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Relativistic Hydrodynamic Interpretation of de Broglie Matter Waves

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Abstract
We present a classical hydrodynamic analog of free relativistic quantum particles inspired by de Broglie's pilot wave theory and recent developments in hydrodynamic quantum analogs. The proposed model couples a periodically forced Klein–Gordon equation with a nonrelativistic particle dynamics equation. The coupled equations may represent both quantum particles and classical particles driven by the gradients of locally excited Faraday waves. Exact stationary solutions of the coupled system reveal a highly nonlinear mechanism responsible for the self-propulsion of free particles, leading to the onset of unsteady motion. Although the model is essentially nonrelativistic, a stabilizing mechanism for any particle traveling close to the wave signaling speed emerges through the coupling with the wavefield. Consequently, inline particle oscillations comparable to de Broglie's wavelength are realized through this fully-classical model, suggesting a new classical interpretation for the motion of relativistic quantum particles.

Keywords Matter waves · De Broglie · Particle–wave interactions · Hydrodynamic analog · Relativistic dynamics

1 Introduction

In 1924, de Broglie proposed that particles may be associated with an intrinsic clock, oscillating at the Compton frequency. He envisaged a particle as an infinite yet localized field guided by a pilot wave while exchanging rest-mass energy with field energy [1]. Interactions between the particle generating the wave-field and, in turn, the wavefield guiding the particle constituted de Broglie's realistic picture of matter throughout his years of research [2]. De Broglie's double-solution pilot wave

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may therefore be regarded as an attempt to reconcile quantum mechanics and relativity using realistic wave mechanics.

In his original work de Broglie did not define a specific guiding wavefield; however, he did suggest several candidates for such waves, including the Klein–Gordon equation [3], Dirac equation, and others. The phase of the waves determined the guiding of particles by the pilot wave. De Broglie's phase waves were generally realized as monochromatic plane waves or at least quasi-monochromatic, as appeared in later notes [3], using optical geometry considerations.

Whether the pilot wave conceptualized by de Broglie describes a realistic picture of matter or not, there is reason to believe that it should constitute much more complex two-way coupled particle–wave interactions that were not realized by de Broglie. Similar nonlinear particle–wave interactions are frequently encountered in fluid dynamics; since the introduction of the Madelung transformation [4], multifold attempts have been made to realize quantum mechanics relying on fluid mechanics principles. De Broglie–Bohm theory [5], Nelson's theory [6], and Stochastic electrodynamics [7] raised the possibility that underlying physics—so-called 'hidden variables'—govern the dynamics of quantum particles giving rise to the quantum statistical signature, somewhat similar to Brownian motion. However, in contrast to Brownian motion, quantum statistics emerge from an unknown physical mechanism [8].

Deterministic hydrodynamic analogies of quantum mechanics were also recently developed. One of these most successful analogies was found by Couder and Fort, who experimentally observed millimetric oil droplets bouncing over a vibrating bath that remarkably feature the statistical behavior of many quantum mechanical systems [9–11]. In this hydrodynamic quantum analogy (HQA), droplets interact in resonance with a quasi-monochromatic wavefield they generate and exhibit a self-propelling mechanism. This analog has extended the range of classical physics to include many features previously thought to be exclusively quantum, including tunneling [12–15], Landau levels [16–18], the quantum harmonic oscillator [19, 20], the quantum corral [21–25], the quantum mirage [24], and Friedel oscillations [26].

For de Broglie's theory, and in particular the notion of *harmony of phases*, which associates the particle oscillations in any frame of reference, relativistic considerations are imperative. Despite the success of HQA in reproducing many quantum mechanical features, the relation between this analog and the relativistic aspect of de Broglie's theory remains elusive. It is, therefore, instructive to follow macroscopic analogies that involve similar wave mechanics from first principles that are described by deterministic classical equations. As we shall see, such similarities may inspire new relativistic interpretations of quantum mechanics based on fully classical hydrodynamic analogs. Notable recent studies deal with classical mechanics and fluid dynamics to interpret nonrelativistic and relativistic quantum mechanical behavior [27–31]. Jamet and Drezet considered classical models in which particles and waves are coupled through a holonomic constraint [32].

Recently, we developed a hydrodynamically-inspired quantum theory [33] (HQFT), a theoretical model of relativistic quantum dynamics inspired by de Broglie's pilot wave theory. In this framework, the particle is assumed as a localized yet infinite oscillating disturbance, externally forcing a Klein–Gordon wave equation. A relativistic dynamic equation couples the motion of the localized particle to the wave. Using this deterministic framework, several features of quantum mechanics are revealed. The most intriguing is probably the particle momentum in this analogy, which is associated with inline oscillations that correspond to the relation $p = \hbar k$, realized through interactions with the wave field. Notably, the particle speed modulations are averaged at the de Broglie wavelength and modulated by the relativistic frequency kc. Although the nonlinear system of free particles in HQFT is chaotic in nature, and inline oscillations may be characterized by multiple modes, de Broglie wavelength is most pronounced and may be realized as *quasi-monochromatic* modulation of the particle motion.

Excitation of motion and the waveform of HQFT at non-relativistic speeds were examined by Durey and Bush [20], who revealed the wave generation and self-propelling mechanism for the coupled wave-particle system, and provided a fundamental analytical validity to the subsequent work on the hydrodynamic field theory.

However, due to the highly nonlinear relativistic terms, no similar analysis has been performed so far for a particle at relativistic speeds. Although the pilot wave theory of de Broglie was somewhat put aside with the astounding success of the Copenhagen Interpretation, its wavelength formula has proved most successful. Note that the relativistic de Broglie's wavelength λ_B may be written in terms of the Compton wavelength, λ_c, the particle velocity, and the Lorentz boost factor, γ, as

$$\lambda_B/\lambda_c = \frac{h/\gamma mV}{h/mc} = \frac{1}{\gamma\beta} \, , \tag{1}$$

where h is the Planck constant, V is the particle speed, m its rest mass, c is the speed of light and $\beta = V/c$.

Any classical relativistic analogy should account for the behavior derived from de Broglie's wavelength. To further explore the extent to which such a theory may be realized as a viable interpretation of de Broglie's theory and relativistic quantum dynamics, a fully classical *non-relativistic* dynamic system is considered in the present work. This is in contrast to our previous study [33], where a relativistic equation of motion is introduced to properly satisfy a Lorentz covariant formulation. The current formulation allows the isolation of the role of classical wave mechanics in producing relativistic quantum signatures. Moreover, this model closely correlates to the hydrodynamic analog, which is essentially non-relativistic.

A simplified analytical model is derived here to reveal the fundamental interactions between a localized particle and waves.

2 Particle-Wave Model

We consider a Klein–Gordon wave equation, forced by a localized periodically oscillating disturbance,

$$\frac{\partial^2\psi}{\partial t^2} - c^2\frac{\partial^2\psi}{\partial x^2} + \omega_c^2\psi = f(t)\delta\left(x - x_p\right) \, , \tag{2}$$

where ψ represents a real one-dimensional wave, ω_c is the Compton frequency, $f(t)$ is a time-dependent function, which as in our previous study [33], will be realized here as a periodic function, with an angular frequency of $2\omega_c$.

The forcing term is localized about the particle location, x_p, by the Dirac delta function, δ. Particle dynamics are modeled as a classical non-relativistic equation of motion driven by wave gradients,

$$m\ddot{x}_p + D\dot{x}_p = -\alpha \left.\frac{\partial \psi}{\partial x}\right|_{x=x_p} , \tag{3}$$

where m is the particle mass, D is a drag coefficient, and the rightmost term represents the wave forcing term, where α is a coupling constant.

The set of Eqs. (2–3) represents the free particle motion driven by wave gradients, as appears in HQA. In both systems, the particle may lock into a quasi-steady motion about a constant average speed [33]. In both systems, in the limit of low particle mass, quantum statistics emerge. We, therefore, assume that the inertial term of Eq. (3) is negligible, and $m \to 0$. Hence, the particle speed in our model is dictated solely by local wave gradients similar to the model of relativistic HQFT [33].

3 Waveform Generated by a Steady Moving Particle

To analytically examine the waveform generated by a particle moving at an arbitrary constant speed, we introduce the following transformation of coordinates of Eq. (2),

$$\xi = \mu(x - Vt); \;\; \tau = \mu ct; \;\; \psi(x,t) = \varphi(\xi,\tau), \tag{4}$$

by which we can write the dimensionless wave equation in a frame of reference (ξ, τ) moving at the particle's speed $V = \dot{x}_p$:

$$\mu^2 c^2 \frac{\partial^2 \varphi}{\partial \tau^2} + \mu^2 (V^2 - c^2) \frac{\partial^2 \varphi}{\partial \xi^2} + \mu^2 c^2 \varphi - 2\mu^2 V c \frac{\partial^2 \varphi}{\partial \xi \partial \tau} = f(\tau)\delta(\xi) , \tag{5}$$

where $\mu = mc/\hbar$. Equation 5 is a function of two variables only, ξ and τ.

We may further simplify the equation by seeking stationary wave solutions in the particle frame of reference. Although much simplified, by strobing the system at the Compton frequency [33] and by strobing the hydrodynamic system [11] at the Faraday frequency, similar stationary solutions were found. Moreover, if the forcing term is realized as a periodic function at the Compton frequency, then it will only depend on the spatial coordinate when strobed. This is also the case for a droplet bouncing at the Faraday frequency in HQA. We shall, therefore, arbitrarily choose $f(\tau)$ to be constant and seek stationary solutions for the coupled system. Hence, after some manipulations, Eq. (5) takes the stationary form

$$\frac{\partial^2 \varphi}{\partial \xi^2} - \gamma^2 \varphi = -\gamma^2 \delta(\xi) , \tag{6}$$

which is a screened Poisson equation. Here, $\gamma = \frac{1}{\sqrt{(1-V^2/c^2)}}$ is a constant, markedly similar to the Lorentz factor for a particle moving at the constant speed V. It should be noted, however, that this Lorentz-like factor was not imposed in this study. This is key for our analogy since the Klein Gordon equation may also represent classical *non-relativistic* water waves. Here, however, the boost factor emerges as a result of the translation of the KG equation reference frame.

Solutions for the stationary wavefield and its gradient are of the form

$$\varphi(\xi) = c_1 e^{\gamma\xi} + c_2 e^{-\gamma\xi} \; ; \tag{7}$$

$$\frac{\partial \varphi}{\partial \xi} = c_1 \gamma e^{\gamma\xi} - c_2 \gamma e^{-\gamma\xi} \; , \tag{8}$$

where c_1 and c_2 are constants that can be determined at $\xi = 0$. We shall now set the wave field at $\xi = 0$ to be the constant φ_0 so that $\varphi(0) = c_1 + c_2 = \varphi_0$.

As in the hydrodynamic walker system in HQA [11], we assume that the particle motion is gradient driven. Hence, since our model is continuous, as opposed to HQA, we may also associate a wave gradient with a particle moving at a specific speed. As such, a wave-gradient corresponding to the velocity V is defined here as the velocity of the moving frame of reference, (ξ, τ), providing the boundary condition at $\xi = 0$,

$$V = -\alpha \frac{\partial \psi}{\partial x}_{x=x_p} = -\alpha\mu \frac{\partial \varphi}{\partial \xi}_{\xi=0} = -\alpha\mu (c_1 \gamma - c_2 \gamma) \; , \tag{9}$$

from which the coefficients,

$$c_1 = \frac{1}{2}\left(\varphi_0 - \frac{\beta}{\tilde{\alpha}\gamma}\right); \;\; c_2 = \frac{1}{2}\left(\varphi_0 + \frac{\beta}{\tilde{\alpha}\gamma}\right) , \tag{10}$$

are extracted. Thus the wavefield takes the form

$$\varphi(\xi) = \varphi_0 \cosh(\gamma\xi) - \frac{\beta}{\tilde{\alpha}\gamma} \sinh(\gamma\xi) \; . \tag{11}$$

Here, $\beta = \frac{V}{c}$ is assumed constant, and all other constants are absorbed into the new unknown coupling coefficient, $\tilde{\alpha}$. Note that in this formulation, for a certain particle speed β, we have two unknowns: the wave coupling constant, and the wave amplitude at $\xi = 0$.

The waveform generated by a localized particle as viewed from the particle's frame of reference is demonstrated in Fig. 1a, and its spatial gradient in Fig. 1b. For these parameters, we can observe different waveforms emerging for different particle speeds.

In the low speed limit, $\beta \to 0$, $\gamma \to 1$, and the first (hyperbolic cosine) term of Eq. (11) is dominant. Thus, the waveform at low speeds is symmetric about the particle location. When increasing the speed, the wave becomes asymmetric, where amplitudes decrease for positive values of ξ. With further increase of the particle

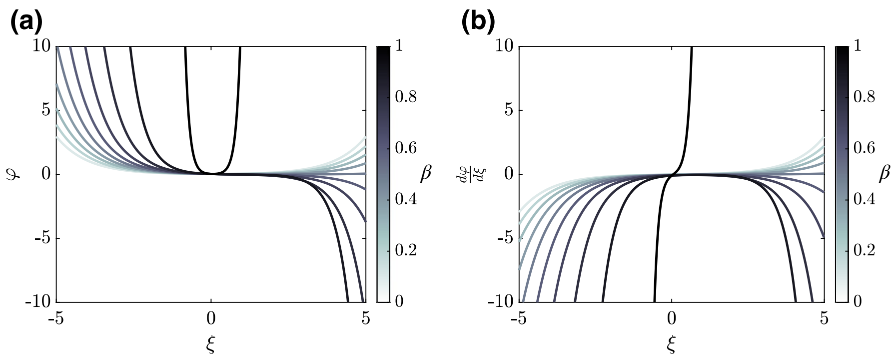

Fig. 1 Waveform (**a**) and gradient of the wave (**b**) at the particle frame of reference, color-coded by particle speed from $\beta = 0$ to $\beta = 0.99$; the coupling constant is set to $\alpha = 10$, and $\varphi_0 = 0.04$

speed, the wave amplitude at large positive values of ξ becomes negative due to the hyperbolic sine term of Eq. (11). However, further increasing β to velocities close to the speed of light, we observe a remarkable change of trend: as a jump in the wavefield appears, the wavefield becomes symmetric again, and its gradients increase significantly. When the particle speed increases enough, eventually, the two terms of Eq. (11) will be equal, which leads to a singular speed at which the wavefield spatially nullifies. The nonlinear interaction leads to a jump in the waveform, and its gradient, which can be clearly seen in Fig. 1b. This highly nonlinear behavior is due to the boost factor γ, which is dominant while the particle's speed is close to the speed of light.

4 Particle Dynamics

4.1 Quasi-Steady Particle Motion

We proceed by analyzing the particle response to the wavefield for a particle traveling at different speeds through the equation of motion (3). Though the assumption so far was a steady moving frame of reference at speed β, we may now extend our analysis to assess the response of a particle to a deflection about its steady trajectory. Denoting the deflection by δ_ξ, which corresponds to a velocity addition of $\dot{\delta}_\xi$, we may now write the equation of motion for a deflected particle in the *steady moving* frame of reference,

$$\beta + \dot{\delta}_\xi = -\tilde{\alpha} \left. \frac{\partial \varphi}{\partial \xi} \right|_{\xi=\delta_\xi} = -\tilde{\alpha}\varphi_0\gamma \sinh(\gamma\delta_\xi) + \beta \cosh(\gamma\delta_\xi), \tag{12}$$

using Eqs. (3) and (8).

Note that this analysis corresponds to the *low memory limit* in HQA, in which the particle is driven by wave gradients at the steady trajectory but is not affected by previously generated waves as it deflects from this trajectory.

In the limit of $\delta_\xi \to 0$, the RHS converges to β, and therefore $\dot{\delta}_\xi \to 0$ as expected, to satisfy the boundary conditions. The phase map diagram in Fig. 2 shows the particle added velocity $\dot{\delta}_\xi$ as a function of the deflection from the steady trajectory ($\xi = 0$). At $\beta = 0$, the particle responds to any deflection from the steady trajectory by an additional velocity in the opposite direction. This suggests a stable behavior. We may also identify this behavior in the hydrodynamic system, where droplets at low memory tend to show stationary stable bouncing.

However, increasing β, the deflection velocity reveals an unstable behavior as gradients become positive for positive values of δ_ξ, which suggests that any deflection of the particle will quickly increase its velocity. Further increasing β, gradients become steeper and slightly asymmetric - and the particle more unstable. Remarkably, close to the speed of light (for example, at $\beta = 0.99$ as shown in Fig. 2), the system is stabilized again due to the nonlinear function γ appearing in the first term of the RHS of Eq. (12). Under these conditions, the hyperbolic sine function is dominant, significantly changing the waveform. Gradients at such high speeds are extremely large, suggesting a quick decay of any perturbation about the steady trajectory; any deflection—positive or negative—will be responded to by a stabilizing deflection speed. Hence, at the limit of $\beta \to 1$, it is expected that the particle will travel exactly at the speed of light without any inline oscillations. Note that this analysis depends on the choice of $\tilde{\alpha}$ and φ_0. However, choosing other values for these coefficients does not seem to change the overall behavior of the system.

Hence, this simplified model demonstrates a complex *classical* dynamic behavior for any particle deflected from its steady trajectory: small deflections result in an increase in deflection speeds until the boost term is dominant.

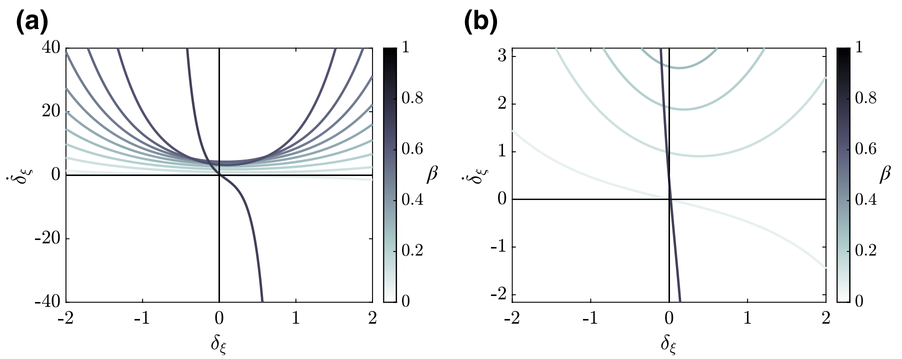

Fig. 2 **a** Phase diagram of a particle deflected from its steady trajectory at zero memory, color-coded by steady particle speeds from $\beta = 0$ to $\beta = 0.99$; the coupling constant is set to $\alpha = 10$, and $\varphi_0 = 0.04$. **b** a closeup view of the phase diagram showing low speeds and the maximal speed $\beta = 0.99$

This may provide a new, fully classical explanation for the origins of the onset of unstable inline oscillations in quantum mechanics [11, 33], and their decay at close to the speed of light.

4.2 Emergence of de Broglie Wavelength

We may now extend our discussion and interpret the results of Figs. 1 and 2 in terms of inline oscillations. As mentioned in Sect. 4.1, as the steady speed of the frame of reference increases, the wave gradients gradually change for relatively low speeds but are inverted drastically for speeds closer to the speed of light. This, in turn, should yield indefinite particle acceleration for low speeds and deceleration at higher speeds. A particle inline oscillation mechanism may thus be realized, for which the turning points would be determined by the deflection δ_ξ at which the added speed $\dot{\delta}_\xi$ is nullified. Mathematically, this condition is satisfied when the LHS of the equation (12) equals β, that is

$$\beta = -\tilde{\alpha}\varphi_0\gamma \sinh(\gamma\delta_\xi) + \beta\cosh(\gamma\delta_\xi)\,. \tag{13}$$

Two distinct solutions of Eq. 13 reveal a trivial solution $\delta_\xi = 0$ at the boundary $\xi = 0$, and a nontrivial solution of the equation

$$-\frac{\beta}{\tilde{\alpha}\gamma} = \frac{\sinh(\gamma\delta_\xi)}{1-\cosh(\gamma\delta_\xi)}\,, \tag{14}$$

which may be written explicitly as

$$\tilde{\delta}_\xi = \frac{2}{\gamma}\coth^{-1}\left(\frac{\beta}{\tilde{\alpha}\gamma}\right). \tag{15}$$

Here $\tilde{\delta}_\xi$ denotes the solution for a 'quasi-steady' deflection, about which we expect the particle to oscillate. It may therefore represent a characteristic length-scale for half the inline particle oscillations we observe in HQFT [33], where a relativistic covariant formulation is assumed. Figure 3 demonstrates how this length-scale, which is extracted here from Eq. (15), is closely related to de Broglie's relativistic wavelength $\frac{\lambda_B}{\lambda_c} = \frac{1}{\gamma\beta}$.

It should be noted that the choice of φ_0 is rather arbitrary, and since we currently do not have a proper physical interpretation for the coupling constant α or the initial wave amplitude φ_0, we set $\tilde{\alpha} = 0.02,\ 0.05$ and $\varphi_0 = 0$, without the loss of generality. However, smaller and larger values of $\tilde{\alpha}$ also seem to reasonably capture the overall trend.

At the limit $\beta \to 1$, $\gamma \to \infty$ and $\tilde{\delta}_\xi \to 0$, the model properly zeros any inline oscillation, as in de Broglie's relativistic theory. On the other hand, at the low-speed limit $\beta \to 0$, $\gamma \to 1$, a real solution for the nonlinear equation is not found. This result is expected since, at this limit of very low velocities, the hyperbolic cosine function of Eq. 11 dominates, and the solution is expected to be stable (see also zero β curve in Fig. 2b). This is in contrast to the runaway solutions predicted by de Broglie, i.e., infinite wavelengths for stationary particles.

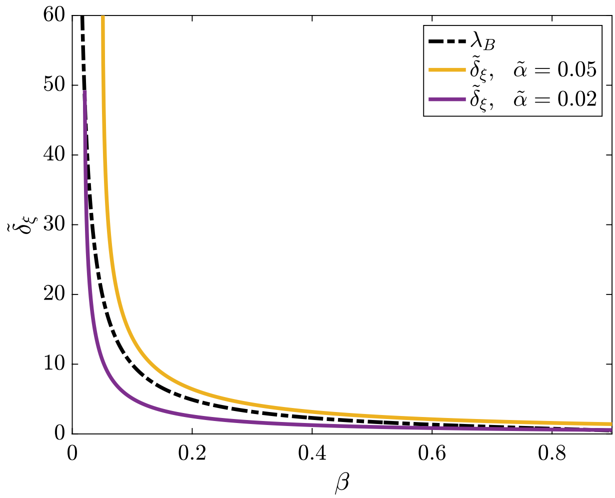

Fig. 3 Classical hydrodynamic interpretation of inline particle oscillations in solid lines showing similar trends as de Broglie's wavelength, $\lambda_B = \frac{1}{\gamma\beta}$ (dashed black line). Coupling constant arbitrarily chosen as $\tilde{\alpha} = 0.02, 0.05$ for comparison

5 Conclusions

A new classical hydrodynamic interpretation for relativistic particle dynamics is suggested. Following our recent study [33], a particle dynamic equation is coupled to a Klein–Gordon wave field. In contrast to our previous study, the particle equation of motion here is fully classical and non-relativistic. Thus, this system may represent classical water waves interacting with gradient-driven particles as in other recent hydrodynamic analogs [11].

By translating both the wave equation and the equation of motion at a constant speed, new exact analytical solutions are found for a waveform as it is observed from the particle frame of reference, assuming a stationary waveform. Notably, although no relativistic restrictions are imposed in this study, a Lorentz-like boost term appears through the translation of coordinates and interactions with the wavefield.

The particle response to a deflection from its steady motion is conceptualized using the new wave solutions, revealing highly nonlinear dynamics and unsteady characteristics at low to intermediate particle speeds. Any deflection at these conditions results in acceleration in the direction of the deflection. However, at high enough speeds, due to the highly nonlinear boost term, this trend is inverted, and at speeds close to the speed of light, a stabilizing mechanism is revealed as a concequence of the boost factor.

This classical dynamic realization suggests a more general oscillatory mechanism for any particle, driven by nonlinear particle–wave interactions yet limited to subluminal speeds. Although simplified, spatial quantum statistics comparable to de Broglie wavelength may also be interpreted from our analytical framework.

Thus, it may suggest a simple deterministic model for a long-sought ‘hidden variable’ mechanism to reconcile relativity and quantum dynamics.

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