



On the Emergent Origin of the Inertial Mass

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Abstract

In the context of a particular framework of emergent quantum mechanics, it is argued the emergent origin of the inertial mass of a physical system. Two main consequences of the theory are discussed: an emergent interpretation of the law of inertia and a derivation of the energy-time uncertainty relation.

Keywords Emergent quantum mechanics · Emergence of mass · Emergent gravity

1 Introduction

The problem of finding a consistent picture unifying quantum mechanics and the gravitational interaction, usually in the form of a quantum theory of gravity, has been the goal of intense investigations and research in the field of theoretical physics during decades. The most common view is that the gravitational interaction must be quantized. The exact meaning of quantum gravity varies greatly from theory to theory, and one has several candidates for an unifying paradigms, among them string theory, loop quantum gravity, non-commutative spacetimes or causal set theory. Just the opposite approach, namely *relativize the quantum theory*, has been advocated by R. Penrose [1–3]. Other points of view towards a unifying theory explore the possibility that gravity is emergent, while other fields are of quantum nature. This is the case of Jacobson’s theory [4], Verlinde’s entropic gravity approach [5, 6] and

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Padmanabhan's approach to classical gravity as a thermodynamical system [7–9], three significant proposals for emergent gravity. Remarkably, these theories rely on quantum mechanical principles to the extent that a version of Unruh's formula relating temperature and acceleration and the weak equivalence principle must hold good.

On the other hand, emergent approaches to quantum mechanics have been the object of investigation by several authors from the point of view of an underlying fundamental theory [10–19]. Other approaches attack the emergent character of quantum theory from a thermodynamical point of view [20–23].

Within one of these emergent approaches to the foundations of quantum mechanics [14, 24], it is discussed in this paper how a given notion of mass can be identified with the inertial mass of the system. The fundamental idea of this approach is that quantum systems are emergent from an underlying level of physical reality, described by deterministic dynamical systems with many degrees of freedom, the dynamics being almost cyclic for free quantum systems. In this context, it is shown how the notion of inertial mass has an emergent character, directly associated with the sub-quantum degrees of freedom of the associated Hamilton–Randers system. In particular, a relation between mass and the semi-period of the fundamental cycle is presented. Furthermore, we show how a simple model for the semi-period implies a direct relation between the mass and the number of degrees of freedom N . Several consequences of the theory are explored. Among them, a new interpretation of the law of inertia of classical dynamics, linked with the notion of complexity and emergence, and a derivation of an energy-time uncertainty relation, that can be interpreted as a sharper relation than the standard energy-time uncertainty of quantum mechanics.

2 Notion of Hamilton–Randers Dynamical Systems

The fundamental notions of Hamilton–Randers theory can be found in [14]. Here we distill the necessary notions of the theory to developed the ideas of this paper. One starts by considering a \tilde{M} a smooth manifold. The state of a Hamilton–Randers system is described by a point of the co-tangent space $T^*\tilde{M}$. In particular, it is assumed a product structure for the configuration manifold of the form

$$\tilde{M} \cong \prod_{k=1}^N T M_4^k, \quad (2.1)$$

where all the manifolds M_4^k , $k = 1, \dots, N$ are diffeomorphic to each other. This construction is consistent with the assumption that all sub-quantum degrees of freedom are identical.

In Hamilton/Randers theory it is assumed that every quantum system can be described in terms of a Hamilton–Randers dynamical system. Indeed, Hamilton–Randers theory is based on generic properties that geometric models of Randers type metrics on tangent bundles T^*TM_k possess. A Randers space is a metric space where a Riemannian or pseudo-Riemannian norm function α has been perturbed by a linear term

β . The Randers metrics provide three fundamental elements: first, a notion of proper time and the corresponding geodesic flow, and second, a notion of measure, that is used below in taking the averaging operation. These two structures play relevant roles in the construction of the theory. Finally, it determines the exact form of the dynamics for the sub-quantum degrees of freedom, in the form of first order ordinary differential equations.

The *emergent degrees of freedom* are described by a dynamics denoted U_τ . The emergent degrees of freedom can be seen as a coarse grained average description of the system of N sub-quantum degrees of freedom of a Hamilton–Randers system. Such emergent degrees of freedom are associated with probability densities and it is indeed possible to show that such probability densities follow Born rule of quantum mechanics [14]. Therefore, the U_τ dynamics is associated with a quantum dynamics.

The notion of 2-time dynamics is of fundamental relevance for the construction of the emergence mechanism proposed in Hamilton–Randers theory. The consequences of adopting the 2-time dynamics and the effective description of the system in terms of the coarse grained description includes a mechanism for the reduction of the wave function [25] and the interpretation of entanglement and quantum non-locality [26], based upon the formal projection from the description in terms of the 2-time dynamics to the description in terms of the 1-time dynamics describing the dynamical evolution of the coarse grained degrees of freedom.

3 Interpretation of the Semi-period T for a Particular Class of t -Parameter

In Hamilton–Randers theory there is no geometric structure defined on the tangent space TM that can be used to determine a natural t -time parameter for the U_t dynamics. Indeed, the Hamilton–Randers dynamical models are invariant under positive oriented t -time re-parameterizations. However, the freedom in the choice of the t -parameter does not preclude the existence of choices of the t -parameter that are particularly enlightening. In particular, it can be useful to identify the semi-period T as a characteristic of the system it describes since, given a fixed t -time parameter, different systems can have different semi-periods. In this context, we postulate the existence of t -time parameters of the form $[0, 2T] \subset \mathbb{R}$ such that for any Hamilton–Randers system corresponding to a free quantum system, the relation

$$\log \left(\frac{T}{T_{min}} \right) = \frac{T_{min} m c^2}{\hbar} \tag{3.1}$$

holds good. T_{min} corresponds to the minimal period of the fundamental cycles for any Hamilton–Randers dynamical system. This minimal period must exist since, by assumption, the sub-quantum degrees of freedom are different from the quantum degrees of freedom. Since by assumption, quantum systems contain a finite number of them, then the limiting case when $N = 1$ imposes a limit on the complexity of the quantum system. This shows the need for the existence of a T_{min} if the period is directly linked to the number of sub-quantum degrees of freedom. On the grounds

of how quantum systems emerge from Hamilton–Randers theory, N_{min} is expected to be much larger than 1. The value of T_{min} depends on the choice of the arbitrary t -parameter but, once the parameter has been chosen, T_{min} achieves the same universal value for all the Hamilton–Randers systems.

One observes that the parameter T is a measure of the complexity of the system, since for simple systems one expects T smaller than for large systems. This is analogous to the interpretation of the parameter m . Later we will compare this interpretation with the usual interpretation of the inertial mass of a system.

4 The Parameter m as a Notion of Inertial Mass

It is useful to re-cast the relation (3.1) as a definition of the mass parameter m in terms of the semi-period T of the fundamental cycle. The mass parameter m of a Hamilton–Randers dynamical system with fundamental semi-period T is postulated to be given by the relation

$$m = \frac{\hbar}{T_{min} c^2} \log \left(\frac{T}{T_{min}} \right). \quad (4.1)$$

Fundamental properties of the m parameter. The following are properties of m that are obtained directly from the relation (4.1):

- (1) Since the semi-period T is an attribute of the physical system under consideration, the mass parameter m is also an attribute that increases with the *complexity of the system*.
- (2) For any Hamilton–Randers system, the mass parameter m is necessarily non-negative, with the minimum value for the parameter m being equal to $m = 0$, that corresponds to $T = T_{min}$.
- (3) As long as the period T is preserved for each fundamental cycle of the U_t dynamics, the mass parameter m is preserved and remains the same for each cycle of the U_t and the U_τ dynamics.
- (4) Since the value of the semi-period is linked to the ergodic properties of the system, T is a measure of the complexity of the system. Hence m is a measure of the complexity of the system as well.

The properties (1)–(3) make it reasonable to identify the parameter m given by the relation (4.1) as the mass parameter of the Hamilton–Randers system. From the above points the emergent origin of the mass parameter m given by the relation (4.1) as a measure of the complexity of the system is also direct.

The above interpretation of the parameter m is associated to the definition of this class of t -parameters. Indeed, given any positive t -parameter for the U_t dynamics, one could obviously define m by the relation (4.1), but in general, such a parameter m will not be associated with the Hamilton–Randers system, none of the properties (1)–(4) above will hold good.

4.1 Inertia as Emergent Phenomenon

Within the above context of associating the mass parameter m with the complexity of the system and the emergence of the notion of mass, it is reasonable to expect that coherent changes in the state of the system will be more difficult to be continuously kept as the complexity of the system increases. In the theory developed above, the mass parameter m as given by the relation (4.1) is related to the complexity of the fundamental cycles, which is related to the semi-period: the greater the complexity of the system, the larger the mass parameter m . Hence m is also becomes a measure of the opposition to coherent changes in the state of the system. Thus the tendency of an object is to *resist* a change in its state, and the resistance of change is greater when the mass m is larger. Therefore, the emergent nature of the inertia law and the identification of m with the inertial mass becomes apparent by the above argument.

5 Periods and Mass Parameter for Composite Hamilton–Randers Systems

Let us consider two arbitrary Hamilton–Randers systems a, b with semi-periods T_a and T_b . When the sub-systems a and b do not interact between each other, or if they interact, the effect of the interactions can be neglected, the *joint Hamilton–Randers system* $a \sqcup b$ describing the joint sub-systems a and b can be defined as follows. From the point of view of the quantum mechanical description, such physical systems are described by elements of the tensor product of the corresponding Hilbert spaces. From the point of view of Hamilton–Randers theory, if the sub-quantum molecules determining the quantum system a do not interact with the sub-quantum molecules defining the quantum system b and the corresponding structures are $(M_a, (\alpha_a, \beta_a))$ and $(M_b, (\alpha_b, \beta_b))$ respectively, then there is a natural Randers structure constructed from the structures a and b which is the product structure,

$$(M_a \times M_b, (\alpha_a \otimes \alpha_b, \beta_a \otimes \beta_b)).$$

This structure defines the Hamilton–Randers system $a \sqcup b$.

If the systems a and b do not interact and the corresponding semi-periods are T_a and T_b , then we further can assume that the semi-period for $a \sqcup b$ is given by the multiplicative rule

$$T_{a \sqcup b} := T_a T_b. \tag{5.1}$$

Note that this rule is not satisfied for a generic t -time parameter, since for the joint system $a \sqcup b$ one can choose an arbitrary t -time parameter to describe the U_t dynamics. Conversely, the condition (5.1) can be taken as the definition of non-interacting systems at the quantum mechanical level.

The above discussion shows two general features of the theory:

- The semi-period associated with a composite system should increase exponentially with the number of independent, non-interacting components of the system.
- From the relation (5.1) the mass M and the semi-period T given by the relation (3.1) and such that the t -parameters such that $T_{min} = 1$, it follows that the mass parameter of a composite non-interacting system $a \sqcup b$ is additive,

$$M_{a \sqcup b} = M_a + M_b. \quad (5.2)$$

5.1 Models for the Semi-periods of the Internal Dynamics

Although the U_t dynamics is manifestly t -time re-parametrization invariant (see [14]), the interesting features of the t -parameters for which the relations (3.1), (5.1) (5.2) hold good invite to consider the construction of specific models for a special class of t -parameters.

A fundamental characteristic of a Hamilton–Randers system is the number of sub-quantum degrees of freedom N . Also, we would like to point out that the dynamics U_t preserves N . Thus $N(t)$ is constant for any choice of the t -parameter. In a process where the system can be subdivided into two parts, the number N is also subdivided: if there is a process of the form $1 \rightarrow 2 \sqcup 3$, then we assume the condition

$$N_1 = N_2 + N_3 - N_{min}. \quad (5.3)$$

Thus the number of degrees of freedom of a composite system is the sum of the independent degrees of freedom; then N_{min} appears as the number of degrees of freedom of a common border and is subtracted in order to do not count twice such degrees of freedom.

In the following, we consider two models relating the semi-period T as function of the number of degrees of freedom N .

Model 1. Let us consider the following model for the semi-period,

$$\frac{T}{T_{min}} := \frac{T(N)}{T_{min}} = N^\epsilon, \quad (5.4)$$

where ϵ is a positive constant that does not depend upon the specific Hamilton–Randers system, nor on the number of sub-quantum degrees of freedom N_a . Then it is clear that T_a is a measure of the complexity of the system. The model (5.4) for the semi-period is consistent with the relation (3.1), but does not reproduce the multiplication rule (5.1) when the relation $N_1 = N_2 + N_3 - N_{min}$ holds good.

Model 2. A model that it is consistent with (5.3) is the exponential model

$$T = T_{min} \exp\left(\frac{\lambda T_{min} c^2}{\hbar}(N - N_{min})\right), \quad (5.5)$$

where λ is a constant relating the mass parameter m and the number of degrees of freedom:

$$m = \lambda(N - N_{min}). \tag{5.6}$$

In the model given by (5.5) N_{min} corresponds to T_{min} . Furthermore, when $T = T_{min}$, then $m = 0$ and in addition if $T_{min} = 1$, then the model (5.5) is such that $T_{a \cup b} = T_a \cdot T_b$. According to this model, the mass parameter m is equivalent to the number N of sub-quantum degrees of freedom, making therefore m a measure of the *quantity of sub-quantum matter*, in analogy of the notion of mass as amount of matter in classical mechanics. Furthermore, from the definition (4.1) of the mass parameter m and the relation (5.6), preservation of the number of lines N in the sense of the relation (5.3) implies additivity of the mass parameter m . These properties makes Model 2 a very appropriate model for the semi-period T .

6 Emergence of an Energy-Time Uncertainty Relation from Emergent Quantum Mechanics

The relation (3.1) is not equivalent to the quantum energy-time uncertainty relation, since the t -parameter is not an external time parameter. This is because the relation between m and T is given by $\log T$ rather than being linear with T^{-1} , as should be expected for an energy-time uncertainty quantum relation. Furthermore, according to the relation (3.1), m increases monotonically with T .

However, if we consider the variation of the parameter m due to a variation of the period $2T$ in the relation (3.1), we have that

$$\Delta(m c^2) = (\Delta m) c^2 = \frac{\hbar}{T_{min}} \frac{\Delta T}{T}.$$

The variation in the mass can be conceived as a variation due to a continuous interaction of the system with the environment; $\Delta T/T_{min}$ is the number of fundamental dynamical cycles that contribute to the stability of the quantum system. In a theory with maximal acceleration spacetime geometry, as we assume to be the case of the models (M_4^k, η, β) for each sub-quantum degree of freedom under consideration, the expression $m c^2$ is the energy of a system measured by an observer instantaneously co-moving with the system [14], when the system has zero proper acceleration. If there is an associated local coordinate system associated with the Hamilton–Randers system in a way which is in some sense instantaneously co-moving with the system, then we can apply the relativistic expression for the energy [27]. The value of the semi-period T is a characteristic of the quantum system associated in such a local coordinate frame. From the definition of T_{min} we have that $\Delta T/T_{min} \geq 1$. Then

$$\Delta E T \geq \hbar \tag{6.1}$$

with the *spread of energy* defined as

$$\Delta E := \Delta(m c^2) = \Delta(m) c^2. \tag{6.2}$$

Therefore, the *uncertainty* in the energy at rest E associated with the system is related to the inverse T^{-1} of the semi-period.

One natural interpretation of the energy-time relation (6.1) is to think the quantity ΔE as the minimal exchange of energy between the Hamilton–Randers system and the environment in such a way that the system is stable at least during a whole cycle of semi-period T . This energy exchange is measured in an instantaneous inertial reference frame in co-motion with the system just before the system changes to another different state or it decays to another different type of quantum system. Identifying the t -time parameter describing the U_t evolution of the sub-quantum degrees of freedom with a macroscopic coordinate time of a co-moving system with the quantum system and assuming that the average lifetime τ is much longer than T , one obtains the standard energy-time quantum uncertainty relation,

$$\Delta E \tau \geq \hbar. \quad (6.3)$$

This interpretation is fully consistent with T as a parameter associated with the U_t evolution and at the same time, an intrinsic parameter associated to the system.

7 Discussion

In this paper the relation between certain t -time parameters and the emergent origin of mass has been discussed in the context of Hamilton–Randers theory [14]. In the discussion, it is essential to consider the 2-time character of the dynamics. Dynamical systems where time is represented by two parameters exist in the literature. In classical dynamics, fast/slow variables are defined, just to split in a convenient way the dynamics of different classes of degrees of freedom. In such models, there is a diffeomorphism between the different time parameters [28], so indeed, there is only one time flow, but labelled by different parameters. More related to our theme, this type of fast/slow dynamics appears in the theory of emergent quantum mechanics according to 't Hooft [15, 16]. However, such a diffeomorphic relation between the t -time parameters and the τ -time parameters is absent in Hamilton–Randers theory, because the time parameters used in the quantum dynamics have an emergent character [14, 29] and are essentially discrete (although approximated by a continuum), while the t -parameters do not have such an emergent origin. Let us also mention that the notion of 2-dimensional time that we use is different from the 2-dimensional time that occurs in $2T$ -time physics models of I. Bars [30]. In Bars' theory, the 2-dimensional character of time is embedded in the signature of the 10-dimensional spacetimes of the underlying string theory models, while in Hamilton–Randers theory such a link with the geometry of the spacetime is in principle absent. This is because of the emergent character of τ -time parameters from the cyclic structure of the underlying U_t dynamics.

The relation between the expression (5.3) and the relation (4.1) leads to an interesting interpretation of massless systems. From one side, the relation (5.3) can be interpreted as associating N_{min} to the number of degrees of freedom in *borders*: when a system S_1 is joined with system S_2 , the degrees of freedom at the topological border of the joint region should be *counted* only once. Hence one needs to subtract the degrees of freedom at the joint region, namely N_{min} . On the other side, the number

of degrees of freedom corresponds to a massless quantum system, according to the relation (4.1) and the model of the semi-period (5.5). This implies that massless quantum particles should be interpreted as *borders* between *bulks* that corresponds to massive systems. Thus massive particles as regions with *non-empty interiors* of T^*TM . Also, that for all massless systems have associated the same number of sub-quantum degrees of freedom, namely N_{min} .

It is also very suggestive to compare the relation (4.1),

$$m = \frac{\hbar}{T_{min} c^2} \log \left(\frac{T}{T_{min}} \right),$$

with the expression of the entropy in terms of microstates,

$$S = k_B \log V,$$

where k_B is the Boltzmann constant and V is the phase space volume compatible with the *coarse grained* description of the system. To relate m and S given by the above expressions implies to relate the quantities T with V . Let us note that while the mass of a quantum system is to be expected to be preserved, if the system evolves freely, then the entropy is only preserved when it is maximal. Therefore, an entropic interpretation of mass based on the relation of T with V is subjected to this constraint. On the other hand, when thinking in terms of sub-quantum degrees of freedom, a dimensional argument suggests that

$$T \sim c \tilde{V}^{1/16}, \tag{7.1}$$

where \tilde{V} is the volume of the phase space of each sub-quantum degree of freedom. A maximum entropy state \tilde{S} corresponds to stable systems, for any Hamilton–Randers system. If (7.1) holds good and because \tilde{V} is a geometric invariant, then there is an emergent and general covariant definition of mass of the system,

$$\tilde{m} = \frac{1}{16} k_B N \log \frac{\tilde{V}}{\tilde{V}_{min}}, \tag{7.2}$$

where V_{min} corresponds to T_{min} . One is then tempted to re-define the entropy of the system in the form

$$\tilde{S} = k_B N \log \tilde{V},$$

providing an entropic character to the mass \tilde{m} . Note that \tilde{S} corresponds to the entropy of N independent sub-quantum degrees of freedom. However, the condition of independent (or non-interacting) sub-quantum degrees of freedom is indeed not adequate, due to the highly non-trivial U_t dynamics.

Finally, we should remark that, given the emergent nature of inertial mass and if the weak equivalence principle holds good, then the passive gravitational mass must be also emergent, providing another argument in favour of the emergent character of gravity, consistently with the general mechanism found in the context of Hamilton–Randers theory [14, 31].

8 Conclusion

An emergent concept of inertial mass has been discussed in the framework of Hamilton–Randers systems. The idea brings to light a new perspective on several fundamental notions, among them the law of inertia and the energy-time uncertainty relation. Furthermore, the emergent nature of inertial mass, together with the inertial mass/gravitational mass equality, implies an additional argument in favour of the emergent character of gravity. Indeed, the ideas described in this paper are part of a more ample program of research where fundamental notions of current physics are viewed as emergent notions from a fundamental theory beneath quantum theory [14, 24].

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