(J_s\cap J_t)} \sigma(y_i) g(\mathbf{v}_A^i)\\
&\quad = \sum_{i\in J_s\cap J_t} \sigma(x_i) g(\mathbf{v}_A^i) + \sum_{i\in J_s\cap J_t} \sigma(y_i) g(\mathbf{v}_A^i) + \sum_{i\in J_s\setminus(J_s\cap J_t)} \sigma(x_i) g(\mathbf{v}_A^i)\\
&\qquad + \sum_{i\in J_t\setminus(J_s\cap J_t)} \sigma(y_i) g(\mathbf{v}_A^i)\\
&\quad = \sum_{i\in J_s} \sigma(x_i) g(\mathbf{v}_A^i) + \sum_{i\in J_t} \sigma(y_i) g(\mathbf{v}_A^i)\\
&\quad = f(\mathbf{s}_A) + f(\mathbf{t}_A)
\end{aligned}$$

Second show that $f(x\mathbf{u}_A) = \sigma(x) f(\mathbf{u}_A)$, for any $x \in K_A$ and $\mathbf{u}_A \in V_A$. Suppose that under the basis $\mathbf{u}_A = \sum_{i\in J} x_i \mathbf{v}_A^i$. Then

$$\begin{aligned}
f(x\mathbf{u}_A) &= f(x \sum_{i\in J} x_i \mathbf{v}_A^i)\\
&= f(\sum_{i\in J} (x x_i) \mathbf{v}_A^i)\\
&= \sum_{i\in J} \sigma(x x_i) g(\mathbf{v}_A^i)\\
&= \sum_{i\in J} \sigma(x)\sigma(x_i) g(\mathbf{v}_A^i)\\
&= \sigma(x) \sum_{i\in J} \sigma(x_i) g(\mathbf{v}_A^i)\\
&= \sigma(x) f(\sum_{i\in J} x_i \mathbf{v}_A^i)\\
&= \sigma(x) f(\mathbf{u}_A)
\end{aligned}$$

[10] A *Hamel basis* of a vector space V is an independent set $B \subseteq V$ such that every vector in V is a (finite) linear combination of vectors in B.

It can now be concluded that f is a quasi-linear map from V_A to V_B with σ as the associated division ring isomorphism. It remains to show that $\mathcal{P}(f) = \mathcal{P}(g)$. Let $\mathbf{u}_A \in V_A$ be arbitrary. Suppose that $\mathbf{u}_A = \sum_{i\in J} x_i \mathbf{v}_A^i$ under the basis. When $e \notin J$ or $x_e = 0$,

$$f(\mathbf{u}_A) = \sum_{i\in J} \sigma(x_i) g(\mathbf{v}_A^i) = \mathbf{0}_B, \qquad g(\mathbf{u}_A) = \sum_{i\in J} \tau(x_i) g(\mathbf{v}_A^i) = \mathbf{0}_B$$

When $e \in J$ and $x_e \neq 0$,

$$\begin{aligned} f(\mathbf{u}_A) &= \sum_{i\in J} \sigma(x_i) g(\mathbf{v}_A^i) = \sigma(x_e) g(\mathbf{v}_A^e) \neq \mathbf{0}_B \\ g(\mathbf{u}_A) &= \sum_{i\in J} \tau(x_i) g(\mathbf{v}_A^i) = \tau(x_e) g(\mathbf{v}_A^e) \neq \mathbf{0}_B, \end{aligned}$$

It follows that $\mathcal{P}(f) = \mathcal{P}(g)$. □

Second we deal with the non-degenerate case. Please note that here we need the commutativity of the multiplication.

Lemma 3.25 *Let V_A and V_B be two vector spaces over two fields $\mathcal{K}_A$ and $\mathcal{K}_B$, respectively, $f, g : V_A \to V_B$ two quasi-linear maps such that both of their images are at least two-dimensional, and σ and τ the division ring isomorphisms associated to f and g, respectively. Then the following are equivalent*:

(i) $\sigma = \tau$;
(ii) *there is a subspace $P_A \subseteq \Sigma_{V_A}$, a projective collineation C_A on $\mathfrak{F}_{V_A}$ and a projective collineation C_B on $\mathfrak{F}_{V_B}$ such that $\mathcal{P}(f)[P_A]$ is not a singleton and* $(C_B \circ \mathcal{P}(f))\restriction_{P_A} = (\mathcal{P}(g) \circ C_A)\restriction_{P_A}$.

Proof **From (i) to (ii):** Since the image of f and that of g are both at least two-dimensional, let $\mathbf{u}_A^1, \mathbf{u}_A^2 \in V_A$ be such that $f(\mathbf{u}_A^1)$ and $f(\mathbf{u}_A^2)$ are linearly independent, $\mathbf{v}_A^1, \mathbf{v}_A^2 \in V_A$ be such that $g(\mathbf{v}_A^1)$ and $g(\mathbf{v}_A^2)$ are linearly independent. It follows that $\mathbf{u}_A^1$ and $\mathbf{u}_A^2$ are linearly independent and $\mathbf{v}_A^1$ and $\mathbf{v}_A^2$ are linearly independent. Extend $\{\mathbf{u}_A^1, \mathbf{u}_A^2\}$ to a Hamel basis $\{\mathbf{u}_A^i \mid i \in I\}$ of V_A such that $1, 2 \in I$. Extend $\{\mathbf{v}_A^1, \mathbf{v}_A^2\}$ to a Hamel basis $\{\mathbf{v}_A^i \mid i \in I\}$ of V_A. We can use the same index set because any two Hamel bases of V_A are of the same cardinality. Extend $\{f(\mathbf{u}_A^1), f(\mathbf{u}_A^2)\}$ to a Hamel basis $\{\mathbf{u}_B^j \mid j \in J\}$ of V_B such that $1, 2 \in J$, $\mathbf{u}_B^1 = f(\mathbf{u}_A^1)$ and $\mathbf{u}_B^2 = f(\mathbf{u}_A^2)$. Extend $\{g(\mathbf{v}_A^1), g(\mathbf{v}_A^2)\}$ to a Hamel basis $\{\mathbf{v}_B^j \mid j \in J\}$ of V_B such that $\mathbf{v}_B^1 = g(\mathbf{v}_A^1)$ and $\mathbf{v}_B^2 = g(\mathbf{v}_A^2)$. We can use the same index set because any two Hamel bases of V_B are of the same cardinality.

Let $P_A = \langle \mathbf{u}_A^1 \rangle \star_A \langle \mathbf{u}_A^2 \rangle$. Since $\mathbf{u}_B^1 = f(\mathbf{u}_A^1)$, $\mathbf{u}_B^2 = f(\mathbf{u}_A^2)$ and $f(\mathbf{u}_A^1)$ and $f(\mathbf{u}_A^2)$ are linearly independent, $\mathcal{P}(f)[P_A]$ is not a singleton.

Moreover, define $c_A : V_A \to V_A$ to be the linear map that maps $\mathbf{u}_A^i$ to $\mathbf{v}_A^i$ for each $i \in I$ and $c_B : V_B \to V_B$ to be the linear map that maps $\mathbf{u}_B^i$ to $\mathbf{v}_B^i$ for each $i \in J$. By Theorem 3.23 $C_A = \mathcal{P}(c_A)$ and $C_B = \mathcal{P}(c_B)$ are two projective collineations on $\mathfrak{F}_{V_A}$ and $\mathfrak{F}_{V_B}$, respectively.

Finally, we verify that $(C_B \circ \mathcal{P}(f)) \restriction_{P_A} = (\mathcal{P}(g) \circ C_A) \restriction_{P_A}$: Let $\mathbf{w}_A \in P_A \backslash \{\mathbf{0}\}$ be arbitrary. Then there are $x, y \in K_A$ such that $\{x, y\} \neq \{0\}$ and $\mathbf{w}_A = x\mathbf{u}_A^1 + y\mathbf{u}_A^2$.

$$\begin{aligned}
(C_B \circ \mathcal{P}(f))(\langle \mathbf{w}_A \rangle) &= (\mathcal{P}(c_B) \circ \mathcal{P}(f))(\langle x\mathbf{u}_A^1 + y\mathbf{u}_A^2 \rangle) \\
&= \mathcal{P}(c_B)(\langle \sigma(x) f(\mathbf{u}_A^1) + \sigma(y) f(\mathbf{u}_A^2) \rangle) \\
&= \mathcal{P}(c_B)(\langle \sigma(x)\mathbf{u}_B^1 + \sigma(y)\mathbf{u}_B^2 \rangle) \\
&= \langle \sigma(x)\mathbf{v}_B^1 + \sigma(y)\mathbf{v}_B^2 \rangle \\
&= \langle \tau(x) g(\mathbf{v}_A^1) + \tau(y) g(\mathbf{v}_A^2) \rangle && \text{(by (i))} \\
&= \mathcal{P}(g)(\langle x\mathbf{v}_A^1 + y\mathbf{v}_A^2 \rangle) \\
&= \mathcal{P}(g)(\langle x c_A(\mathbf{u}_A^1) + y c_A(\mathbf{u}_A^2) \rangle) \\
&= (\mathcal{P}(g) \circ \mathcal{P}(c_A))(\langle x\mathbf{u}_A^1 + y\mathbf{u}_A^2 \rangle) \\
&= (\mathcal{P}(g) \circ C_A)(\langle \mathbf{w}_A \rangle)
\end{aligned}$$

From (ii) to (i): Let $x \in K_A$ be arbitrary. If $x = 0$, $\sigma(x) = 0 = \tau(x)$ by the definition of field isomorphisms. It remains to consider the case when $x \neq 0$.

By (ii) and Theorem 3.23 there are two bijective linear maps $c_A : V_A \to V_A$ and $c_B : V_B \to V_B$ such that $C_A = \mathcal{P}(c_A)$ and $C_B = \mathcal{P}(c_B)$.

Also by (ii) there are $\mathbf{u}_A^1, \mathbf{u}_A^2 \in V_A \backslash \{\mathbf{0}_A\}$ such that $\langle \mathbf{u}_A^1 \rangle, \langle \mathbf{u}_A^2 \rangle \in P_A$, $f(\mathbf{u}_A^1) \neq \mathbf{0}_B$, $f(\mathbf{u}_A^2) \neq \mathbf{0}_B$, $\langle f(\mathbf{u}_A^1) \rangle$ and $\langle f(\mathbf{u}_A^2) \rangle$ are two different elements in $\mathcal{P}(f)[P_A]$. Then $\langle \mathbf{u}_A^1 + \mathbf{u}_A^2 \rangle, \langle \mathbf{u}_A^1 + x\mathbf{u}_A^2 \rangle \in P_A$. We calculate

$$
\begin{aligned}
(C_B \circ \mathcal{P}(f))(\langle \mathbf{u}_A^1 \rangle) &= (\mathcal{P}(c_B) \circ \mathcal{P}(f))(\langle \mathbf{u}_A^1 \rangle) \\
&= \mathcal{P}(c_B)(\langle f(\mathbf{u}_A^1) \rangle) \\
&= \langle (c_B \circ f)(\mathbf{u}_A^1) \rangle \\
(\mathcal{P}(g) \circ C_A)(\langle \mathbf{u}_A^1 \rangle) &= (\mathcal{P}(g) \circ \mathcal{P}(c_A))(\langle \mathbf{u}_A^1 \rangle) \\
&= \mathcal{P}(g)(\langle c_A(\mathbf{u}_A^1) \rangle) \\
&= \langle (g \circ c_A)(\mathbf{u}_A^1) \rangle \\
(C_B \circ \mathcal{P}(f))(\langle \mathbf{u}_A^2 \rangle) &= (\mathcal{P}(c_B) \circ \mathcal{P}(f))(\langle \mathbf{u}_A^2 \rangle) \\
&= \mathcal{P}(c_B)(\langle f(\mathbf{u}_A^2) \rangle) \\
&= \langle (c_B \circ f)(\mathbf{u}_A^2) \rangle \\
(\mathcal{P}(g) \circ C_A)(\langle \mathbf{u}_A^2 \rangle) &= (\mathcal{P}(g) \circ \mathcal{P}(c_A))(\langle \mathbf{u}_A^2 \rangle) \\
&= \mathcal{P}(g)(\langle c_A(\mathbf{u}_A^2) \rangle) \\
&= \langle (g \circ c_A)(\mathbf{u}_A^2) \rangle \\
(C_B \circ \mathcal{P}(f))(\langle \mathbf{u}_A^1 + \mathbf{u}_A^2 \rangle) &= (\mathcal{P}(c_B) \circ \mathcal{P}(f))(\langle \mathbf{u}_A^1 + \mathbf{u}_A^2 \rangle) \\
&= \mathcal{P}(c_B)(\langle f(\mathbf{u}_A^1) + f(\mathbf{u}_A^2) \rangle) \\
&= \langle (c_B \circ f)(\mathbf{u}_A^1) + (c_B \circ f)(\mathbf{u}_A^2) \rangle \\
(\mathcal{P}(g) \circ C_A)(\langle \mathbf{u}_A^1 + \mathbf{u}_A^2 \rangle) &= (\mathcal{P}(g) \circ \mathcal{P}(c_A))(\langle \mathbf{u}_A^1 + \mathbf{u}_A^2 \rangle) \\
&= \mathcal{P}(g)(\langle c_A(\mathbf{u}_A^1) + c_A(\mathbf{u}_A^2) \rangle) \\
&= \langle (g \circ c_A)(\mathbf{u}_A^1) + (g \circ c_A)(\mathbf{u}_A^2) \rangle \\
(C_B \circ \mathcal{P}(f))(\langle \mathbf{u}_A^1 + x\mathbf{u}_A^2 \rangle) &= (\mathcal{P}(c_B) \circ \mathcal{P}(f))(\langle \mathbf{u}_A^1 + x\mathbf{u}_A^2 \rangle) \\
&= \mathcal{P}(c_B)(\langle f(\mathbf{u}_A^1) + \sigma(x) f(\mathbf{u}_A^2) \rangle) \\
&= \langle (c_B \circ f)(\mathbf{u}_A^1) + \sigma(x)(c_B \circ f)(\mathbf{u}_A^2) \rangle \\
(\mathcal{P}(g) \circ C_A)(\langle \mathbf{u}_A^1 + x\mathbf{u}_A^2 \rangle) &= (\mathcal{P}(g) \circ \mathcal{P}(c_A))(\langle \mathbf{u}_A^1 + x\mathbf{u}_A^2 \rangle) \\
&= \mathcal{P}(g)(\langle c_A(\mathbf{u}_A^1) + x c_A(\mathbf{u}_A^2) \rangle) \\
&= \langle (g \circ c_A)(\mathbf{u}_A^1) + \tau(x)(g \circ c_A)(\mathbf{u}_A^2) \rangle
\end{aligned}
$$

As $(C_B \circ \mathcal{P}(f)) \restriction_{P_A} = (\mathcal{P}(g) \circ C_A) \restriction_{P_A}$, there are non-zero $k, l, m, n \in K_B$ such that

$$
\begin{aligned}
k(c_B \circ f)(\mathbf{u}_A^1) &= (g \circ c_A)(\mathbf{u}_A^1) \\
l(c_B \circ f)(\mathbf{u}_A^2) &= (g \circ c_A)(\mathbf{u}_A^2) \\
m\big((c_B \circ f)(\mathbf{u}_A^1) + (c_B \circ f)(\mathbf{u}_A^2)\big) &= (g \circ c_A)(\mathbf{u}_A^1) + (g \circ c_A)(\mathbf{u}_A^2) \\
n\big((c_B \circ f)(\mathbf{u}_A^1) + \sigma(x)(c_B \circ f)(\mathbf{u}_A^2)\big) &= (g \circ c_A)(\mathbf{u}_A^1) + \tau(x)(g \circ c_A)(\mathbf{u}_A^2)
\end{aligned}
$$

Hence

$$
\begin{aligned}
m(c_B \circ f)(\mathbf{u}_A^1) + m(c_B \circ f)(\mathbf{u}_A^2) &= k(c_B \circ f)(\mathbf{u}_A^1) + l(c_B \circ f)(\mathbf{u}_A^2) \\
n(c_B \circ f)(\mathbf{u}_A^1) + n\sigma(x)(c_B \circ f)(\mathbf{u}_A^2) &= k(c_B \circ f)(\mathbf{u}_A^1) + \tau(x) l(c_B \circ f)(\mathbf{u}_A^2)
\end{aligned}
$$

and thus

$$(m - k)(c_B \circ f)(\mathbf{u}_A^1) + (m - l)(c_B \circ f)(\mathbf{u}_A^2) = 0$$
$$(n - k)(c_B \circ f)(\mathbf{u}_A^1) + (n\sigma(x) - \tau(x)l)(c_B \circ f)(\mathbf{u}_A^2) = 0$$

Since $f(\mathbf{u}_A^1)$ and $f(\mathbf{u}_A^2)$ are linearly independent, we have $n = k = l = m$ and $n\sigma(x) = \tau(x)l$, so $\sigma(x) = n^{-1}\tau(x)n$. Since $\mathcal{K}_B$ is a field, $\sigma(x) = \tau(x)$. ☐

Now we define the relation of being colleagues, and use it to characterize the arguesian continuous homomorphisms each induced by a quasi-linear map with the same associated field isomorphism.

Definition 3.26 Let $\mathfrak{F}_A$ and $\mathfrak{F}_B$ be two quantum Kripke frames. For any two continuous homomorphisms F and F' from $\mathfrak{F}_A$ to $\mathfrak{F}_B$, F is a *colleague* of F', denoted by $F \fallingdotseq F'$, if at least one of the following holds:

- either $F[\Sigma_A]$ or $F'[\Sigma_A]$ is a singleton;
- there is a closed $P_A \subseteq \Sigma_A$ and two projective collineations $C_A : \mathfrak{F}_A \to \mathfrak{F}_A$ and $C_B : \mathfrak{F}_B \to \mathfrak{F}_B$ such that $F[P_A]$ is not a singleton and $(C_B \circ F)\upharpoonright_{P_A} = (F' \circ C_A)\upharpoonright_{P_A}$.

Theorem 3.27 *Let V_A and V_B be two generalized Hilbert spaces over a field $\mathcal{K}$. Let $h : V_A \to V_B$ be a linear map such that $h[V_A]$ is at least two-dimensional. For each arguesian continuous homomorphism F from $\mathfrak{F}_{V_A}$ to $\mathfrak{F}_{V_B}$, the following are equivalent*:

(i) $F \fallingdotseq \mathcal{P}(h)$;
(ii) *there is a linear map $f : V_A \to V_B$ such that $F = \mathcal{P}(f)$.*

Moreover, if $F[\Sigma_{V_A}]$ is not a singleton and the f in (ii) exists, f is unique up to scalar multiplication.

Proof **From (ii) to (i):** If $F[\Sigma_{V_A}]$ is a singleton, by definition $F \fallingdotseq \mathcal{P}(h)$. It remains to consider the case when $F[\Sigma_{V_A}]$ is not a singleton. Then $f[V_A]$ is at least two-dimensional. Since both f and h are linear maps, by Lemma 3.25 there is a subspace $Q_A \subseteq \Sigma_{V_A}$ and two projective collineations $C_A : \mathfrak{F}_{V_A} \to \mathfrak{F}_{V_A}$ and $C_B : \mathfrak{F}_{V_B} \to \mathfrak{F}_{V_B}$ such that $F[Q_A]$ is not a singleton and $(C_B \circ \mathcal{P}(f))\upharpoonright_{Q_A} = (\mathcal{P}(h) \circ C_A)\upharpoonright_{Q_A}$. Since $F[Q_A]$ is not a singleton, pick two distinct $s_A, t_A \in Q_A$ such that $F(s_A) \neq F(t_A)$. By Proposition 2.17 $s_A \star_A t_A$ is closed. Then $F[s_A \star_A t_A]$ is not a singleton and $(C_B \circ \mathcal{P}(f))\upharpoonright_{s_A \star_A t_A} = (\mathcal{P}(h) \circ C_A)\upharpoonright_{s_A \star_A t_A}$. By definition $F = \mathcal{P}(f) \fallingdotseq \mathcal{P}(h)$.

From (i) to (ii): If $F[\Sigma_{V_A}]$ is a singleton, by Lemma 3.24 there is a linear map $f : V_A \to V_B$ such that $F = \mathcal{P}(f)$. It remains to consider the case when $F[\Sigma_{V_A}]$ is not a singleton. By (ii) and the definition of colleagues there is a $P_A \subseteq \Sigma_{V_A}$ and two projective collineations $C_A : \mathfrak{F}_{V_A} \to \mathfrak{F}_{V_A}$ and $C_B : \mathfrak{F}_{V_B} \to \mathfrak{F}_{V_B}$ such that $\sim_A \sim_A P_A = P_A$, $F[P_A]$ is not a singleton and $(C_B \circ F)\upharpoonright_{P_A} = (\mathcal{P}(h) \circ C_A)\upharpoonright_{P_A}$. Since P_A is closed, it is a subspace by Lemma A.4. Since F is an arguesian continuous homomorphism, by Theorem 3.18 there is a quasi-linear map $f : V_A \to V_B$ such that $F = \mathcal{P}(f)$. By

Lemma 3.25 the division ring isomorphisms associated to f and h are the same, so f is a linear map.

Uniqueness: It follows from the uniqueness part of Theorem 3.18. □

With this theorem, we finally know how to define Σ: **We take all arguesian continuous homomorphisms which are the colleagues of a particular non-degenerate continuous homomorphism**.

4 Linear Maps of Trace 0

Please recall that our question is the following:

> Given two quantum Kripke frames $\mathfrak{F}_A = (\Sigma_A, \rightarrow_A)$ and $\mathfrak{F}_B = (\Sigma_B, \rightarrow_B)$ describing two quantum systems, respectively, what is the quantum Kripke frame $\mathfrak{F} = (\Sigma, \rightarrow)$ that describes the bipartite quantum system consisting of the two given quantum systems?

In the previous section, we define Σ by finding the counterparts of linear maps. In this section, we work towards defining a non-orthogonality relation $\rightarrow$ on Σ.

From the definitions, we see that the non-orthogonality relation between one-dimensional subspaces is defined via the inner product of two vectors being zero; and in the tensor product constructed from linear maps the inner product is defined via the trace function. Therefore, to define the non-orthogonality relation $\rightarrow$ we need a characterization of linear maps of trace 0. In this section our goal is such a characterization.

Throughout this section, we fix a *finite-dimensional* vector space V over a *field* $\mathcal{F} = (F, +, \cdot, 0, 1)$ equipped with an *anisotropic* Hermitian form Φ[11]. Moreover, we use $(\cdot)^*$ to denote the accompanying involution of Φ.

Such a V is a generalization of a finite-dimensional generalized Hilbert space over a field and thus that of a finite-dimensional Hilbert space over $\mathbb{C}$.

We start from proving two lemmas about $\mathcal{F}$ which will be used later.

Lemma 4.1 *Suppose that the dimension of V is at least 2 and each linear map on a subspace of V has at least one eigenvector. For each $x \in F$, there is a $y \in F$ such that $x = yy$.*

Proof Let $x \in F$ be arbitrary. Since V is of dimension at least 2, there are $\mathbf{u}, \mathbf{v} \in V$ which are linearly independent. Consider the following function defined on $L(\{\mathbf{u}, \mathbf{v}\})$: for any $c, d \in F$,

$$f(c\mathbf{u} + d\mathbf{v}) = c\mathbf{v} + dx\mathbf{u}$$

[11] An Hermitian form Φ is *anisotropic* if, for each $\mathbf{v} \in V$, $\Phi(\mathbf{v}, \mathbf{v}) = 0$ implies $\mathbf{v} = \mathbf{0}$. Every orthomodular Hermitian form is anisotropic. A proof can be found below Definition 1.2 on page 206 of [45].

It can be verified that this is a linear map on $L(\{\mathbf{u}, \mathbf{v}\})$. By the assumption it has an eigenvector $a\mathbf{u} + b\mathbf{v}$, where $a \neq 0$ or $b \neq 0$. Denote the corresponding eigenvalue by y. Then

$$y(a\mathbf{u} + b\mathbf{v}) = f(a\mathbf{u} + b\mathbf{v}) = a\mathbf{v} + bx\mathbf{u}$$

and thus

$$(ya - bx)\mathbf{u} + (yb - a)\mathbf{v} = 0$$

Since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, $ya - bx = 0$ and $yb - a = 0$. Hence $yyb - bx = 0$. Since $\mathcal{F}$ is a field, $b(yy - x) = 0$.

Note that $b \neq 0$; otherwise, we have $a \neq 0$ and $ya\mathbf{u} - a\mathbf{v} = 0$, and thus $y\mathbf{u} - \mathbf{v} = 0$, contradicting that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.

Therefore, $x = yy$. □

Definition 4.2 *V admits normalization*, if, for each $\mathbf{v} \in V \backslash \{\mathbf{0}\}$, there is an $x \in F$ such that $\Phi(x\mathbf{v}, x\mathbf{v}) = 1$.

Lemma 4.3 *Suppose that V admits normalization and, for each $x \in F$, there is a $y \in F$ such that $x = yy$. For each $\mathbf{v} \in V$, there is a $\sqrt{\Phi(\mathbf{v}, \mathbf{v})} \in F$ such that $\sqrt{\Phi(\mathbf{v}, \mathbf{v})} = (\sqrt{\Phi(\mathbf{v}, \mathbf{v})})^*$ and $\Phi(\mathbf{v}, \mathbf{v}) = \sqrt{\Phi(\mathbf{v}, \mathbf{v})}\sqrt{\Phi(\mathbf{v}, \mathbf{v})}$.*

Proof Let $\mathbf{v} \in V$ be arbitrary. If $\mathbf{v} = \mathbf{0}$, by definition $\Phi(\mathbf{v}, \mathbf{v}) = 0$, so $0 \in F$ has the required property. It remains to deal with the case when $\mathbf{v} \neq \mathbf{0}$. Since V admits normalization, there is an $x \in F$ such that $\Phi(x\mathbf{v}, x\mathbf{v}) = 1$. Then there is a $y \in F$ such that $x = yy$. Let $\sqrt{\Phi(\mathbf{v}, \mathbf{v})} = y^{-1}(y^{-1})^*$. It is obvious that $\sqrt{\Phi(\mathbf{v}, \mathbf{v})} = (\sqrt{\Phi(\mathbf{v}, \mathbf{v})})^*$. Moreover,

$$\begin{aligned}
\sqrt{\Phi(\mathbf{v}, \mathbf{v})}\sqrt{\Phi(\mathbf{v}, \mathbf{v})} &= y^{-1}(y^{-1})^* y^{-1}(y^{-1})^* \\
&= y^{-1}y^{-1}(y^{-1})^*(y^{-1})^* && (\mathcal{F} \text{ is a field}) \\
&= (yy)^{-1}(y^{-1}y^{-1})^* \\
&= x^{-1}(x^{-1})^* \\
&= x^{-1}(x^{-1})^*\Phi(x\mathbf{v}, x\mathbf{v}) \\
&= \Phi(x^{-1}x\mathbf{v}, x^{-1}x\mathbf{v}) \\
&= \Phi(\mathbf{v}, \mathbf{v})
\end{aligned}$$

□

Proposition 4.4 *Suppose that V admits normalization and each linear map on a subspace of V has an eigenvector. For each linear map $f : V \to V$, there is a $\mathbf{w} \in V \backslash \{\mathbf{0}\}$ such that $Tr(f) = \Phi(f(\mathbf{w}), \mathbf{w})$.*

Proof Use induction on the dimension n of V.

Base Step: $n = 1$. Take a unit vector $\mathbf{w} \in V$. By the definition of trace $Tr(f) = \Phi(f(\mathbf{w}), \mathbf{w})$.

Induction Step: $n = k$. By the supposition f has at least one eigenvector $\mathbf{v}$. Since V admits normalization, without loss of generality, we assume that $\mathbf{v}$ is a unit vector and $f(\mathbf{v}) = a\mathbf{v}$. Then $U = \{\mathbf{v}\}^{\perp}$ is an $(n-1)$-dimensional subspace of V. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_{k-1}\}$ be an orthonormal basis of U. Then $\{\mathbf{v}, \mathbf{u}_1, \dots, \mathbf{u}_{k-1}\}$ is an orthonormal basis of V. Hence $Tr(f) = \Phi(f(\mathbf{v}), \mathbf{v}) + \sum_{i=1}^{k-1} \Phi(f(\mathbf{u}_i), \mathbf{u}_i)$.

Now define two functions $\pi_0 : V \to V$ and $\pi_1 : V \to V$ as follows: for each $x, x_1, \dots, x_{k-1} \in F$,

$$\pi_0\left(x\mathbf{v} + \sum_{i=1}^{k-1} x_i\mathbf{u}_i\right) = x\mathbf{v}, \qquad \pi_1\left(x\mathbf{v} + \sum_{i=1}^{k-1} x_i\mathbf{u}_i\right) = \sum_{i=1}^{k-1} x_i\mathbf{u}_i$$

It can be verified that:

1. both π_0 and π_1 are idempotent and self-adjoint linear maps, and thus they are orthogonal projections onto $\{\mathbf{v}\}$ and $U = \{\mathbf{v}\}^{\perp}$, respectively;
2. $\pi_0 + \pi_1 = id_V$, and thus $f = id_V \circ f = (\pi_0 + \pi_1) \circ f = \pi_0 \circ f + \pi_1 \circ f$;
3. $(\pi_1 \circ f) \restriction_U$ is a linear map on U.

Hence by the IH there is a $\mathbf{u} \in U \backslash \{\mathbf{0}\}$ such that

$$Tr((\pi_1 \circ f) \restriction_U) = \Phi((\pi_1 \circ f) \restriction_U(\mathbf{u}), \mathbf{u}) = \Phi((\pi_1 \circ f)(\mathbf{u}), \mathbf{u})$$

Since $\{\mathbf{u}_1, \dots, \mathbf{u}_{k-1}\}$ is an orthonormal basis of U,

$$Tr((\pi_1 \circ f) \restriction_U) = \sum_{i=1}^{k-1} \Phi(((\pi_1 \circ f) \restriction_U)(\mathbf{u}_i), \mathbf{u}_i) = \sum_{i=1}^{k-1} \Phi((\pi_1 \circ f)(\mathbf{u}_i), \mathbf{u}_i)$$

Therefore, $\Phi((\pi_1 \circ f)(\mathbf{u}), \mathbf{u}) = \sum_{i=1}^{k-1} \Phi((\pi_1 \circ f)(\mathbf{u}_i), \mathbf{u}_i)$. Moreover,

$$\begin{aligned} Tr(f) &= \Phi(f(\mathbf{v}), \mathbf{v}) + \sum_{i=1}^{k-1} \Phi(f(\mathbf{u}_i), \mathbf{u}_i) \\ &= \Phi(f(\mathbf{v}), \mathbf{v}) + \sum_{i=1}^{k-1} \Phi((\pi_0 \circ f + \pi_1 \circ f)(\mathbf{u}_i), \mathbf{u}_i) \\ &= \Phi(f(\mathbf{v}), \mathbf{v}) + \sum_{i=1}^{k-1} \Phi((\pi_0 \circ f)(\mathbf{u}_i) + (\pi_1 \circ f)(\mathbf{u}_i), \mathbf{u}_i) \\ &= \Phi(f(\mathbf{v}), \mathbf{v}) + \sum_{i=1}^{k-1} \Phi((\pi_1 \circ f)(\mathbf{u}_i), \mathbf{u}_i) \\ &= \Phi(f(\mathbf{v}), \mathbf{v}) + \Phi((\pi_1 \circ f)(\mathbf{u}), \mathbf{u}) \end{aligned}$$

Let

$$f(\mathbf{v}) = a\mathbf{v}, \qquad f(\mathbf{u}) = b\mathbf{v} + c\mathbf{u} + \mathbf{r}$$

where $\mathbf{r} \in \{\mathbf{v}, \mathbf{u}\}^{\perp}$. Then $\Phi(f(\mathbf{v}), \mathbf{v}) = \Phi(a\mathbf{v}, \mathbf{v}) = a$ and $\Phi((\pi_1 \circ f)(\mathbf{u}), \mathbf{u}) = \Phi(c\mathbf{u} + \mathbf{r}, \mathbf{u}) = c\Phi(\mathbf{u}, \mathbf{u})$. We consider three cases.

Case 1: $a = 0$. We let $\mathbf{w}$ be $\mathbf{u}$. Since $\mathbf{u} \neq \mathbf{0}$, $\mathbf{w} \neq \mathbf{0}$. Moreover,

$$\begin{aligned}
\Phi(f(\mathbf{w}), \mathbf{w}) &= \Phi(f(\mathbf{u}), \mathbf{u}) \\
&= \Phi(b\mathbf{v} + c\mathbf{u} + \mathbf{r}, \mathbf{u}) \\
&= c\Phi(\mathbf{u}, \mathbf{u}) \\
&= a + c\Phi(\mathbf{u}, \mathbf{u}) && (\text{by } a = 0) \\
&= \Phi(f(\mathbf{v}), \mathbf{v}) + \Phi((\pi_1 \circ f)(\mathbf{u}), \mathbf{u}) \\
&= Tr(f)
\end{aligned}$$

Case 2: $b = 0$. We let $\mathbf{w}$ be $\mathbf{v} + \mathbf{u}$. Obviously $\mathbf{w} \neq \mathbf{0}$. Moreover,

$$\begin{aligned}
\Phi(f(\mathbf{w}), \mathbf{w}) &= \Phi(f(\mathbf{v} + \mathbf{u}), \mathbf{v} + \mathbf{u}) \\
&= \Phi(f(\mathbf{v}), \mathbf{v}) + \Phi(f(\mathbf{u}), \mathbf{v}) + \Phi(f(\mathbf{v}), \mathbf{u}) + \Phi(f(\mathbf{u}), \mathbf{u}) \\
&= \Phi(a\mathbf{v}, \mathbf{v}) + \Phi(c\mathbf{u} + \mathbf{r}, \mathbf{v}) + \Phi(a\mathbf{v}, \mathbf{u}) + \Phi(c\mathbf{u} + \mathbf{r}, \mathbf{u}) \\
&= a + c\Phi(\mathbf{u}, \mathbf{u}) \\
&= \Phi(f(\mathbf{v}), \mathbf{v}) + \Phi((\pi_1 \circ f)(\mathbf{u}), \mathbf{u}) \\
&= Tr(f)
\end{aligned}$$

Case 3: $a \neq 0$ and $b \neq 0$.

Since $n \geq 2$, there are $\mathbf{s}, \mathbf{t} \in V$ such that $\Phi(\mathbf{s}, \mathbf{t}) = 0$. Since V admits normalization, without loss of generality we assume that both $\mathbf{s}$ and $\mathbf{t}$ are unit vectors. Consider the vector $\frac{a}{b}\mathbf{s} + \frac{1}{2}\mathbf{t}$,[12] and we have

$$\begin{aligned}
&\Phi\left(\frac{a}{b}\mathbf{s} + \frac{1}{2}\mathbf{t}, \frac{a}{b}\mathbf{s} + \frac{1}{2}\mathbf{t}\right) \\
&= \Phi\left(\frac{a}{b}\mathbf{s}, \frac{a}{b}\mathbf{s}\right) + \Phi\left(\frac{a}{b}\mathbf{s}, \frac{1}{2}\mathbf{t}\right) + \Phi\left(\frac{1}{2}\mathbf{t}, \frac{a}{b}\mathbf{s}\right) + \Phi\left(\frac{1}{2}\mathbf{t}, \frac{1}{2}\mathbf{t}\right) \\
&= \frac{aa^*}{bb^*}\Phi(\mathbf{s}, \mathbf{s}) + \frac{a}{b}\frac{1}{2}\Phi(\mathbf{s}, \mathbf{t}) + \frac{1}{2}\frac{a^*}{b^*}\Phi(\mathbf{t}, \mathbf{s}) + \frac{1}{2}\frac{1}{2}\Phi(\mathbf{t}, \mathbf{t}) \\
&= \frac{aa^*}{bb^*} + \frac{1}{4}
\end{aligned}$$

By the supposition Lemma 4.1 holds for V. Hence by Lemma 4.3 there is an $x \in F$ such that $x = x^*$ and $\frac{aa^*}{bb^*} + \frac{1}{4} = xx$.

Consider the vector

$$\mathbf{w} = \frac{b}{a}(x - \frac{1}{2})\mathbf{v} + \mathbf{u}$$

[12] Since $\mathcal{K}$ is a field, the fraction notation like $\frac{a}{b}$ is legitimate and intuitive.

Obviously, $\mathbf{w} \neq \mathbf{0}$. We calculate

$$\begin{aligned}
&\Phi(f(\mathbf{w}), \mathbf{w}) \\
&= \Phi\Big(f\Big(\frac{b}{a}(x-\frac{1}{2})\mathbf{v}+\mathbf{u}\Big), \frac{b}{a}(x-\frac{1}{2})\mathbf{v}+\mathbf{u}\Big) \\
&= \Phi\Big(\frac{b}{a}(x-\frac{1}{2})f(\mathbf{v})+f(\mathbf{u}), \frac{b}{a}(x-\frac{1}{2})\mathbf{v}+\mathbf{u}\Big) \\
&= \frac{b}{a}(x-\frac{1}{2})\Big(\frac{b}{a}(x-\frac{1}{2})\Big)^*\Phi(f(\mathbf{v}),\mathbf{v}) + \Big(\frac{b}{a}(x-\frac{1}{2})\Big)^*\Phi(f(\mathbf{u}),\mathbf{v}) \\
&\quad + \frac{b}{a}(x-\frac{1}{2})\Phi(f(\mathbf{v}),\mathbf{u}) + \Phi(f(\mathbf{u}),\mathbf{u}) \\
&= \frac{b}{a}(x-\frac{1}{2})\Big(\frac{b}{a}(x-\frac{1}{2})\Big)^*\Phi(a\mathbf{v},\mathbf{v}) + \Big(\frac{b}{a}(x-\frac{1}{2})\Big)^*\Phi(b\mathbf{v}+c\mathbf{u}+\mathbf{r},\mathbf{v}) \\
&\quad + \frac{b}{a}(x-\frac{1}{2})\Phi(a\mathbf{v},\mathbf{u}) + \Phi(b\mathbf{v}+c\mathbf{u}+\mathbf{r},\mathbf{u}) \\
&= \frac{b}{a}(x-\frac{1}{2})\Big(\frac{b}{a}(x-\frac{1}{2})\Big)^* a + \Big(\frac{b}{a}(x-\frac{1}{2})\Big)^* b + c\Phi(\mathbf{u},\mathbf{u}) \\
&= \frac{bb^*}{aa^*}\Big(x-\frac{1}{2}\Big)\Big(x^*-\frac{1}{2}\Big)a + \frac{b^*}{a^*}\Big(x^*-\frac{1}{2}\Big)b + c\Phi(\mathbf{u},\mathbf{u}) \\
&= \frac{bb^*}{aa^*}\Big(x-\frac{1}{2}\Big)\Big(x-\frac{1}{2}\Big)a + \frac{b^*}{a^*}\Big(x-\frac{1}{2}\Big)b + c\Phi(\mathbf{u},\mathbf{u}) \qquad (\text{by } x = x^*) \\
&= \frac{bb^*}{a^*}\Big(xx - x + \frac{1}{4}\Big) + \frac{bb^*}{a^*}\Big(x-\frac{1}{2}\Big) + c\Phi(\mathbf{u},\mathbf{u}) \\
&= \frac{bb^*}{a^*}\Big(\frac{aa^*}{bb^*} + \frac{1}{2} - x\Big) + \frac{bb^*}{a^*}\Big(x-\frac{1}{2}\Big) + c\Phi(\mathbf{u},\mathbf{u}) \qquad (\text{by } xx = \frac{aa^*}{bb^*} + \frac{1}{4}) \\
&= \frac{bb^*}{a^*}\frac{aa^*}{bb^*} + c\Phi(\mathbf{u},\mathbf{u}) \\
&= a + c\Phi(\mathbf{u},\mathbf{u}) \\
&= Tr(f)
\end{aligned}$$

□

Theorem 4.5 *Suppose that V admits normalization and each linear map on a subspace of V has an eigenvector. For each linear map $f : V \to V$, the following are equivalent*:

(i) $Tr(f) = 0$;
(ii) *there is an orthonormal basis* $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ *of V such that* $\Phi(f(\mathbf{v}_i), \mathbf{v}_i) = 0$ *holds for each* $i \in \{1, \dots, n\}$.

Proof **From (i) to (ii):** We use induction on the dimension n of V.

Base Step: $n = 1$. In this case, both (i) and (ii) are equivalent to that f is the zero map.

Induction Step: $n = k$. Assume that $Tr(f) = 0$. By the above proposition $\Phi(f(\mathbf{v}_1), \mathbf{v}_1) = 0$, for some $\mathbf{v}_1 \in V \setminus \{\mathbf{0}\}$.

Since V admits normalization, without loss of generality, we assume that $\mathbf{v}_1$ is a unit vector. Extend the set $\{\mathbf{v}_1\}$ to an orthonormal basis $\{\mathbf{v}_1, \mathbf{u}_2, \dots \mathbf{u}_k\}$ of V and let $U = \{\mathbf{v}_1\}^\perp = \{\mathbf{u}_2, \dots \mathbf{u}_k\}^{\perp\perp}$. Define two functions π_0 and π_1 on V as follows: for any $x_1, \dots, x_k \in F$,

$$\pi_0(x_1\mathbf{v}_1 + \sum_{i=2}^{k} x_i\mathbf{u}_i) = x_1\mathbf{v}_1, \qquad \pi_1(x_1\mathbf{v}_1 + \sum_{i=2}^{k} x_i\mathbf{u}_i) = \sum_{i=2}^{k} x_i\mathbf{u}_i$$

It can be verified that:

1. both π_0 and π_1 are idempotent and self-adjoint linear maps, and thus they are orthogonal projections onto $U^\perp = \{\mathbf{v}_1\}$ and U, respectively;
2. $\pi_0 + \pi_1 = id_V$, and thus $f = id_V \circ f = (\pi_0 + \pi_1)\circ f = \pi_0 \circ f + \pi_1 \circ f$;
3. $(\pi_1 \circ f)\restriction_U$ is a linear map on U.

Hence

$$\begin{aligned} Tr(f) &= \Phi(f(\mathbf{v}_1), \mathbf{v}_1) + \sum_{i=2}^{k} \Phi(f(\mathbf{u}_i), \mathbf{u}_i) = 0 + \sum_{i=2}^{k} \Phi(f(\mathbf{u}_i), \mathbf{u}_i) \\ &= \sum_{i=2}^{k} \Phi((\pi_0 \circ f + \pi_1 \circ f)(\mathbf{u}_i), \mathbf{u}_i) \\ &= \sum_{i=2}^{k} \Big(\Phi((\pi_0 \circ f)(\mathbf{u}_i), \mathbf{u}_i) + \Phi((\pi_1 \circ f)(\mathbf{u}_i), \mathbf{u}_i) \Big) \\ &= \sum_{i=2}^{k} \Phi((\pi_1 \circ f)(\mathbf{u}_i), \mathbf{u}_i) \\ &= \sum_{i=2}^{k} \Phi((\pi_1 \circ f)\restriction_U(\mathbf{u}_i), \mathbf{u}_i) = Tr((\pi_1 \circ f)\restriction_U) \end{aligned}$$

Then $Tr((\pi_1 \circ f)\restriction_U) = Tr(f) = 0$. Since the dimension of U is $k-1$, the IH applies. Hence there is an orthonormal basis $\{\mathbf{v}_2, \dots, \mathbf{v}_k\}$ of U such that $\Phi((\pi_1 \circ f)\restriction_U(\mathbf{v}_i), \mathbf{v}_i) = 0$ holds for each $i \in \{2, \dots, k\}$. Since $\{\mathbf{v}_1, \mathbf{u}_2, \dots \mathbf{u}_k\}$ is an orthonormal basis of V and $\{\mathbf{v}_2, \dots, \mathbf{v}_k\}$ is an orthonormal basis of $U = \{\mathbf{v}_1\}^\perp = \{\mathbf{u}_2, \dots \mathbf{u}_k\}^{\perp\perp}$, $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is an orthonormal basis of V.

Consider this basis. We already know $\Phi(f(\mathbf{v}_1), \mathbf{v}_1) = 0$. Moreover, for each $i \in \{2, \dots, k\}$,

$$\begin{aligned}\Phi(f(\mathbf{v}_i), \mathbf{v}_i) &= \Phi((\pi_0 \circ f)(\mathbf{v}_i) + (\pi_1 \circ f)(\mathbf{v}_i), \mathbf{v}_i)\\ &= \Phi((\pi_0 \circ f)(\mathbf{v}_i), \mathbf{v}_i) + \Phi((\pi_1 \circ f)(\mathbf{v}_i), \mathbf{v}_i)\\ &= \Phi((\pi_0 \circ f)(\mathbf{v}_i), \mathbf{v}_i) + \Phi((\pi_1 \circ f)\upharpoonright_U(\mathbf{v}_i), \mathbf{v}_i)\\ &= \Phi((\pi_0 \circ f)(\mathbf{v}_i), \mathbf{v}_i) + 0\\ &= \Phi(x\mathbf{v}_1, \mathbf{v}_i), \text{ for some } x \in F\\ &= x\Phi(\mathbf{v}_1, \mathbf{v}_i), \text{ for some } x \in F\\ &= 0\end{aligned}$$

As a result, the orthonormal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ satisfies that $\Phi(f(\mathbf{v}_i), \mathbf{v}_i) = 0$ holds for each $i = 1, \dots, k$.

From (ii) to (i): Since $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is an orthonormal basis of V, by the definition of trace $Tr(f) = \sum_{i=1}^{n} \Phi(f(\mathbf{v}_i), \mathbf{v}_i) = 0$. □

With this theorem, we finally know how to define an orthogonality relation between linear maps: **Two linear maps f and g from V_A to V_B are orthogonal, if there is an orthonormal basis of V_A such that, for every element $\mathbf{v}_A$ in this basis,** $\Phi_B(f(\mathbf{v_A}), g(\mathbf{v_A})) = 0$. This idea can be easily lifted to the level of continuous homomorphisms.

5 Tensor Product of Two Quantum Kripke Frames

Given two finite-dimensional Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ over $\mathbb{C}$, we can construct the tensor product of them based on $Hom(\overline{\mathcal{H}_A}, \mathcal{H}_B)$. In this section, based on the results obtained before, we mimic this construction in the framework of quantum Kripke frames. We will only deal with a special case when two quantum Kripke frames $\mathfrak{F}_A = (\Sigma_A, \rightarrow_A)$ and $\mathfrak{F}_B = (\Sigma_B, \rightarrow_B)$ satisfy the following 5 assumptions:

- **Assumption 1:**
  Both $\mathfrak{F}_A$ and $\mathfrak{F}_B$ are Pappian.
- **Assumption 2:**
  For each $i \in \{A, B\}$ and $P_i \subseteq \Sigma_i$ satisfying $P_i = {\sim_i}{\sim_i}P_i$, each arguesian continuous homomorphism F_i on $(P_i, \rightarrow_i \cap (P_i \times P_i))$ has at least one fixed point.
- **Assumption 3:**
  For each $i \in \{A, B\}$, $\mathfrak{F}_i$ satisfies that, for any $s_i, t_i \in \Sigma_i$, if $s_i \nrightarrow t_i$, then there is an isomorphism F_i on $\mathfrak{F}_i$ such that $F_i(s_i) = t_i$, $F_i(t_i) = s_i$ and $F_i\upharpoonright_{\sim\{s_i,t_i\}}$ is the identity.
- **Assumption 4:**
  There is a non-degenerate arguesian continuous homomorphism H from $\mathfrak{F}_A$ to $\mathfrak{F}_B$.
- **Assumption 5:**
  Both $\mathfrak{F}_A$ and $\mathfrak{F}_B$ are finite-dimensional.

We are going to use the analytic method. The following proposition reveals the analytic consequences of these five assumptions:

Proposition 5.1 *Let $\mathfrak{F}_A = (\Sigma_A, \rightarrow_A)$ and $\mathfrak{F}_B = (\Sigma_B, \rightarrow_B)$ be two quantum Kripke frames satisfying the five assumptions.*

1. *There is a field $\mathcal{K}$ and, for each $i \in \{A, B\}$, there is a generalized Hilbert space V_i such that*

(a) *V_i is over the field $\mathcal{K}$;*
(b) $\mathfrak{F}_i \cong \mathfrak{F}_{V_i}$;
(c) *V_i is of finite dimension at least* 3;
(d) *V_i admits normalization*;
(e) *Every linear map on a subspace of V_i has at least one eigenvector.*

2. *There is a linear map $h : V_A \to V_B$ such that $H = \mathcal{P}(h)$.*

In particular, both V_A and V_B satisfy the supposition in Theorem 4.5.

Proof By Assumption 1 and Theorem 2.25, for each $i \in \{A, B\}$, there is a generalized Hilbert space V_i over a field $\mathcal{K}_i$ such that $\mathfrak{F}_i \cong \mathfrak{F}_{V_i}$.

By Assumption 5, for each $i \in \{A, B\}$, V_i is finite-dimensional.

By Assumption 4 and Theorem 3.18 there is a quasi-linear map $h : V_A \to V_B$ such that $H = \mathcal{P}(h)$ and the dimension of V_A and that of V_B are both at least 3. Moreover, without loss of generality, we can assume that $\mathcal{K}_A = \mathcal{K}_B$ and the field isomorphism associated to h is the identity, so h is a linear map. Thus we can drop the subscripts for the field. Therefore, Items 1(a) to 1(c) and Item 2 are proved.

It remains to prove Items 1(d) and 1(e). Let $i \in \{A, B\}$ be arbitrary.

For Item 1(d), by Assumption 3 and the proof of Theorem 2.13 in [26], especially the proof of the claim in the direction from (ii) to (i), for any $\mathbf{s}_i, \mathbf{t}_i \in V_i \backslash \{\mathbf{0}_i\}$, if $\Phi_i(\mathbf{s}_i, \mathbf{t}_i) = 0$, then there is an $x \in K$ such that $\Phi_i(\mathbf{s}_i, \mathbf{s}_i) = \Phi_i(x\mathbf{t}_i, x\mathbf{t}_i)$. For any $\mathbf{s}_i, \mathbf{t}_i \in V_i \backslash \{\mathbf{0}_i\}$, since the dimension of V_i is at least 3, we can always find a $\mathbf{u}_i \in V_i \backslash \{\mathbf{0}_i\}$ such that $\Phi_i(\mathbf{s}_i, \mathbf{u}_i) = 0$ and $\Phi_i(\mathbf{u}_i, \mathbf{t}_i) = 0$, so there are $y, z \in K$ such that $\Phi_i(\mathbf{s}_i, \mathbf{s}_i) = \Phi_i(y\mathbf{u}_i, y\mathbf{u}_i)$ and $\Phi_i(\mathbf{u}_i, \mathbf{u}_i) = \Phi_i(z\mathbf{t}_i, z\mathbf{t}_i)$, and thus letting $x = yz$ we have $\Phi_i(\mathbf{s}_i, \mathbf{s}_i) = \Phi_i(x\mathbf{t}_i, x\mathbf{t}_i)$. Finally, by Theorem 2.25 Φ_i is fixed only up to a constant multiple, so we can rescale Φ_i such that $\Phi(\mathbf{s}_i, \mathbf{s}_i) = 1$ for some $\mathbf{s}_i \in V_i$. As a result, without loss of generality, we can assume that V_i admits normalization.

For Item 1(e), for each subspace W_i of V_i, it can be verified that $\Sigma_{W_i} \subseteq \Sigma_{V_i}$, $\Sigma_{W_i} = \sim_i \sim_i \Sigma_{W_i}$ and $(\Sigma_{W_i}, \rightarrow_{W_i}) = (\Sigma_{W_i}, \rightarrow_i \cap (\Sigma_{W_i} \times \Sigma_{W_i}))$. By Assumption 2 and Theorem 3.18 every linear map on W_i has at least one eigenvector. □

Theorem 5.2 *Let $\mathfrak{F}_A = (\Sigma_A, \rightarrow_A)$ and $\mathfrak{F}_B = (\Sigma_B, \rightarrow_B)$ be two quantum Kripke frames satisfying the five assumptions. $\mathfrak{F} = (\Sigma, \rightarrow)$ defined as follows is a quantum Kripke frame*:

1. *$\Sigma = \{F \mid F$ is an arguesian continuous homomorphism from $\mathfrak{F}_A$ to $\mathfrak{F}_B$ such that $F \eqsim H\}$;*
2. *for any $F, G \in \Sigma$, $F \to G$, if and only if, for each maximal orthogonal set P_A in $\mathfrak{F}_A$, there is an $s_A \in P_A$ such that both $F(s_A)$ and $G(s_A)$ are defined and $F(s_A) \to_B G(s_A)$.*

Proof Since $\mathfrak{F}_A$ and $\mathfrak{F}_B$ satisfy the five assumptions, by Proposition 5.1 Items 1 and 2 in the proposition hold. We will also use the same notations.

It can be verified that $Hom(V_A, V_B)$ can be organized into a generalized Hilbert space over $\mathcal{K}$ in a way similar to that in Theorem 2.9, so $\mathfrak{F}_{Hom(V_A,V_B)} = (\Sigma_{Hom(V_A,V_B)}, \to_{Hom(V_A,V_B)})$ is a quantum Kripke frame.

To prove that $\mathfrak{F}$ is a quantum Kripke frame, it suffices to prove that $\mathfrak{F} \cong \mathfrak{F}_{Hom(V_A,V_B)}$. Define that

$$\mathbf{I} : \Sigma_{Hom(V_A,V_B)} \to \Sigma :: \langle f \rangle \mapsto \mathcal{P}(f), \text{ where } f \in Hom(V_A, V_B) \text{ is not the zero map}$$

By linearity $\mathbf{I}(\langle f \rangle)$ does not depend on the choice of f, so $\mathbf{I}$ is a well-defined function.

For injectivity, assume that $\mathbf{I}(\langle f \rangle) = \mathbf{I}(\langle g \rangle)$ for some $f, g \in Hom(V_A, V_B)$ neither of which is the zero map. By definition $\mathcal{P}(f) = \mathcal{P}(g)$. If the ranges of both $\mathcal{P}(f)$ and $\mathcal{P}(g)$ are at most one-dimensional, since both are linear maps, $f \in \langle g \rangle$ and thus $\langle f \rangle = \langle g \rangle$. If the ranges of both $\mathcal{P}(f)$ and $\mathcal{P}(g)$ are at least two-dimensional, by the uniqueness part of Theorem 3.27 f is unique up to scalar multiplication such that $\mathcal{P}(f) = \mathcal{P}(g)$. Then $f \in \langle g \rangle$ and thus $\langle f \rangle = \langle g \rangle$.

For surjectivity, let $F \in \Sigma$. Then F is an arguesian continuous homomorphism from $\mathfrak{F}_A$ to $\mathfrak{F}_B$ such that $F \eqsim H$. Since $\mathfrak{F}_i \cong \mathfrak{F}_{V_i}$ for $i \in \{A, B\}$, without loss of generality we can consider F as an arguesian continuous homomorphism from $\mathfrak{F}_{V_A}$ to $\mathfrak{F}_{V_B}$ such that $F \eqsim H$. By Theorem 3.27 there is an $f \in Hom(V_A, V_B)$ such that $F = \mathcal{P}(f)$. Then $\mathbf{I}(\langle f \rangle) = \mathcal{P}(f) = F$.

Therefore, $\mathbf{I}$ is a bijection.

Finally, for any $s, t \in \Sigma_{Hom(V_A,V_B)}$, by definition there are $f, g \in Hom(V_A, V_B)$ such that $s = \langle f \rangle$ and $t = \langle g \rangle$. By Theorem 4.5

$$
\begin{aligned}
& s \nrightarrow_{Hom(V_A,V_B)} t \\
\Leftrightarrow\ & \langle f \rangle \nrightarrow_{Hom(V_A,V_B)} \langle g \rangle \\
\Leftrightarrow\ & \Phi(f,g) = 0 \\
\Leftrightarrow\ & Tr(g^\dagger \circ f) = 0 \\
\Leftrightarrow\ & \text{there is an orthonormal basis } \{\mathbf{v}_A^i \mid i = 1,\dots,n\} \text{ of } V_A \text{ such that} \\
& \Phi_A((g^\dagger \circ f)(\mathbf{v}_A^i), \mathbf{v}_A^i) = 0 \text{ holds for each } i = 1,\dots,n \\
\Leftrightarrow\ & \text{there is an orthonormal basis } \{\mathbf{v}_A^i \mid i = 1,\dots,n\} \text{ of } V_A \text{ such that} \\
& \Phi_B(f(\mathbf{v}_A^i), g(\mathbf{v}_A^i)) = 0 \text{ holds for each } i = 1,\dots,n \\
\Leftrightarrow\ & \text{there is a maximal orthogonal set } \{v_A^i \mid i = 1,\dots,n\} \text{ of } \mathfrak{F}_A \text{ such that,} \\
& \text{for each } i = 1,\dots,n, \text{ at least one of the following holds: } \mathcal{P}(g)(v_A^i) \text{ is undefined,} \\
& \mathcal{P}(f)(v_A^i) \text{ is undefined and } \mathcal{P}(f)(v_A^i) \nrightarrow_B \mathcal{P}(g)(v_A^i) \\
\Leftrightarrow\ & \text{there is a maximal orthogonal set } \{v_A^i \mid i = 1,\dots,n\} \text{ of } \mathfrak{F}_A \text{ such that,} \\
& \text{for each } i = 1,\dots,n, \text{ at least one of the following holds: } \mathbf{I}(s)(v_A^i) \text{ is undefined,} \\
& \mathbf{I}(t)(v_A^i) \text{ is undefined and } \mathbf{I}(s)(v_A^i) \nrightarrow_B \mathbf{I}(t)(v_A^i) \\
\Leftrightarrow\ & \mathbf{I}(s) \nrightarrow \mathbf{I}(t)
\end{aligned}
$$

As a result, **I** is an isomorphism. □

According to this theorem, we see that the non-orthogonality between the states of a bipartite system is related to the non-orthogonality relations of the two subsystems in a natural way. Let two quantum systems A and B be described by two quantum Kripke frames $\mathfrak{F}_A = (\Sigma_A, \rightarrow_A)$ and $\mathfrak{F}_B = (\Sigma_B, \rightarrow_B)$, respectively, satisfying the five assumptions; and F and G two arguesian continuous homomorphisms from $\mathfrak{F}_A$ to $\mathfrak{F}_B$ modelling two states of the bipartite system consisting of A and B. Then **F and G are non-orthogonal, if and only if there is no (local) measurement on A and (local) measurement on B such that preforming each of these two measurements once can always distinguish between F and G .** To be precise, if F and G are non-orthogonal, then, for any measurement on A, even if after performing it once we know that the state of A is s_A, it is possible (depending on what is s_A) that there is no measurement on B such that one performance of it can distinguish between $F(s_A)$ and $G(s_A)$, i.e. the system in either states before the measurement may yield the same result in the measurement; Similarly, for any measurement on B, even if after performing it once we know that the state of B is s_B, it is possible (depending on what is s_B) that there is no measurement on A such that one performance of it can distinguish between $F^\dagger(s_B)$ and $G^\dagger(s_B)$.

6 Conclusion and Discussion

In this paper, we manage to mimic the tensor product construction of two finite-dimensional Hilbert spaces in the framework of quantum Kripke frames under five assumptions. Despite of the complicated and tedious technical details and the restrictive assumptions, the definition of the non-orthogonality relation in the 'tensor product' of two quantum Kripke frames is intuitive and natural. This shows the hope that quantum Kripke frames can model bipartite quantum systems and thus the phenomenon of quantum entanglement. Besides those in [26], this is another piece of evidence that quantum Kripke frames are good qualitative models of quantum systems.

In the following, we discuss some issues related to this construction.

6.1 Discussion About the Five Assumptions

None of the assumptions seems to have clear intuitive meaning, so we only discuss their role in the technical aspect. For **Assumption 1**, the significance is three-fold. First, it guarantees that the quantum Kripke frames under consideration are isomorphic to those induced by vector spaces, and thus we can use the analytic method. Second, it restricts our attention to only vector spaces of dimension at least 3. Third, it restricts our consideration to only vector spaces over *fields* and thus excludes those over division rings like that of the quaternions (Theorem 2.25). The third significance is crucial in characterizing continuous homomorphisms with the same associated division ring isomorphism and thus finding the right mathematical objects to model the states of bipartite systems (Lemma 3.25); it is also needed in characterizing trace-zero linear maps and thus defining the orthogonality relation between arguesian continuous homomorphisms (Lemmas 4.1 and 4.3 and Proposition 4.4). For **Assumption 2**, it guarantees that, in the vector spaces involved, each linear map has at least one eigenvector; so it excludes vector spaces over the real numbers. It is used in characterizing trace-zero linear maps and thus defining the orthogonality relation between arguesian continuous homomorphisms (Lemma 4.1 and Proposition 4.4). If Assumptions 1 and 2 are proved to be necessary, then it helps to argue from a mathematical perspective that we do not use real numbers or quaternions in quantum theory. **Assumption 3** guarantees that the vector spaces involved admit normalization (the proof of Theorem 2.13 in [26]). It is used in characterizing trace-zero linear maps and thus defining the orthogonality relation between arguesian continuous homomorphisms (Lemma 4.3 and Proposition 4.4). **Assumption 4** is used to guarantee that the two fields corresponding to the two quantum Kripke frames are isomorphic (Theorem 3.18), which seems to be natural and necessary. **Assumption 5** makes the vector spaces involved be finite-dimensional. It is desirable to drop this assumption. First of all, note that, once we consider infinite-dimensional vector spaces, according to Theorem 2.13 in [26] which is a corollary of Solèr's Theorem [46], Assumption 3 forces the vector spaces involved to be infinite-dimensional Hilbert spaces over the real numbers, the complex numbers and the quaternions. (Considering this, Assumption 3 is acceptable.) Combining with Assumptions 1 and 2, we are only dealing with (infinite-dimensional) Hilbert spaces over $\mathbb{C}$. Then we will succeed to drop Assumption 5, if we manage to finish two tasks. The first task is to characterize Hilbert-Schmidt operators between Hilbert spaces, which are used in the tensor product construction involving infinite-dimensional Hilbert spaces over $\mathbb{C}$. Recall that each of them is a bounded linear map f from a Hilbert space $\mathcal{H}_A$ to

$\mathcal{H}_B$ such that, for each orthonormal basis $\{\mathbf{v}_A^i \mid i \in I\}$ of $\mathcal{H}_A$, $\sum_{i\in I}\langle f(\mathbf{v}_A^i), f(\mathbf{v}_A^i)\rangle$ is finite. The question is whether arguesian continuous homomorphisms correspond to such operators in the infinite-dimensional case, and, if not, whether we need to add structure on quantum Kripke frames such that these operators can be characterized. The second task is to generalize Theorem 4.5 to the infinite-dimensional case. Given the nice properties of Hilbert-Schmidt operators, this may not be difficult; but it will still be left to future work. In a word, it will be interesting to see how far the five assumptions used in our construction can be relaxed; such a study will improve our understanding of quantum theory from the perspective of describing composition of quantum systems in terms of the orthogonality relation.

6.2 Application in Analysing Quantum Entanglement

Theorem 5.2 reveals a natural and intuitive relation between the orthogonality relation of a bipartite system and those of its subsystems. When it comes to measuring the degree of entanglement, on the one hand, since the framework of quantum Kripke frames is purely qualitative, there is little hope that we can have something like (full-fledge) von-Neumann entropy to evaluate the degree of entanglement. On the other hand, our framework adheres some ways of evaluating the degree of entanglement from the idea of modelling entanglement using linear maps. For example, consider a state of a bipartitle quantum system modelled by an arguesian continuous homomorphism F from $\mathfrak{F}_A$ to $\mathfrak{F}_B$ satisfying the five assumptions. If the range of F is a singleton s_B, then the state is in fact a separable state in which A is in the state in the singleton $\sim\mathrm{Ker}(F)$ and B is in the state s_B. If F is an isomorphism from $\mathfrak{F}_A$ to $\mathfrak{F}_B$, then the state is of the maximal degree of entanglement like a Bell state. Further application of our results in analysing quantum entanglement is left to future work.

6.3 Extension to Multi-partite Quantum Systems in General

On the one hand, in principle there is no technical difficulty to apply the construction in this paper to quantum systems consisting of more than two subsystems. We take a quantum system consisting of three parts, A, B and C, as an example. Let the Hilbert spaces $\mathcal{H}_A$, $\mathcal{H}_B$ and $\mathcal{H}_C$ describe these three systems, respectively, such that $\mathfrak{F}_{\mathcal{H}_A}$, $\mathfrak{F}_{\mathcal{H}_B}$ and $\mathfrak{F}_{\mathcal{H}_C}$ satisfy the five assumptions; moreover, we use the symbol $\otimes$ to denote the tensor product of Hilbert spaces and our construction on quantum Kripke frames. Then by Theorem 5.2 $\mathfrak{F}_{\mathcal{H}_A\otimes\mathcal{H}_B} \cong \mathfrak{F}_{\mathcal{H}_A} \otimes \mathfrak{F}_{\mathcal{H}_B}$ and $\mathfrak{F}_{(\mathcal{H}_A\otimes\mathcal{H}_B)\otimes\mathcal{H}_C} \cong (\mathfrak{F}_{\mathcal{H}_A\otimes\mathcal{H}_B}) \otimes \mathfrak{F}_{\mathcal{H}_C} \cong (\mathfrak{F}_{\mathcal{H}_A} \otimes \mathfrak{F}_{\mathcal{H}_B}) \otimes \mathfrak{F}_{\mathcal{H}_C}$. Hence, from $\mathfrak{F}_{\mathcal{H}_A}$, $\mathfrak{F}_{\mathcal{H}_B}$ and $\mathfrak{F}_{\mathcal{H}_C}$, we construct $(\mathfrak{F}_{\mathcal{H}_A} \otimes \mathfrak{F}_{\mathcal{H}_B}) \otimes \mathfrak{F}_{\mathcal{H}_C}$ which can describe the tripartite system. On the other hand, however, we have to confess that this simple-minded extension loses the intuitive feature of the construction. Now an element $\mathcal{F}$ in $(\mathfrak{F}_{\mathcal{H}_A} \otimes \mathfrak{F}_{\mathcal{H}_B}) \otimes \mathfrak{F}_{\mathcal{H}_C}$ is an arguesian continuous homomorphism that maps an arguesian continuous homomorphism from $\mathfrak{F}_{\mathcal{H}_A}$ to $\mathfrak{F}_{\mathcal{H}_B}$ in the domain of $\mathcal{F}$ to an element in $\Sigma_{\mathcal{H}_C}$. This picture is complicated, and it will become more complicated if the number of systems is even higher. Moreover, the frame $(\mathfrak{F}_{\mathcal{H}_A} \otimes \mathfrak{F}_{\mathcal{H}_B}) \otimes \mathfrak{F}_{\mathcal{H}_C}$ is not handy to use in calculation in some scenarios. Suppose that we know that the state of the tripartite system is modelled by $\mathcal{F}$ in $(\mathfrak{F}_{\mathcal{H}_A} \otimes \mathfrak{F}_{\mathcal{H}_B}) \otimes \mathfrak{F}_{\mathcal{H}_C}$; we first do a measurement on A and get to know that the state of A after the measurement is s_A; then we do a measurement on B and get to know that the state of B after the measurement is s_B. Now we hope to calculate the state of C after these two measurements. We will have to proceed as follows: First, we calculate an element in $\mathfrak{F}_{\mathcal{H}_C} \otimes \mathfrak{F}_{\mathcal{H}_B}$ which maps every $t_C \in \Sigma_{\mathcal{H}_C}$ to $\mathcal{F}^{\dagger}(t_C)(s_A) \in \Sigma_{\mathcal{H}_B}$, if $\mathcal{F}^{\dagger}(t_C)(s_A)$ is defined; second we take the adjoint of this continuous homomorphism and get an element in $\mathfrak{F}_{\mathcal{H}_B} \otimes \mathfrak{F}_{\mathcal{H}_C}$; finally, we apply it to s_B, and then the output is the state of C after these two measurements. Therefore, compared to the Hilbert space formalism, the framework of quantum Kripke frames is more operational and has less and simpler primitive structure, but we sacrifice the quantitative part and thus some convenience in modelling and calculation. It is left to future work whether there is a tensor product construction on quantum Kripke frames which can elegantly and operationally model multipartite quantum systems and interesting states of them like the GHZ states and the W states.

6.4 Relation to Similar Works

There are two works in the literature which are most similar to that in this paper. **One** is the paper [34] by Baltag and Smets. Although they also use Kripke frames to model quantum systems, their work is not about constructing a Kripke frame modelling a composite system from the Kripke frames modelling the subsystems. Instead, they start from a Kripke frame which is abstracted from a Hilbert space over $\mathbb{C}$ modelling a quantum system composing of finitely many subsystems. (For convenience, let N be the set of indices of the subsystems; and thus an $I \subseteq N$ indexes the part of the system formed by the subsystems whose indices are in I.) Then they introduce a propositional variable w whose interpretation is exactly a fixed fully separable state of the system and a modal operator K_I for each part $I \subseteq N$. Roughly speaking, $K_I P$ means that at the current (global) state, after any unitary evolutions local at the part I, the resulting state has property P. Observing that local unitary evolutions never change a separable state to an entangled state and vice versa, they characterize entanglement as follows: at the current state the part I is entangled with the other subsystems, if and only if $K_{N \setminus I} K_I w$ is false. An advantage of their approach is that they use Hilbert spaces directly and thus they can conveniently handle entanglement of more than two subsystems, although they do not give an example of this in the paper. In this paper this is not easy as is mentioned above. However, if only bipartite systems are under consideration, then their analysis can be carried out in our abstract Kripke frames. As is mentioned above, a separable state can be modelled by a constant arguesian continuous homomorphism. According to Wigner's Theorem [47], unitary evolutions local at system A can be modelled by isomorphisms on $(\Sigma_A, \to_A)$ and thus the modal operator K_A can be properly interpreted; the same observation also applies to K_B. Hence in our abstract framework we can also characterize the separable states by $K_A K_B w$, where w is interpreted by exactly one constant arguesian continuous homomorphism. Moreover, in our framework, as is mentioned above, we can measure the degree of entanglement by the ranges of arguesian continuous homomorphisms. In [34] they do not discuss this, but we guess that, for each entanglement between certain parts and of a certain degree, to characterize the states of the system entangled in such a way, they have to introduce a propositional variable to denote a fixed state entangled in such a way. In one word, considering only bipartite systems, Baltag and Smets use an abstract and simple formal language to talk about concrete models of entanglement, i.e. Hilbert spaces; in this paper, we propose abstract models of entanglement which can interpret their formal language. **The other** is the work on categorical quantum mechanics by Abramsky, Coecke and others. We think that, based on a class of quantum Kripke frames each pair of which satisfies the five assumptions and the arguesian continuous homomorphisms between them, we will be able to construct a symmetric monoidal dagger category and even categories with richer structures. The details still have to be checked. If this is true, we will have symmetric monoidal dagger categories based on mathematical structures a bit more general than finite-dimensional Hilbert spaces.

Appendix A

In this appendix, we are going to prove Proposition 2.17 which is crucial to the dimension theory of quantum Kripke frames.

We start from reviewing three results about projective geometry.

The first one is about a way of constructing the linear closure of a set.

Proposition A.1 *Let $\mathfrak{G} = (G, \star)$ be a projective geometry. For each $A \subseteq G$, define a sequence $\{A_i\}_{i \in \mathbb{N}}$ of subsets of G as follows:*

- $A_0 = A$;
- $A_{n+1} = \bigcup\{a \star b \mid a, b \in A_n\}$.

Then $\mathcal{C}(A) = \bigcup_{i \in \mathbb{N}} A_i$.

Proof First we prove by induction that $A_i \subseteq \mathcal{C}(A)$, for every $i \in \mathbb{N}$.

Base Step: $i = 0$. By the definition of linear closures $A_0 = A \subseteq \mathcal{C}(A)$.

Induction Step: $i = n + 1$. Let $c \in A_{n+1}$ be arbitrary. By the definition of A_{n+1} there are $a, b \in A_n$ such that $c \in a \star b$. By the induction hypothesis $a, b \in A_n \subseteq \mathcal{C}(A)$, so $c \in a \star b \subseteq \mathcal{C}(A)$ since $\mathcal{C}(A)$ is a subspace.

This finishes the proof by induction. Therefore, $\bigcup_{i \in \mathbb{N}} A_i \subseteq \mathcal{C}(A)$.

Second we prove that $\bigcup_{i \in \mathbb{N}} A_i$ is a subspace including A, and thus $\mathcal{C}(A) \subseteq \bigcup_{i \in \mathbb{N}} A_i$. By definition $A = A_0 \subseteq \bigcup_{i \in \mathbb{N}} A_i$. Now let $a, b \in \bigcup_{i \in \mathbb{N}} A_i$ be arbitrary. Then there are $n, n' \in \mathbb{N}$ such that $a \in A_n$ and $b \in A_{n'}$. Note that by definition $A_i \subseteq A_{i+1}$, for every $i \in \mathbb{N}$. Hence $a, b \in A_m$, where $m = max\{n, n'\}$. Therefore, $a \star b \subseteq A_{m+1} \subseteq \bigcup_{i \in \mathbb{N}} A_i$. As a result, $\bigcup_{i \in \mathbb{N}} A_i$ is a subspace. □

The second one is a very important and useful result in projective geometry called *the projective law*.

Theorem A.2 (Corollary 2.4.5 in [38]) *Let* $\mathfrak{G} = (G, \star)$ *be a projective geometry. For any non-empty sets* $A, B \subseteq G$,

$$\mathcal{C}(A \cup B) = \bigcup \{a \star b \mid a \in \mathcal{C}(A),\ b \in \mathcal{C}(B)\}.$$

The third one is a corollary of the projective law to be used later.

Corollary A.3 *Let* $\mathfrak{G} = (G, \star)$ *be a projective geometry. For any* $a \in G$ *and* $A \subseteq G$, $\mathcal{C}(\{a\} \cup \mathcal{C}(A)) = \mathcal{C}(\{a\} \cup A)$.

Proof By definition $\mathcal{C}(A)$ is a subspace, and thus $\mathcal{C}(\mathcal{C}(A)) = \mathcal{C}(A)$. Using the projective law,

$$\begin{aligned}\mathcal{C}(\{a\} \cup A) &= \bigcup \{c \star d \mid c \in \mathcal{C}(\{a\}),\ d \in \mathcal{C}(A)\} \\ &= \bigcup \{c \star d \mid c \in \mathcal{C}(\{a\}),\ d \in \mathcal{C}(\mathcal{C}(A))\} \\ &= \mathcal{C}(\{a\} \cup \mathcal{C}(A))\end{aligned}$$

□

Now we are going to prove that in a quantum Kripke frame, for any $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$, $\mathcal{C}(\{s_1, ..., s_n\}) = {\sim\sim}\{s_1, ..., s_n\}$. Please remind that, according to Theorem 2.16, in a quantum Kripke frame $\mathfrak{F} = (\Sigma, \to)$, for any $s, t \in \Sigma$, ${\sim\sim}\{s, t\}$ is the line $s \star t$ in the projective geometry corresponding to $\mathfrak{F}$.

Lemma A.4 *In a quantum Kripke frame* $\mathfrak{F} = (\Sigma, \to)$, *if* $P \subseteq \Sigma$ *is closed, it is a subspace of* $\mathbf{G}(\mathfrak{F})$.

Proof Assume that $s, t \in P$. Then $\{s, t\} \subseteq P$. Applying *2* of Lemma 2.15 twice, one can obtain ${\sim\sim}\{s, t\} \subseteq {\sim\sim}P$. Since P is closed, ${\sim\sim}\{s, t\} \subseteq {\sim\sim}P = P$. □

Lemma A.5 *In a quantum Kripke frame* $\mathfrak{F} = (\Sigma, \to)$, ${\sim}Q = {\sim}\mathcal{C}(Q)$, *for every* $Q \subseteq \Sigma$.

Proof By definition $Q \subseteq \mathcal{C}(Q)$, so ${\sim}\mathcal{C}(Q) \subseteq {\sim}Q$ by *2* of Lemma 2.15. It remains to show that ${\sim}Q \subseteq {\sim}\mathcal{C}(Q)$.

We define a sequence of sets $\{Q_i\}_{i \in \mathbb{N}}$ in the same way as in Proposition A.1. Then by the proposition $\mathcal{C}(Q) = \bigcup_{i \in \mathbb{N}} Q_i$. It is easy to see from the definition that ${\sim}\mathcal{C}(Q) = {\sim} \bigcup_{i \in \mathbb{N}} Q_i = \bigcap_{i \in \mathbb{N}} {\sim}Q_i$. We prove ${\sim}Q \subseteq \bigcap_{i \in \mathbb{N}} {\sim}Q_i = {\sim}\mathcal{C}(Q)$ by showing that ${\sim}Q \subseteq {\sim}Q_i$, for every $i \in \mathbb{N}$. Use induction on i.

Base Step: $i = 0$. ${\sim}Q \subseteq {\sim}Q = {\sim}Q_0$ obviously holds.

Induction Step: $i = n + 1$. Let $s \in {\sim}Q$ and $t \in Q_{n+1}$ be arbitrary. By definition there are $u, v \in Q_n$ such that $t \in {\sim\sim}\{u, v\}$. By IH $s \in {\sim}Q \subseteq {\sim}Q_n$. Hence $s \not\to u$ and

$s \nrightarrow v$, i.e. $s \in \sim\{u, v\}$. Since $t \in \sim\sim\{u, v\}$, $s \nrightarrow t$. Since t is arbitrary, $s \in \sim Q_{n+1}$. Therefore, $\sim Q \subseteq \sim Q_{n+1}$. □

The following lemma suggests a way to get a bigger closed set from a smaller one using linear closures. It is a special case of Proposition 14.2.5 in [38]. Since a direct proof is not long, we give it here to avoid introducing general terminologies.

Lemma A.6 *In a quantum Kripke frame* $\mathfrak{F} = (\Sigma, \rightarrow)$, *let* $s \in \Sigma$ *and* $P \subseteq \Sigma$ *be closed. Then* $\mathcal{C}(\{s\} \cup P)$ *is closed.*

Proof If $s \in P$, then $\mathcal{C}(\{s\} \cup P) = \mathcal{C}(P) = P$ is closed by Lemma A.4 and the definition of subspaces. It remains to show the case when $s \notin P$. Since P is closed, $s \notin \sim\sim P$, so there is a $u \in \sim P$ such that $s \rightarrow u$.

By Lemma 2.15 $\mathcal{C}(\{s\} \cup P) \subseteq \sim\sim\mathcal{C}(\{s\} \cup P)$. It remains to show that $\sim\sim\mathcal{C}(\{s\} \cup P) \subseteq \mathcal{C}(\{s\} \cup P)$. By Lemma A.5 it suffices to show that $\sim\sim(\{s\} \cup P) \subseteq \mathcal{C}(\{s\} \cup P)$.

Let $w \in \sim\sim(\{s\} \cup P)$ be arbitrary. If $w = s$, then $w \in \mathcal{C}(\{s\} \cup P)$; so it remains to deal with the case when $w \neq s$. By Lemma 4.11 in [41] there is a $v \in \sim\sim\{w, s\}$ such that $u \nrightarrow v$. Since $s \rightarrow u$ and $u \nrightarrow v$, $s \neq v$. Hence by Lemma 4.12 in [41] $w \in \sim\sim\{s, v\}$. To show that $w \in \mathcal{C}(\{s\} \cup P)$ and thus finish the proof, it remains to show that $v \in P = \sim\sim P$.

Let $x \in \sim P$ be arbitrary. When $x = u$, then $v \nrightarrow u$ follows from the construction of v. When $x \neq u$, by Lemma 4.11 in [41] there is a $y \in \sim\sim\{x, u\}$ such that $y \nrightarrow s$. Since $x, u \in \sim P$, $y \in \sim\sim\{x, u\} \subseteq \sim\sim\sim P = \sim P$. Together with $y \nrightarrow s$, we have $y \in \sim\{s\} \cap \sim P = \sim(\{s\} \cup P)$. Hence $w \nrightarrow y$. Since $v \in \sim\sim\{w, s\}$, $v \nrightarrow y$. Since $s \rightarrow u$ and $y \nrightarrow s$, $y \neq u$. Hence by Lemma 4.12 in [41] $x \in \sim\sim\{y, u\}$. Since $v \nrightarrow u$ and $v \nrightarrow y$, $v \in \sim\{u, y\}$, so $x \nrightarrow v$. Since x is arbitrary, $v \in \sim\sim P = P$. □

Finally we are ready to prove Proposition 2.17.

Proposition A.7 (Proposition 2.17) *In a quantum Kripke frame* $\mathfrak{F} = (\Sigma, \rightarrow)$, *for any* $n \in \mathbb{N}$ *and* $s_1, ..., s_n \in \Sigma$, $\mathcal{C}(\{s_1, ..., s_n\}) = \sim\sim\{s_1, ..., s_n\}$.

Proof We prove by induction that, for every $n \in \mathbb{N}$, $\mathcal{C}(\{s_1, ..., s_n\})$ is closed.

Base Step: $n = 0$. By convention $\{s_1, ..., s_n\} = \emptyset$, so $\mathcal{C}(\emptyset) = \emptyset$ is closed by Lemma 2.15.

Induction Step: $n = k + 1$. By the induction hypothesis $\mathcal{C}(\{s_1, ..., s_k\})$ is closed. Then by Lemma A.6 $\mathcal{C}(\{s_{k+1}\} \cup \mathcal{C}(\{s_1, ..., s_k\}))$ is closed. By Corollary A.3

$$\mathcal{C}(\{s_{k+1}\} \cup \mathcal{C}(\{s_1, ..., s_k\})) = \mathcal{C}(\{s_{k+1}\} \cup \{s_1, ..., s_k\}) = \mathcal{C}(\{s_1, ..., s_k, s_{k+1}\})$$

Hence $\mathcal{C}(\{s_1, ..., s_k, s_{k+1}\})$ is closed. This finishes the proof by induction.

Now by Lemma A.5 $\sim\sim\{s_1, ..., s_n\} = \sim\sim\mathcal{C}(\{s_1, ..., s_n\})$. As $\mathcal{C}(\{s_1, ..., s_n\})$ is closed, $\sim\sim\{s_1, ..., s_n\} = \mathcal{C}(\{s_1, ..., s_n\})$. □

Appendix B

In this appendix, we list the results in projective geometry in the literature which are used in this paper. The following results with the notations are from [38], and the footnotes are added by me.

Proposition 2.3.3 (Page 34) For every subset A of a projective geometry G there exists a smallest subspace $\mathcal{C}(A)$ containing A and the so obtained operator $\mathcal{C}$ is a closure operator on G.

Proposition 3.1.13 (Page 59) Let G be a projective geometry and $\mathcal{C}$ the closure operator associated to it. Then the set G together with the closure operator $\mathcal{C}$ is a geometry.[13]

Lemma 6.3.4 (Page 137) Let V_1, V_2 be vector spaces over K_1, K_2 respectively, $f : V_1 \to V_2$ and $h : V_1 \to V_2$ two additive maps[14] such that $h(x) \in K_2 \cdot f(x)$ for all $x \in V_1$. We suppose that $f(V_1)$ contains at least two linearly independent vectors. Then there exists a unique $\mu \in K_2$ such that $h(x) = \mu \cdot f(x)$ for all $x \in V_1$.

Theorem 10.3.1 (Page 243) For a partial map $g : G \dashrightarrow G'$ between arguesian geometries the following conditions are equivalent:

(1) if homogeneous coordinates $u : \mathcal{P}V \to G$ and $u' : \mathcal{P}V' \to G'$ are given, there exists a semilinear map $f : V \to V'$ such that $g = u' \circ \mathcal{P}f \circ u^{-1}$,
(2) g is described by a semilinear map in some homogeneous coordinates,
(3) g is the composite of two non-degenerate morphisms,
(4) g is the composite of finitely many non-degenerate morphisms.

Definition 10.3.2 (Page 243) A morphism $g : G \dashrightarrow G'$ between arguesian geometries is called **arguesian** if it satisfies the equivalent conditions of the preceding theorem.

Proposition 11.2.5 (Page 259) For any vector space V one has a natural isomorphism $\Theta_V : \mathcal{P}(V^*) \to (\mathcal{P}V)^*$.[15] It is induced by the map $\Omega : (V^*)^\bullet \to (\mathcal{P}V)^*, \varphi \mapsto \mathcal{P}(\ker \varphi)$.[16]

Definition 13.4.1(Page 310) A **dualized (projective) geometry** is a projective geometry G together with a subspace $\Gamma \subseteq G^*$ of the dual geometry satisfying $\bigcap \Gamma = \emptyset$.

Example 13.4.2 (Page 310) Let V be a **dualized vector space**, i.e. a vector space with a vector subspace $V' \subseteq V^*$ of the algebraic dual which is separating. This means that for every $x \neq 0$ there exists $l \in V'$ such that $l(x) = 1$. Then $\mathcal{P}(V)$ together with the subspace $\Theta_V(\mathcal{P}(V'))$ (cf. 11.2.5) is a dualized geometry.

[13] A geometry is a simple matroid.
[14] A map $f : V_1 \to V_2$ is additive, if $f(x + y) = f(x) + f(y)$ for any $x, y \in V_1$.
[15] For a vector space V, its algebraic dual V^* is the vector space of the linear functionals on V. For a projective geometry $(G, \star)$, its dual geometry is the ordered pair $(G^*, \star^*)$ such that G^* is the set of hyperplanes of $(G, \star)$ and, for any $A, B \in G^*$, $A \star^* B$ is the set of hyperplanes each of which includes $A \cap B$.
[16] For a vector space V, $V^\bullet$ denotes the set of non-zero vectors in V.

Proposition 13.5.3 (Page 316) Let V_1 and V_2 be dualized vector spaces. Then for every non-degenerate continuous homomorphism $g : \mathcal{P}(V_1) \dashrightarrow \mathcal{P}(V_2)$ there exists a continuous quasilinear map $f : V_1 \to V_2$ such that $g = \mathcal{P}f$.

Proposition 14.1.4 (Page 324) Let G be an orthogeometry. Then G together with the set $\Gamma := \{a^\perp \mid a \in G\}$ is a dualized geometry.

Definition 14.4.1 (Page 334) An **adjunction** between two orthogeometries G_1 and G_2 consists of two partial maps $g_1 : G_1 \dashrightarrow G_2$ and $g_2 : G_2 \dashrightarrow G_1$ satisfying

1) Ker $g_1 = (\text{Im } g_2)^\perp$,
2) Ker $g_2 = (\text{Im } g_1)^\perp$,
3) for all $a \in \text{Dom } g_1$ and $b \in \text{Dom } g_2$ one has $g_1 a \perp b$ iff $a \perp g_2 b$.

Proposition 14.4.4 (Page 335) For a partial map $g_1 : G_1 \dashrightarrow G_2$ between orthogeometries the following conditions are equivalent:

(1) there exists a partial map $g_2 : G_2 \dashrightarrow G_1$ such that (g_1, g_2) is an adjunction,
(2) g_1 is a continuous homomorphism.

Lemma 14.4.9 (Page 337) A quasilinear map $f : V_1 \to V_2$ is continuous if and only if there exists a quasilinear map $f^\circ : V_2 \to V_1$ such that $\Phi_2(fx, y) = \tau(\Phi_1(x, f^\circ y))$ for all $x \in V_1$ and $y \in V_2$. The map f° is unique and called the **adjoint** of f.

The following results with the notations are from [39].

Definition (Page 62) Let **P** be a projective space.[17] We say that in **P** the **theorem of Pappus** holds if any two intersecting lines g and h with $g \neq h$ satisfy the following condition. If A_1, A_2, A_3 are distinct points on g and B_1, B_2, B_3 are distinct points on h all different from $g \cap h$ then the points

$$Q_{12} := A_1B_2 \cap B_1A_2,\ Q_{23} := A_2B_3 \cap B_2A_3 \text{ and } Q_{31} := A_3B_1 \cap B_3A_1$$

lie on a common line.

Theorem 2.2.2 (Page 62) Let V be a vector space over a division ring F. Then the theorem of Pappus holds in $\mathbf{P}(V)$ if and only if F is commutative (in other words, if F is a field).

Theorem 2.2.3 Hessenberg's Theorem (Page 65) Let **P** be an arbitrary projective space. If the theorem of Pappus holds in **P** then the theorem of Desargues is also true in **P**.

Theorem 3.6.7 (Page 132) The projective collineations of $\mathbf{P}(V)$ are precisely the products of central collineations.

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[17] In [39] each projective space is assumed to have at least two lines (the sentences in a box on Page 7).

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