Foundations of Physics (2023) 53:6
https://doi.org/10.1007/s10701-022-00635-0

Beyond the Born Rule in Quantum Gravity

Antony Valentini[1,2]

Received: 23 September 2022 / Accepted: 28 September 2022 / Published online: 26 November 2022

Abstract

We have recently developed a new understanding of probability in quantum gravity. In this paper we provide an overview of this new approach and its implications. Adopting the de Broglie–Bohm pilot-wave formulation of quantum physics, we argue that there is no Born rule at the fundamental level of quantum gravity with a non-normalisable Wheeler–DeWitt wave functional Ψ. Instead the universe is in a perpetual state of quantum nonequilibrium with a probability density $P \neq |\Psi|^2$. Dynamical relaxation to the Born rule can occur only after the early universe has emerged into a semiclassical or Schrödinger approximation, with a time-dependent and normalisable wave functional ψ, for non-gravitational systems on a classical spacetime background. In that regime the probability density ρ can relax towards $|\psi|^2$ (on a coarse-grained level). Thus the pilot-wave theory of gravitation supports the hypothesis of primordial quantum nonequilibrium, with relaxation to the Born rule taking place soon after the big bang. We also show that quantum-gravitational corrections to the Schrödinger approximation allow quantum nonequilibrium $\rho \neq |\psi|^2$ to be created from a prior equilibrium ($\rho = |\psi|^2$) state. Such effects are very tiny and difficult to observe in practice.

Keywords de Broglie–Bohm theory · Pilot-wave theory · Born rule · Quantum nonequilibrium · Quantum gravity

1 Introduction

It has long been known that the pilot-wave theory of de Broglie and Bohm provides us with an objective and deterministic account of quantum physics [1–7]. Historically, the theory was constructed by de Broglie in a series of papers from 1922 to

✉ Antony Valentini
antonyv@clemson.edu

1 Augustus College, 14 Augustus Road, London SW19 6LN, UK

2 Department of Physics and Astronomy, Kinard Laboratory, Clemson University, Clemson, SC 29634-0978, USA

1927, culminating in a pilot-wave dynamics for a many-body system.[1] The theory was revived by Bohm in 1952, who extended the dynamics to field theory and, crucially, showed in detail how the theory accounts for the general quantum theory of measurement [3, 4]. Despite this success, there remains a long-standing controversy concerning the status of the Born probability rule in this theory. In recent work we have argued that the status of the Born rule in pilot-wave theory changes radically when we consider a regime in which quantum-gravitational effects are important [8]. In this paper we provide an overview of these new ideas and results, with a minimum of technicalities, and with an emphasis on the conceptual implications.

In pilot-wave theory a system with configuration-space wave function $\psi(q,t)$ has an actual trajectory $q(t)$ whose velocity $v = dq/dt$ is determined by de Broglie's law of motion or 'guidance equation', where for systems with conventional Hamiltonians v is proportional to the gradient $\partial_q S$ of the phase S of $\psi = |\psi|e^{iS}$. For an ensemble of systems with the same wave function $\psi(q,t)$, the ensemble distribution $\rho(q,t)$ of configurations is usually assumed to be given by the Born rule

$$\rho = |\psi|^2. \tag{1}$$

It is a simple consequence of the equations of motion that if (1) holds at an initial time $t = 0$ it will hold for all t. On these grounds in the 1920s de Broglie simply took (1) as an assumption with no further explanation or justification [2]. This stance was however criticised by Pauli and by Keller, in 1953, who argued that such an initial condition was unjustified in a deterministic theory and should be derived from the dynamics [9, 10]. This criticism was partially met by Bohm in the same year when he argued that an ensemble of two-level molecules would relax to the state (1) when subjected to random collisions [11]. However, no general argument for relaxation was given. In 1954, citing difficulties with understanding relaxation to the Born rule, Bohm and Vigier abandoned the original deterministic theory and introduced random (subquantum) 'fluid fluctuations' that drive relaxation to the Born rule for a general system [12]. Since then, most authors have simply adopted de Broglie's original position, with (1) in effect taken as an additional postulate (alongside the Schrödinger equation for ψ and de Broglie's law for v) [5, 13–15].

This author has long argued that simply postulating (1) is a mistake, akin to artificially restricting classical mechanics to a state of thermal equilibrium [16–26]. In pilot-wave theory the Born rule (1) really describes a state of statistical equilibrium, or 'quantum equilibrium', analogous for example to the Maxwell distribution of molecular speeds for a gas in thermal equilibrium. Just as a classical ensemble can be in thermal nonequilibrium, with a distribution of velocities different from that of Maxwell, in pilot-wave theory an ensemble can be in quantum nonequilibrium with a distribution of configurations

$$\rho \neq |\psi|^2 \tag{2}$$

[1] By de Broglie's own account his ideas originated in a paper of 1922 on blackbody radiation, although his first paper on pilot-wave theory proper did not appear until 1923, culminating in his theory of a many-body system presented at the 1927 Solvay conference (see Ref. [2, Chap. 2]).

different from that of Born. For such an ensemble the statistical predictions of textbook quantum mechanics would fail—raising the question of why such nonequilibrium phenomena have never been observed in the laboratory. The answer, at least as proposed by this author in 1991, is that all the systems we have access to have a long and violent history that traces back ultimately to the big bang. During that time there has been ample opportunity for dynamical relaxation $\rho \to |\psi|^2$ to take place (on a coarse-grained level)—a process of 'quantum relaxation' that is broadly analogous to classical thermal relaxation, and which presumably occurred in the early universe. This process has been studied in general terms and has been observed to take place efficiently in a wide range of numerical simulations [16–18, 20, 23, 26–34] . We can then understand why the Born rule holds to high accuracy today. At the same time, we understand that pilot-wave theory also contains a wider nonequilibrium physics, which may have been active in the early universe, and which could have left discernible traces today—in the form of anomalies in the cosmic microwave background (CMB), as well as in relic cosmological particles that might today still display violations of the Born rule [20, 35–43].

On this view, quantum physics is merely a special case of a much wider physics in which the Born rule is broken. That wider physics allows violations of the uncertainty principle as well as practical nonlocal signalling [17, 21, 22, 24]. According to pilot-wave theory, at least when correctly interpreted, textbook quantum mechanics is merely an effective theory that emerges in the state of quantum equilibrium. Many supposedly fundamental quantum constraints are really peculiarities of equilibrium and are broken for more general ensembles.

An alternative view has, however, long been championed by the 'Bohmian mechanics' school of de Broglie–Bohm theory—a distinctive approach to the subject first proposed by Dürr et al. in 1992 [13–15]. In this approach, the wave function Ψ of the whole universe is used to define a (supposedly) fundamental probability (or 'typicality') measure $|\Psi|^2$, from which one can readily derive the Born rule (1) for subsystems with an effective wave function ψ. On this view the Born rule is built into the theory and there is no prospect of ever finding nonequilibrium violations, not even in the early universe. While this approach has been influential among some philosophers [44, 45], the argument is essentially circular: the Born rule is derived for subsystems only by assuming the Born rule for the whole universe at the initial time $t = 0$. There is no reason why our universe should have started with those particular initial conditions—whether or not the universe began in equilibrium or nonequilibrium is ultimately an empirical question to be decided by observation and experiment, not by theoretical or philosophical fiat [19, 20, 26].

As we will see in this paper, the controversy over the Born rule in pilot-wave theory changes drastically when we consider a regime where quantum gravity is important. For in that regime there simply is no normalisable physical probability (or typicality) measure $|\Psi|^2$ for the whole universe, and the (circular) argument employed by the Bohmian mechanics school can no longer even be formulated. Instead, normalisable wave functions ψ emerge only in the semiclassical regime—for systems evolving on a classical spacetime background—and in that regime the Born rule (1) can emerge by a dynamical process of quantum relaxation [8]. In this way, considerations from quantum gravity vindicate the hypothesis of quantum nonequilibrium at the big bang, with relaxation to the Born rule taking place only afterwards.

Before presenting technical details of how all this works, let us first sketch the key ideas in simple terms. In canonical quantum gravity the geometry of 3-space is described by a metric tensor g_{ij}.[2] If we include a matter field ϕ we might expect the system to have a wave function (or functional) $\Psi[g_{ij}, \phi, t]$ obeying a time-dependent Schrödinger equation $i\partial\Psi/\partial t = \hat{H}\Psi$ (with an appropriate Hamiltonian $\hat{H}$ and time parameter t). Instead, when we quantise the gravitational field we obtain a wave functional $\Psi[g_{ij}, \phi]$ obeying a time-independent Wheeler–DeWitt equation [47, 48]

$$\hat{\mathcal{H}}\Psi = 0 \tag{3}$$

(with an appropriate Hamiltonian density operator $\hat{\mathcal{H}}$).[3] Time makes no appearance in the equations. After more than half a century since it was first written down, the physical interpretation of this 'timeless' theory remains controversial.

Most workers in the field agree that a time-dependent Schrödinger equation $i\partial\psi/\partial t = \hat{H}\psi$ for a conventional wave functional $\psi[\phi, t]$ can emerge only in a semi-classical regime for a quantum field ϕ propagating on a classical background space-time. There is, however, controversy over precisely how an effective time parameter t can emerge from a fundamentally timeless theory. This question is known in the quantum gravity literature as the 'problem of time' [49–55].

Quantum-gravitational effects are expected to be significant at sufficiently early times in our cosmological history (certainly within a Planck time $t_P \sim 10^{-43}$ *sec* after the beginning). In such a deep quantum-gravity regime, the Wheeler–DeWitt equation (3) must be applied. Soon afterwards we expect the universe to emerge into a semiclassical or 'Schrödinger' regime, in which a conventional time-dependent wave equation can be applied. Previous discussion of the Born rule and of quantum relaxation in pilot-wave theory has taken place within the semiclassical or Schrödinger approximation. Outside that approximation, however, the discussion must be carefully revised.

A pilot-wave theory of quantum gravity can be written down by supplementing the Wheeler–DeWitt equation (3) with de Broglie–Bohm trajectories $g_{ij}(t)$ whose velocity $\partial g_{ij}/\partial t$ is proportional to a generalised phase gradient (see Sect. 5) [56–60].[4] We might then expect $\left|\Psi[g_{ij}, \phi]\right|^2$ to define an equilibrium Born-rule probability density [56, 62]. But, as we will see, this cannot be correct. The mathematical structure of (3) ensures that the density $\left|\Psi[g_{ij}, \phi]\right|^2$ cannot be normalised and so cannot be a physical probability distribution. This point has caused controversy and confusion in the literature. In our view, from a pilot-wave perspective, the implication is clear: in quantum gravity there simply is no physical Born-rule equilibrium state [8, 63]. A physical probability density $P[g_{ij}, \phi, t]$ (for a

[2] In this paper we employ the traditional metric representation of the gravitational field. We expect similar conclusions to hold in loop quantum gravity [46].

[3] As we will see there is also a constraint on Ψ guaranteeing coordinate invariance.

[4] For a recent review see Ref. [61].

theoretical ensemble) must be normalisable by definition. Therefore it must differ from $\left|\Psi[g_{ij},\phi]\right|^2$ at all times. We may say that the deep quantum-gravity regime is in a perpetual state of quantum nonequilibrium

$$P \neq |\Psi|^2. \tag{4}$$

In this regime there can be no quantum relaxation and no state of quantum equilibrium.

As we will see there are two immediate implications. First, as the early universe emerges from the deep quantum-gravity regime and settles into a semiclassical regime described by the Schrödinger approximation, we can expect fields ϕ propagating on the classical background to be in a state of quantum nonequilibrium $\rho[\phi,t] \neq |\psi[\phi,t]|^2$—where ψ is the effective (normalisable) Schrödinger wave functional for ϕ and the probability ρ emerges from P as a conditional probability. Second, quantum relaxation as previously understood can begin to take place only after the universe has settled into a conventional Schrödinger regime. Thus, even though there is no fundamental Born-rule equilibrium state in quantum gravity, we still recover quantum relaxation in the Schrödinger approximation—and so we can still explain the Born rule as we see it today.

There is another remarkable result of this analysis. It is well known that the emergent Schrödinger equation is subject to small quantum-gravitational corrections appearing in the effective Hamiltonian $\hat{H}$. Perhaps surprisingly, some of the correction terms are non-Hermitian [64–68]. Such terms are of course inconsistent with standard quantum mechanics, since the norm of ψ is no longer conserved and $|\psi|^2$ cannot be interpreted as a probability density in the usual way. For this reason, in previous studies such terms have been dropped, with no clear justification. In pilot-wave theory, in contrast, there is no inconsistency: such terms simply generate a gravitational instability of the Born rule, whereby an initial density $\rho = |\psi|^2$ can evolve into a final density $\rho \neq |\psi|^2$. As we will see, such effects are extremely small, but observable at least in principle. This means that, when quantum gravity is taken into account, it is no longer the case that once quantum equilibrium is reached we are trapped in that state forever. There is a way out, at least in principle.

To summarise, in this paper we have three new ideas to present:

1. In quantum gravity there is no Born rule for a timeless Wheeler–DeWitt wave functional Ψ and the system is in a perpetual state of quantum nonequilibrium $P \neq |\Psi|^2$.
2. The Born rule $\rho = |\psi|^2$ can emerge by quantum relaxation only in a semiclassical or Schrödinger approximation, for systems with an effective time-dependent wave function ψ on a classical background spacetime.
3. Tiny quantum-gravitational corrections to the Schrödinger approximation can make the Born rule unstable, with initial distributions $\rho = |\psi|^2$ evolving to final distributions $\rho \neq |\psi|^2$.

To develop the details, we begin with a brief outline of some essential formalism.

2 Quantum Gravity and Quantum Cosmology

In this section we provide a brief summary of the essential formalism of canonical quantum gravity [47, 48], together with a simple model of quantum cosmology.

2.1 Canonical Quantum Gravity

The canonical quantisation of the gravitational field begins with a '3+1' foliation of classical spacetime by spacelike slices $\Sigma(t)$ labelled by a time parameter t. This can always be done (generally nonuniquely) for a spacetime that is 'globally-hyperbolic'. The line element then takes the form

$$d\tau^2 = (N^2 - N_i N^i)dt^2 - 2N_i dx^i dt - g_{ij} dx^i dx^j, \tag{5}$$

where N, N^i are respectively the 'lapse function' and 'shift vector', while g_{ij} is the 3-metric on $\Sigma(t)$. For simplicity we can take $N^i = 0$—so that lines of constant x^i are normal to the slices—provided such lines do not encounter singularities. The object to be quantised is then a spatial 3-geometry represented by g_{ij}.

Beginning with the usual Einstein–Hilbert action, standard quantisation methods lead to the Wheeler–DeWitt equation, which for the pure gravitational field reads[5]

$$\left(-G_{ijkl}\frac{\delta^2}{\delta g_{ij}\delta g_{kl}} - \sqrt{g}R\right)\Psi = 0, \tag{6}$$

where $\Psi = \Psi[g_{ij}]$ and

$$G_{ijkl} = \frac{1}{2}g^{-1/2}(g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl}). \tag{7}$$

We have written the kinetic term with a specific operator ordering but it should be understood that the ordering is ambiguous. The wave functional is also subject to a constraint

$$-2D_j\frac{\delta\Psi}{\delta g_{ij}} = 0 \tag{8}$$

associated with spatial coordinate invariance (where D_j is a spatial covariant derivative). This constraint ensures that Ψ is a function of the coordinate-independent 3-geometry and not of the coordinate-dependent 3-metric. Writing (6) and (8) as

$$\hat{\mathcal{H}}\Psi = 0, \quad \hat{\mathcal{H}}^i\Psi = 0,$$

the total Hamiltonian operator is given by

[5] For a 'functional' $\Psi[\phi]$ (mapping a function $\phi(x)$ to a complex number Ψ) the functional derivative $\delta\Psi/\delta\phi(x)$ at a spatial point x is defined by $\delta\Psi = \int d^3x \left[\delta\Psi/\delta\phi(x)\right]\delta\phi(x)$ for arbitrary infinitesimal variations $\delta\phi(x)$.

$$\hat{H} = \int d^3x \, (N\hat{\mathcal{H}} + N_i\hat{\mathcal{H}}^i). \tag{9}$$

In the presence of a scalar matter field ϕ with potential $\mathcal{V}(\phi)$ we have an extended Wheeler–DeWitt equation

$$(\hat{\mathcal{H}}_g + \hat{\mathcal{H}}_\phi)\Psi = 0 \tag{10}$$

for the wave functional $\Psi = \Psi[g_{ij}, \phi]$, where the gravitational term

$$\hat{\mathcal{H}}_g = -G_{ijkl}\frac{\delta^2}{\delta g_{ij}\delta g_{kl}} - \sqrt{g}R \tag{11}$$

in $\hat{\mathcal{H}} = \hat{\mathcal{H}}_g + \hat{\mathcal{H}}_\phi$ is supplemented by a matter term

$$\hat{\mathcal{H}}_\phi = \frac{1}{2}\sqrt{g}\left(-\frac{1}{g}\frac{\delta^2}{\delta\phi^2} + g^{ij}\partial_i\phi\partial_j\phi\right) + \sqrt{g}\mathcal{V}, \tag{12}$$

while the constraint $\hat{\mathcal{H}}^i\Psi = 0$ corresponding to (8) takes the form

$$-2D_j\frac{\delta\Psi}{\delta g_{ij}} + \partial^i\phi\frac{\delta\Psi}{\delta\phi} = 0\,. \tag{13}$$

2.2 A Simple Model of Quantum Cosmology

It will be helpful to illustrate our ideas with a simple model of quantum cosmology.

Consider an expanding flat and homogeneous universe with scale factor $a(t)$ and spacetime line element[6]

$$d\tau^2 = dt^2 - a^2dx^2. \tag{14}$$

We assume that the universe contains a homogeneous matter field ϕ with a potential $\mathcal{V}(\phi)$. We then have a 'mini-superspace' model with two degrees of freedom (a, ϕ).

This system has a Lagrangian [67–69]

$$L = -\frac{1}{2}m_{\mathrm{P}}^2a\dot{a}^2 + \frac{1}{2}a^3\dot{\phi}^2 - a^3\mathcal{V}, \tag{15}$$

where $m_{\mathrm{P}}^2 = 3/4\pi G$ is the square of a (rescaled) Planck mass. This implies canonical momenta

$$p_a = -m_{\mathrm{P}}^2a\dot{a}, \quad p_\phi = a^3\dot{\phi} \tag{16}$$

and a Hamiltonian

6 Here $dx^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$.

$$H = -\frac{1}{2m_{\mathrm{P}}^2}\frac{1}{a}p_a^2 + \frac{1}{2a^3}p_\phi^2 + a^3\mathcal{V} \tag{17}$$

(noting the sign difference between the kinetic terms).

Promoting the canonical momenta to operators $\hat{p}_a = -i\partial/\partial a$ and $\hat{p}_\phi = -i\partial/\partial\phi$, and choosing the factor ordering $\frac{1}{a^2}\hat{p}_a a \hat{p}_a$, the Wheeler–DeWitt equation $\hat{H}\Psi = 0$ for $\Psi(a, \phi)$ reads [67, 68]

$$\frac{1}{2m_{\mathrm{P}}^2}\frac{1}{a}\frac{\partial}{\partial a}\left(a\frac{\partial\Psi}{\partial a}\right) - \frac{1}{2a^2}\frac{\partial^2\Psi}{\partial\phi^2} + a^4\mathcal{V}\Psi = 0. \tag{18}$$

3 Difficulties with the Born Rule in Quantum Gravity

In this section we discuss some of the key difficulties with trying to interpret $|\Psi|^2$ as a probability density for a Wheeler–DeWitt wave funtional Ψ.

3.1 Why $|\Psi|^2$ is Non-normalisable

We have said that, for solutions Ψ of the Wheeler–DeWitt equation, the quantity $|\Psi|^2$ cannot be a physical probability density because it is non-normalisable. To see why $|\Psi|^2$ cannot be normalised, consider for simplicity the case of pure gravitation. The Wheeler–DeWitt equation (6) for the wave functional $\Psi[g_{ij}]$ on the space of 3-metrics g_{ij} is mathematically analogous to the single-particle Klein–Gordon equation

$$-\frac{\partial^2\psi}{\partial t^2} + \delta^{ij}\frac{\partial^2\psi}{\partial x^i\partial x^j} - m^2\psi = 0 \tag{19}$$

for a wave $\psi(x, t)$ on Minkowski spacetime. The analogy can be traced to the indefinite character of the 'DeWitt metric' G_{ijkl} [47]. This means that (6) is formally analogous to an infinite-dimensional Klein–Gordon equation with a 'mass-squared' term $g^{1/2}R$. As a result, the integral $\int Dg\, \left|\Psi[g_{ij}]\right|^2$ over the whole space of 3-metrics necessarily diverges, just as the integral $\int d^3x \int dt\, |\psi(x, t)|^2$ over the whole of spacetime necessarily diverges. It might be thought that the divergence could be removed by an appropriate regularisation. But the divergence is deeper than that, reflecting a basic fact about wave propagation. Solutions $\psi(x, t)$ of the wave equation (19) can be localised with respect to x but not with respect to t, and mathematically the same phenomenon occurs for solutions $\Psi[g_{ij}]$ of the wave equation (6).

What we have just said is slightly simplified. We have not mentioned the constraint (8), which ensures that $\Psi[g_{ij}]$ is not really a function on the space of coordinate-dependent 3-metrics g_{ij} but in fact a function on the space of coordinate-independent 3-geometries (a space commonly referred to as 'superspace'). It might then be thought that the non-normalisability of $\left|\Psi[g_{ij}]\right|^2$ could just be an artifact of having to integrate over an infinite 'gauge volume' of 3-metrics representing the same

3-geometry. But in fact the result still diverges even if we perform a physical integral over the space of 3-geometries (perhaps by factoring out the gauge volume in some way).

The simplest way to see this is to consider our mini-superspace model of quantum cosmology (Sect. 2.2). Each spacelike slice of constant t has a simple coordinate-independent representation as a flat Euclidean 3-space with scale factor $a(t)$. If we rewrite the Wheeler–DeWitt equation (18) for $\Psi(a, \phi)$ in terms of $\alpha = \ln a$ we find

$$\frac{1}{m_{\mathrm{P}}^2}\frac{\partial^2\Psi}{\partial\alpha^2} - \frac{\partial^2\Psi}{\partial\phi^2} + 2e^{6\alpha}\mathcal{V}\Psi = 0. \tag{20}$$

This is a two-dimensional Klein–Gordon equation with a potential term. The free part (ignoring the potential) has the general solution

$$\Psi_{\mathrm{free}} = f(\phi - m_{\mathrm{P}}\alpha) + g(\phi + m_{\mathrm{P}}\alpha), \tag{21}$$

where f and g are packets travelling with the ‘wave speed’ $c = m_{\mathrm{P}}$ in the two-dimensional ‘spacetime’ (α, ϕ). Thus

$$\int\int d\alpha d\phi\ |\Psi_{\mathrm{free}}|^2 = \infty, \tag{22}$$

just as for a Klein–Gordon solution $\psi(x, t)$ we have

$$\int d^3x \int dt\ |\psi(x, t)|^2 = \infty. \tag{23}$$

Clearly the non-normalisability of Ψ has nothing to do with the (technically delicate) issue of unphysical coordinate degrees of freedom. It is simply a consequence of the Klein–Gordon-like character of the Wheeler–DeWitt equation and the resulting wave-like propagation in the mini-superspace (α, ϕ). Similar conclusions must hold in the full theory.

3.2 Naive Schrödinger Interpretation. I

It is in fact well known in quantum-gravity circles that $|\Psi|^2$ cannot be a physical probability density [49–54]. While interpreted as such by Hawking and collaborators in the 1980s [70–73], this came to be known as the ‘naive Schrödinger interpretation’ [49]. Even leaving aside the question of non-normalisability, the interpretation is problematic because the putative probability density $|\Psi|^2$ is time-independent. Attempts were made to repair this by some form of conditioning on a subset of the degrees of freedom, but this ‘conditional probability interpretation’ led to other problems [51, 52]. In any case, in our view the interpretation fails from the outset simply because a non-normalisable density cannot represent a physical probability distribution.

Historically, however, it has been more common to cite another reason for the failure of the naive interpretation of $|\Psi|^2$. It is claimed that treating $|\Psi|^2$ as a conventional probability is incorrect because 'time' is in effect hidden in the metric degrees of freedom g_{ij}. For example, in quantum cosmology with a wave function $\Psi(a, \phi)$, it is commonplace to treat the scale factor a as an effective time parameter. We can then try to recover a Born-rule-type probability for the remaining degrees of freedom at a given value of 'time'. This approach has a long history, dating back to the pioneering work of DeWitt and Wheeler in the 1960s [47, 74], but to this day it remains controversial. Some authors have raised concerns about the *bona fide* temporal properties of gravitational degrees of freedom [49]. For example, if the scale factor a plays the role of time, what happens to time in a universe that expands and recontracts? On the other hand, some supporters of quantum gravity argue that at the deepest level physics is genuinely timeless, and that our common-sense notions of 'time' emerge only approximately and in certain conditions [47, 75–82]. It is however not entirely clear whether quantum mechanics can be properly applied in a fundamentally timeless theory [46, 76, 80–84]. A relatively recent and exhaustive review of the 'problem of time' in quantum gravity runs to nearly a thousand pages and draws no definite conclusions [54], suggesting that the problem has yet to be satisfactorily resolved (though some may disagree).

In this paper we offer a new explanation for the failure of the naive Schrödinger interpretation. Our explanation is that, in the deep quantum-gravity regime, there is no such thing as the Born rule. As we shall see, we can discuss (time-dependent) probability densities such as $P[g_{ij}, \phi, t]$, but these are not tied to the Born rule and can never be. Necessarily, $P[g_{ij}, \phi, t] \neq \left|\Psi[g_{ij}, \phi]\right|^2$ always, since the left-hand side is normalisable (by definition) and the right-hand side is not. We may say that, at the Planck scale, a quantum-gravitational universe is in a perpetual state of quantum nonequilibrium. This, in our view, is the true physical significance of the non-normalisability of the Wheeler–DeWitt wave functional Ψ.

To make sense of this idea, however, we need to look more closely at pilot-wave theory.

4 Pilot-Wave Theory and the Born Rule

When interpreted correctly, pilot-wave theory shows us that the Born rule is not an axiom or law, but instead represents a statistical state of quantum equilibrium (analogous to classical thermal equilibrium) [16–26]. At least, that is the case in non-gravitational physics. As we shall see, in pilot-wave gravitation, in contrast, there is no state of quantum equilibrium and no possibility of obtaining the Born rule, except in the semiclassical regime.

Let us first consider pilot-wave theory for a general system with configuration q and wave function $\psi(q, t)$ on a background classical spacetime with global time parameter t (corresponding to a foliation by spacelike slices $\Sigma(t)$). Here q could represent particle or field configurations on the spacelike slice at time t. The wave function obeys a time-dependent Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = \hat{H}\psi \tag{24}$$

with some Hamiltonian operator $\hat{H}$. This implies a continuity equation for $|\psi|^2$,

$$\frac{\partial|\psi|^2}{\partial t} + \partial_q \cdot j = 0, \tag{25}$$

where ∂_q is a gradient on configuration space and j satisfies

$$\partial_q \cdot j = 2\mathrm{Re}\big(i\psi^*\hat{H}\psi\big). \tag{26}$$

The 'current' j can be written in terms of ψ and its functional form depends on $\hat{H}$ [85].[7] Given the expression $j = j[\psi] = j(q,t)$ we can define a velocity field

$$v(q,t) = \frac{j(q,t)}{|\psi(q,t)|^2} \tag{27}$$

and write down a de Broglie guidance equation

$$\frac{dq}{dt} = v(q,t) \tag{28}$$

for the trajectory $q(t)$. Equations (24), (27) and (28) define a deterministic dynamics for a general system with wave function $\psi(q,t)$.[8] Note that ψ is regarded as a physical field (or 'pilot wave') on configuration space that guides the trajectory $q(t)$ of a single system. At the fundamental dynamical level there is no such thing as probability (as in classical mechanics).

If $\hat{H}$ happens to be quadratic in the momenta, we find that v is proportional to a phase gradient. For example, for a single low-energy particle of mass m we find

$$\mathbf{v} = \frac{1}{m}\mathrm{Im}\frac{\boldsymbol{\nabla}\psi}{\psi} = \frac{1}{m}\boldsymbol{\nabla}S, \tag{29}$$

where $\psi = |\psi|e^{iS}$, while for a many-body system the nth particle has velocity $\mathbf{v}_n = (1/m_n)\boldsymbol{\nabla}_n S$.

We can now consider an ensemble of systems with the same wave function ψ. The systems evolve according to the velocity field v. By construction, then, the distribution $\rho(q,t)$ of configurations evolves by the continuity equation

$$\frac{\partial\rho}{\partial t} + \partial_q \cdot (\rho v) = 0. \tag{30}$$

This matches the continuity equation (25) for $|\psi|^2$. It follows immediately that if $\rho = |\psi|^2$ initially then $\rho = |\psi|^2$ at later times. An ensemble obeying the Born rule is

[7] The continuity equation (25) can also be derived from Noether's theorem as the local conservation law associated with a global phase symmetry $\psi \to \psi e^{i\theta}$ on configuration space [85].

[8] We are assuming the wave function has a single component ψ. The method can be readily extended to spin systems with multi-component wave functions [6].

in a state of 'quantum equilibrium'. For such ensembles we recover the usual statistical predictions of textbook quantum mechanics [3–6].

There is, however, no reason of principle why we could not begin with a 'quantum nonequilibrium' ensemble with $\rho \neq |\psi|^2$ [16–18]. What happens then? In general we will find violations of the usual statistical predictions. For example, for single particles incident on a two-slit screen with incoming wave function ψ, an incident ensemble with $\rho \neq |\psi|^2$ will yield an anomalous distribution $\rho(\mathbf{x}, t) \neq |\psi(\mathbf{x}, t)|^2$ at the backstop, breaking the usual interference pattern. Similarly, atomic transitions will have non-standard probabilities, and so on. And yet, such nonequilibrium phenomena have never been observed in the laboratory. Why not?

4.1 Quantum Relaxation

In pilot-wave theory there is a straightforward answer. At some time in the remote past there took place a process of 'quantum relaxation'—by which we mean the time evolution of ρ towards $|\psi|^2$ (on a coarse-grained level). Quantum equilibrium was already reached, at least to a very good approximation, long before any of our experiments were carried out. That is why we see the Born rule today [16–20, 23–26]. Bearing in mind the long and violent astrophysical and cosmological history of all known systems, quantum relaxation probably occurred in the very early universe.

Quantum relaxation can be understood, by analogy with thermal relaxation for an isolated classical system, in terms of the decrease of a coarse-grained H-function [16, 18]

$$\bar{H}(t) = \int dq\, \bar{\rho} \ln(\bar{\rho}/\overline{|\psi|^2}), \tag{31}$$

where the overbars indicate coarse-graining over small cells in configuration space. This quantity is equal to minus the relative entropy of $\bar{\rho}$ with respect to $\overline{|\psi|^2}$. It is bounded below by zero, $\bar{H} \geq 0$, and $\bar{H} = 0$ if and only if $\bar{\rho} = \overline{|\psi|^2}$. If we begin at $t = 0$ with $\bar{\rho} \neq \overline{|\psi|^2}$ then $\bar{H}(0) > 0$. As the ensemble relaxes towards equilibrium, $\bar{H}(t) \to 0$ and $\bar{\rho} \to \overline{|\psi|^2}$. This relaxing behaviour has been demonstrated in a wide variety of numerical simulations, yielding an approximately exponential decay [23, 28, 30]

$$\bar{H}(t) \approx \bar{H}(0)e^{-t/\tau} \tag{32}$$

on a timescale τ that is (very roughly) comparable to the quantum timescale $\Delta t = \hbar/\Delta E$ (though τ also depends on the coarse-graining length) [28]. Moreover the quantity (31) obeys a general coarse-graining H-theorem [16, 18]

$$\bar{H}(t) \leq \bar{H}(0), \tag{33}$$

assuming no initial fine-grained structure in ρ and $|\psi|^2$ at $t = 0$. Closer analysis shows that $\bar{H}(t)$ strictly decreases when ρ develops fine-grained structure—as tends to happen for velocity fields that vary over the coarse-graining cells [18, 26].

We have said that quantum relaxation probably took place in the early universe, soon after the big bang. This idea is potentially testable. According to inflationary cosmology, primordial quantum fluctuations in a scalar inflaton field were the ultimate source of primordial inhomogeneities, which later grew by gravitational clumping to form large-scale structure, as well as seeding the small temperature anisotropies we see today in the cosmic microwave background (CMB) [86–88]. This means that the statistical properties of the CMB sky ultimately depend on the Born rule for quantum field fluctuations in the very early universe. If the Born rule was broken at sufficiently early times, this could show up as anomalies in the CMB today [35, 37]. Careful analysis shows that on expanding space quantum relaxation is suppressed for long-wavelength (super-Hubble) field modes, suggesting that a pre-inflationary era will end with a power deficit at long wavelengths, which could then carry over to an inflationary phase yielding a large-scale power deficit in the CMB [36, 38]. This scenario has been studied numerically, with some simplifying assumptions [39, 41]. A large-scale power deficit has in fact been reported in the CMB data [89], though its status remains controversial. Fitting the data to a quantum relaxation model has yielded some tantalising results, but the data are too noisy for clear conclusions to be drawn [37, 90].

4.2 Trapped Forever in Quantum Death?

According to pilot-wave theory, at least when correctly interpreted, we are currently trapped in a state of ‘quantum death’ that is broadly analogous to the state of classical thermodynamic ‘heat death’ which was much discussed in the nineteenth century as a seemingly inevitable future end state of our world (in which all systems have reached the same temperature and it is no longer possible to convert heat into work) [6, 16–18]. According to pilot-wave theory, a subquantum analogue of the classical heat death *has already happened* (and a long time ago). In this state, all systems are subject to the same quantum noise, as described by the Born rule—just as, in the classical heat death, all systems are subject to the same thermal noise (at the same global temperature). In the state of quantum death, the uncertainty principle prevents us from observing and controlling the underlying details of de Broglie–Bohm trajectories. As a consequence, we are unable to control the underlying nonlocal dynamics of entangled systems, and in particular we are unable to employ entanglement for nonlocal signalling. But these limitations are not fundamental, they are merely peculiarities of the state of quantum death (just as the inability to convert heat into work is a peculiarity of the state of heat death). Locality emerges at the statistical level only if the Born rule holds exactly. As has been shown explicitly, nonequilibrium entangled systems generally allow instantaneous signalling [17, 21, 22]—which can be understood as defining a preferred foliation of spacetime with a global time parameter t [91].

Here we are interested specifically in the status of the Born rule. Clearly, in pilot-wave theory, the Born rule is not a law of nature, but holds only because we are confined to a certain state of statistical equilibrium. Moreover we seem to be forever trapped in this state, which appears to be *stable*. The equations of motion of pilot-wave theory guarantee that once quantum equilibrium is reached there is no way out (barring extremely rare fluctuations [18], analogous to putting a kettle of water on ice and waiting for the water to boil). There is one caveat, however. Our discussion applies to quantum systems on a classical spacetime background. What happens if spacetime itself is quantised? To answer that, we must turn to the pilot-wave theory of gravity.

5 Pilot-Wave Gravitation

The pilot-wave theory of gravity appears to have been first written down and studied for a mini-superspace model by Vink [56] and in general terms by Horiguchi [57]. It has since been developed, and extensively applied to cosmology, in particular by Pinto-Neto and collaborators [59–61].

Beginning for simplicity with the case of pure gravitation, the Wheeler–DeWitt equation (6) for the wave functional $\Psi[g_{ij}]$ is supplemented by a de Broglie guidance equation for the time evolution $g_{ij}(t)$ of the 3-metric,

$$\frac{\partial g_{ij}}{\partial t} = 2NG_{ijkl}\frac{\delta S}{\delta g_{kl}}, \tag{34}$$

where S is the phase of Ψ and for simplicity we have set $N_i = 0$.[9] The equation of motion (34) can be justified in two ways. We might simply identify the classical canonical momentum density p^{ij} (conjugate to g_{ij}) with the phase gradient $\delta S/\delta g_{ij}$ and then use the well-known classical relation between p^{ij} and $\dot{g}_{ij}$ to yield (34). Alternatively, (34) can be justified as the natural velocity field appearing in the equation

$$\frac{\delta}{\delta g_{ij}}\left(|\Psi|^2 G_{ijkl}\frac{\delta S}{\delta g_{kl}}\right) = 0, \tag{35}$$

which follows from the Wheeler–DeWitt equation (6) (with an appropriate choice of operator ordering,[10] inserting $\Psi = |\Psi|e^{iS}$ and taking the imaginary part), and which can be rewritten as

$$\frac{\delta}{\delta g_{ij}}\left(|\Psi|^2\frac{\partial g_{ij}}{\partial t}\right) = 0 \tag{36}$$

with $\dot{g}_{ij}$ given by (34). Equations (34) and (6), together with the constraint (8), are taken to define the dynamics of a single system.

These equations are readily extended to include a matter field ϕ. The Wheeler–DeWitt equation (10) for $\Psi[g_{ij}, \phi]$ then includes a matter term (12) and Ψ is subject to the constraint (13). We still have the same guidance equation (34) for g_{ij} and in addition a guidance equation

[9] For $N_i \neq 0$ there are additional terms $D_i N_j + D_j N_i$ on the right-hand side of (34).

[10] Specifically, with the ordering $(\delta/\delta g_{ij})G_{ijkl}(\delta/\delta g_{kl})$ in the kinetic term.

$$\frac{\partial \phi}{\partial t} = \frac{N}{\sqrt{g}} \frac{\delta S}{\delta \phi} \tag{37}$$

for ϕ (again taking $N_i = 0$).[11] As before the guidance equations can be justified by identifying the classical canonical momenta with a phase gradient, or by identifying the natural velocity fields appearing in the equation

$$\frac{\delta}{\delta g_{ij}} \left(|\Psi|^2 \frac{\partial g_{ij}}{\partial t} \right) + \frac{\delta}{\delta \phi} \left(|\Psi|^2 \frac{\partial \phi}{\partial t} \right) = 0 \tag{38}$$

(which now follows from the extended Wheeler–DeWitt equation (10)).

Before proceeding we should point out that, in the above dynamics, the status of the spacetime foliation is perhaps not fully understood. The arbitrary functions N, N^i should not affect the 4-geometry that is traced out by the evolving 3-metric (for given initial conditions). Shtanov [58] argued that the 4-geometry depends on N, suggesting that the theory breaks foliation invariance. In that case one might include a specific choice for N as part of the theory. On the other hand, work by Pinto-Neto and Santini [92] suggests that the 4-geometry is in fact independent of N, N^i. By writing the dynamics as a Hamiltonian system, it is argued that the time evolution of an initial 3-geometry yields the same 4-geometry for all N, N^i—with the caveat that the 4-geometry is non-Lorentzian. Local Lorentz invariance is broken (as expected in a nonlocal theory). If this argument is correct it seems to imply that, for a given solution Ψ and for a given initial 3-geometry, the resulting spacetime has an effective preferred foliation. Intuitively, this seems consistent with the first-order (or 'Aristotelian') structure of pilot-wave dynamics [93]. And, as already noted, for nonequilibrium ensembles of entangled systems we obtain statistical nonlocal signals [17, 21, 22], which arguably also define a preferred foliation of spacetime [91]. It would be of interest to study these matters in more detail.

The above dynamics has been applied extensively by numerous authors, in particular to quantum cosmology. Such applications have focussed on properties of the trajectories (such as singularity avoidance) without attempting to construct a theory of a quantum equilibrium ensemble [59–61]. In fact previous workers have avoided discussing ensembles, owing to the pathological (non-normalisable) nature of the density $|\Psi|^2$. By a curious twist, we then find ourselves in a position opposite to that of textbook quantum mechanics: we have a theory of single systems with trajectories, but no theory of ensembles or of probabilities. It has been suggested that this is understandable because (as argued by the Bohmian mechanics school [13, 14]) the notion of probability is (supposedly) meaningless for a single universe [61]. And yet theoretical cosmologists routinely discuss probabilities for primordial cosmological perturbations, and observational cosmologists employ measurements of the CMB to constrain the primordial power spectrum. In practice, by assuming statistical isotropy and statistical homogeneity for a theoretical ensemble, we can and do discuss probabilities for our universe and constrain them by observation [26]. How, then, can we proceed with a theory of probability in pilot-wave gravitation?

[11] For $N_i \neq 0$ there is an additional term $N^i \partial_i \phi$ on the right-hand side of (37).

5.1 Gravity Without the Born Rule

Our suggested answer is to accept that at the fundamental level there is no such thing as the Born rule [8]. An arbitrary theoretical ensemble with the same Wheeler–DeWitt wave functional $\Psi[g_{ij}, \phi]$ will have an arbitrary initial probability distribution $P[g_{ij}, \phi, t_i]$ at time t_i. By definition $P[g_{ij}, \phi, t_i]$ will be normalisable and so cannot be equal to $\left|\Psi[g_{ij}, \phi]\right|^2$ under any circumstances. The initial distribution $P[g_{ij}, \phi, t_i]$ is a contingency unconstrained by any law but which can, at least in principle, be constrained by empirical observation. Furthermore, we can straightforwardly study its time evolution. Each element of the ensemble evolves by the de Broglie velocity field (34) and (37), and so $P[g_{ij}, \phi, t]$ necessarily evolves by the continuity equation[12]

$$\frac{\partial P}{\partial t} + \int d^3x \, \frac{\delta}{\delta g_{ij}} \left(P \frac{\partial g_{ij}}{\partial t} \right) + \int d^3x \, \frac{\delta}{\delta \phi} \left(P \frac{\partial \phi}{\partial t} \right) = 0. \tag{39}$$

We then have a theory for a general ensemble of gravitational systems evolving in time. One of our key claims is that this theory has no Born-rule equilibrium state [8, 63]. At the deepest level of gravitational physics, the universe is in a perpetual state of quantum nonequilibrium

$$P[g_{ij}, \phi, t] \neq \left|\Psi[g_{ij}, \phi]\right|^2. \tag{40}$$

In Sect. 6 we shall illustrate these ideas with our simple model of quantum cosmology.

5.2 Naive Schrödinger Interpretation. II

Some workers instead interpret $|\Psi|^2$ as a probability density (as first suggested by Vink [56] and recently advocated by Dürr and Struyve [62]). This amounts to applying the naive Schrödinger interpretation to pilot-wave gravitation. However, while the presence of trajectories adds a new element, the interpretation remains unworkable because $|\Psi|^2$ is non-normalisable.

It might be thought that equation (38) can be employed to motivate $|\Psi|^2$ is an equilibrium probability density. But (38) is not a continuity equation but an infinity of equations (one per spatial point x). Following Dürr and Struyve [62], if we integrate (38) over x we can write down what we call a 'pseudo-continuity equation'[13]

[12] We might also append a constraint of the form (13) on P, to ensure that it is a function on the space of coordinate-independent 3-geometries. We can avoid this complication by simply working with one representation of the 3-geometry by one (coordinate-dependent) metric g_{ij}.

[13] For a single particle with a static density ρ and a current $\mathbf{j}$, this is analogous to summing the equations $\partial_x j_x = \partial_y j_y = \partial_z j_z = 0$ to yield $\partial \rho / \partial t + \nabla \cdot \mathbf{j} = 0$.

$$\frac{\partial|\Psi|^2}{\partial t}+\int d^3x\,\frac{\delta}{\delta g_{ij}}\left(|\Psi|^2\frac{\partial g_{ij}}{\partial t}\right)+\int d^3x\,\frac{\delta}{\delta\phi}\left(|\Psi|^2\frac{\partial\phi}{\partial t}\right)=0 \tag{41}$$

(where for completeness we have inserted the vanishing term $\partial|\Psi|^2/\partial t=0$). This is formally the same as the physical continuity equation (39) for P. We might then 'deduce' that $P=|\Psi|^2$ is an equilibrium state, as usually done for non-gravitational systems. On this basis Dürr and Struyve claim that $|\Psi|^2$ can be employed as a quantum equilibrium measure of 'typicality' for initial configurations of the universe (from which follows the Born rule for subsystems, along lines already advocated by the Bohmian mechanics school). But this novel application of the naive Schrödinger interpretation again founders on the fact that $|\Psi|^2$ is not normalisable and cannot define a physical probability (or typicality) measure.

We should be wary of artificial attempts to make $|\Psi|^2$ appear like a conventional density. In our view, to interpret $|\Psi|^2$ as a Born-rule measure is a category mistake. At the fundamental level there is no Born rule. As we will see the usual Born-rule measure emerges only in the Schrödinger approximation (on a classical spacetime background).

6 Pilot-Wave Cosmology

Recall our quantum-cosmological model with degrees of freedom (a,ϕ) and wave function $\Psi(a,\phi)$ satisfying the Wheeler–DeWitt equation (18). Inserting $\Psi=|\Psi|e^{iS}$ and taking the imaginary part yields what we call a 'pseudo-continuity equation'

$$\frac{\partial}{\partial a}\left(a^2|\Psi|^2\dot{a}\right)+\frac{\partial}{\partial\phi}\left(a^2|\Psi|^2\dot{\phi}\right)=0 \tag{42}$$

for a density $a^2|\Psi|^2$ and with a velocity field

$$\dot{a}=-\frac{1}{m_{\mathrm{P}}^2}\frac{1}{a}\frac{\partial S}{\partial a},\qquad \dot{\phi}=\frac{1}{a^3}\frac{\partial S}{\partial\phi}. \tag{43}$$

We can identify (43) as the natural de Broglie guidance equations for this system.[14]

A general theoretical ensemble of systems with the same wave function Ψ will have a probability distribution $P(a,\phi,t)$ (with density defined with respect to $dad\phi$). Since each element of the ensemble evolves according to the velocity field (43), the distribution $P(a,\phi,t)$ necessarily evolves according to the continuity equation

$$\frac{\partial P}{\partial t}+\frac{\partial}{\partial a}(P\dot{a})+\frac{\partial}{\partial\phi}\left(P\dot{\phi}\right)=0 \tag{44}$$

(with $\dot{a}$, $\dot{\phi}$ given by (43)). We then have a theory for a general ensemble of cosmological systems evolving in time.

[14] The same equations follow by identifying the canonical momenta (16) with a phase gradient.

6.1 Failure of the Born Rule

At this point we must be careful. In standard quantum theory equation (42) would be regarded as a continuity equation for a Born-rule-like probability density $a^2|\Psi|^2$. The unusual factor a^2 arises simply from the structure of our minisuperspace and if preferred could be eliminated by defining a density with respect to $a^2 dad\phi$ instead of $dad\phi$. In any case, equation (42) seemingly suggests that the quantum probability to find the system in a range $dad\phi$ is given by $a^2|\Psi|^2 dad\phi$. This is the naive Schrödinger interpretation in a quantum-cosmological setting. As we saw in Sect. 3.2 this interpretation fails not only because the putative probability density $a^2|\Psi|^2$ has no explicit time dependence but also because it is non-normalisable.

In pilot-wave theory it might be argued that, because the respective evolution equations (44) and (42) for P and $a^2|\Psi|^2$ are identical (noting that $\partial(a^2|\Psi|^2)/\partial t = 0$), if $P = a^2|\Psi|^2$ initially then $P = a^2|\Psi|^2$ at later times and we may identify this as a state of quantum equilibrium. This is again the naive Schrödinger interpretation applied to pilot-wave theory (Sect. 5.2), and again it is as untenable in pilot-wave theory as it is in standard quantum theory. For a general solution Ψ of the wave equation (18) we inevitably have

$$\int\int dad\phi \, a^2|\Psi(a,\phi)|^2 = \infty. \tag{45}$$

The equality $P = a^2|\Psi|^2$ is mathematically nonsensical since the left-hand side is (by definition) normalisable while the right-hand side is not.

To understand this theory we need to accept that in the deep quantum-gravity regime there is *no* Born-rule-like equilibrium state. There simply is no Born rule and no state of quantum equilibrium. In the context of our quantum-cosmological model, we must have

$$P \neq a^2|\Psi|^2 \tag{46}$$

always (both initially and at later times).

6.2 Impossibility of Quantum Relaxation

In terms of quantum relaxation, for this system the coarse-grained H-function

$$\bar{H}(t) = \int\int dad\phi \, \bar{P} \ln(\bar{P}/\overline{a^2|\Psi|^2}) \tag{47}$$

(minus the relative entropy of $\bar{P}$ with respect to $\overline{a^2|\Psi|^2}$) still obeys a coarse-graining H-theorem (33) but now has no lower bound. If we had

$$\int\int dad\phi \, \overline{a^2|\Psi|^2} = N \tag{48}$$

for some finite N, it is easy to show that (47) would be bounded below by $-\ln N$, with the lower bound attained if and only if $\bar{P} = \frac{1}{N}a^2|\Psi|^2$ [8]. For $N \to \infty$ there is no lower bound. The function $\bar{H}(t)$ can decrease indefinitely without ever reaching a minimum. In this sense $\bar{P}$ is always infinitely far away from the putative 'equilibrium' state $a^2|\Psi|^2$. Limited local relaxation might take place in some regions of configuration space, but to attain global equilibrium is mathematically impossible. Similar reasoning applies to the full gravitational theory [8].

7 Quantum Relaxation in the Schrödinger Approximation

We have argued that, in the deep quantum-gravity regime, quantum relaxation cannot take place because there is no physical equilibrium state to relax to. In effect a quantum-gravitational system is perpetually in nonequilibrium. How are we then to understand the ubiquity of the Born rule in our world today?

Our proposed answer is that quantum relaxation can take place in the semiclassical regime, where the system propagates on an approximately classical spacetime background. In this 'Schrödinger approximation' we have an effective time-dependent Schrödinger equation (24) for a conventional wave function $\psi(q,t)$, where q might represent for example a field configuration ϕ on a background classical curved space and t is the time function associated with a preferred foliation. Since ψ is now normalisable, $|\psi|^2$ can correspond to a physical probability distribution, which is attainable after appropriate relaxation.

To see how this works, we need to outline how the Schrödinger approximation is derived from the underlying quantum-gravitational theory (details are given in Sect. 9.1). Consider again the deep quantum-gravity regime with a matter field ϕ and a Wheeler–DeWitt wave functional $\Psi[g_{ij},\phi]$. We obtain an effective time-dependent wave function ψ, on an approximately classical spacetime background, when the solution $\Psi[g_{ij},\phi]$ of the Wheeler–DeWitt equation takes the approximate form

$$\Psi[g_{ij},\phi] \approx \Psi_{\mathrm{WKB}}[g_{ij}]\psi[\phi,g_{ij}], \tag{49}$$

where $\Psi_{\mathrm{WKB}}[g_{ij}]$ is a WKB wave functional for the 3-metric. The phase $S_{\mathrm{WKB}} = \mathrm{Im}\ln\Psi_{\mathrm{WKB}}$ satisfies a classical Hamilton-Jacobi equation and generates classical trajectories $g_{ij} = g_{ij}(t)$ for the background. If we evaluate $\psi[\phi,g_{ij}]$ along a specific trajectory $g_{ij}(t)$, we can define an effective time-dependent wave functional

$$\psi_{\mathrm{eff}}[\phi,t] = \psi[\phi,g_{ij}(t)] \tag{50}$$

for the matter field ϕ, with a time derivative

$$\frac{\partial}{\partial t} = \int d^3x\, \dot{g}_{ij}\frac{\delta}{\delta g_{ij}}. \tag{51}$$

It can then be shown that ψ_{eff} satisfies an approximate time-dependent Schrödinger equation

$$i\frac{\partial \psi_{\rm eff}}{\partial t} = \hat{H}_{\rm eff}\psi_{\rm eff} \tag{52}$$

for the field ϕ on the classical background, where $\hat{H}_{\rm eff}$ is an effective Hamiltonian (which of course depends on the background).

This method of deriving the Schrödinger approximation has a long history. The WKB trajectories for the classical background allow us to define an effective time parameter t, which historically has often been called 'WKB time' [94]. The origin of such trajectories is unclear in standard quantum mechanics, where they are really being inserted by hand. In pilot-wave theory, in contrast, the WKB trajectories are simply de Broglie–Bohm trajectories evaluated in the WKB approximation, and so the above construction is conceptually clear.

We can now return to the question of quantum relaxation and the Born rule. Once the very early universe enters the semiclassical or Schrödinger regime, fields and particles propagating on the (approximate) classical background will satisfy a time-dependent Schrödinger equation of the form (52), with a conventional and normalisable wave function $\psi_{\rm eff}$. As we will see in Sect. 9.1, the de Broglie guidance equation also takes the standard form (in terms of $\psi_{\rm eff}$). We then find ourselves in the domain which has already been much studied in pilot-wave theory as briefly summarised in Sect. 4.1. If at the beginning of the Schrödinger regime we have a nonequilibrium probability distribution $\rho \neq \left|\psi_{\rm eff}\right|^2$, then in appropriate circumstances quantum relaxation will ensure that $\rho \to \left|\psi_{\rm eff}\right|^2$ on a coarse-grained level, at least to a good approximation and in particular for short-wavelength (sub-Hubble) field modes. In this way, despite the complete absence of a Born rule in the deep quantum-gravity regime, we can nevertheless understand the emergence of the Born rule in the semiclassical or Schrödinger approximation, at scales relevant to laboratory physics, after appropriate quantum relaxation.

We have said that, once we have an approximate time-dependent Schrödinger equation, conventional quantum relaxation can take place. But is there any reason to expect nonequilibrium $\rho \neq \left|\psi_{\rm eff}\right|^2$ at the start of the semiclassical regime? Indeed there is. Fundamentally we have a perpetual nonequilibrium ensemble with distribution $P[g_{ij}, \phi, t] \neq \left|\Psi[g_{ij}, \phi]\right|^2$. As we enter the semiclassical regime, say at some 'initial' time t_i (approximately marking the beginning of that regime), the field ϕ will have a conditional probability density

$$\rho[\phi, t_i] = \frac{P[g_{ij}, \phi, t_i]}{\left(\int P[g_{ij}, \phi, t_i] D\phi\right)}, \tag{53}$$

where on the right-hand side it is understood that we have inserted the actual value of the classical background 3-metric g_{ij} at time t_i. Because here

$$\begin{aligned} P[g_{ij}, \phi, t_i] \neq \left|\Psi[g_{ij}, \phi]\right|^2 &\approx \left|\Psi_{\rm WKB}[g_{ij}]\right|^2 \left|\psi[\phi, g_{ij}]\right|^2 \\ &= \left|\Psi_{\rm WKB}[g_{ij}]\right|^2 \left|\psi_{\rm eff}[\phi, t_i]\right|^2, \end{aligned} \tag{54}$$

it follows that

$$\rho[\phi, t_i] \neq \left|\psi_{\text{eff}}[\phi, t_i]\right|^2 \tag{55}$$

(unless it so happens that $P[g_{ij}, \phi, t_i] = \Pi[g_{ij}]\left|\psi_{\text{eff}}[\phi, t_i]\right|^2$ for some $\Pi[g_{ij}]$). We then expect to find quantum nonequilibrium at the start of the semiclassical or Schrödinger regime, with the Born rule emerging only later after an appropriate period of quantum relaxation.

8 Instability of the Born Rule in Quantum Gravity

We have outlined how the time-dependent Schrödinger equation (52) for the effective wave function ψ_{eff} emerges in the semiclassical approximation. We then have a normalisable wave function and quantum relaxation to equilibrium $\rho \to \left|\psi_{\text{eff}}\right|^2$ can proceed in the usual way. The Schrödinger equation (52) is, however, subject to small quantum-gravitational corrections to the effective Hamiltonian $\hat{H}_{\text{eff}}$. Remarkably, some of the correction terms are non-Hermitian [64–68]. These terms have no consistent interpretation in standard quantum mechanics as they violate the conservation of probability. In pilot-wave theory, in contrast, probability is by construction conserved and (as we shall see) the non-Hermitian terms simply render the Born rule unstable.

The derivation of the correction terms will be presented in the next section. Here we first show how pilot-wave theory is able to accommodate such terms consistently.

The corrections are calculated by performing a 'semiclassical expansion' of the Wheeler–DeWitt equation (Sect. 9). Dropping for simplicity the subscript 'eff', we find an effective Hamiltonian of the form

$$\hat{H} = \hat{H}_\phi + \hat{H}_a + i\hat{H}_b, \tag{56}$$

where $\hat{H}_\phi$ is the usual field Hamiltonian and the Hermitian operators $\hat{H}_a$, $\hat{H}_b$ represent tiny quantum-gravitational corrections. There is a Hermitian correction $\hat{H}_a$ and a non-Hermitian correction $i\hat{H}_b$. Writing $\hat{H}_1 = \hat{H}_\phi + \hat{H}_a$ and $\hat{H}_2 = \hat{H}_b$, the effective Schrödinger equation for $\psi[\phi, t]$ takes the form

$$i\frac{\partial\psi}{\partial t} = (\hat{H}_1 + i\hat{H}_2)\psi \ . \tag{57}$$

Applying the same semiclassical expansion to the de Broglie guidance equation, we find an effective guidance equation of the form

$$\frac{\partial\phi}{\partial t} = \frac{j_1}{|\psi|^2}, \tag{58}$$

where j_1 is the usual current associated with the Hermitian part $\hat{H}_1$ only. Thus, while the Schrödinger equation (57) has a non-Hermitian correction $i\hat{H}_2$, this does not affect the guidance equation (58). As we show in the next section, these results follow directly and without ambiguity from the underlying quantum-gravitational equations in a semiclassical expansion.

To see the consequences note that (57) implies a continuity equation (writing $j_1 = |\psi|^2\dot{\phi}$)

$$\frac{\partial|\psi|^2}{\partial t} + \partial_\phi \cdot (|\psi|^2\dot{\phi}) = s, \tag{59}$$

where $\partial_\phi \cdot (...) = \int d^3x\, \delta/\delta\phi(x)\, (...)$ is a divergence in field configuration space and

$$s = 2\mathrm{Re}\big(\psi^* \hat{H}_2 \psi\big). \tag{60}$$

For an ensemble of systems with the same wave function ψ, each element of the ensemble evolves by the de Broglie velocity field (58). The probability density $\rho[\phi, t]$ then evolves by the usual continuity equation

$$\frac{\partial\rho}{\partial t} + \partial_\phi \cdot (\rho\dot{\phi}) = 0. \tag{61}$$

For $s \neq 0$ there is a mismatch between Eqs. (59) and (61). It follows that an initial distribution $\rho = |\psi|^2$ can evolve into a final distribution $\rho \neq |\psi|^2$. The Born rule is unstable. This can be quantified in terms of the ratio $f = \rho/|\psi|^2$, which is no longer conserved along trajectories. From (59) and (61) we find

$$\frac{df}{dt} = -\frac{sf}{|\psi|^2} \tag{62}$$

(where $d/dt = \partial/\partial t + \int d^3x\, \dot{\phi}.\delta/\delta\phi(x)$ is the time derivative along a trajectory in field configuration space). It is also worth noting that the squared-norm $\int D\phi\, |\psi|^2$ of ψ changes with time,

$$\frac{d}{dt}\int D\phi\, |\psi|^2 = \int dq\, s = 2\big\langle \hat{H}_2 \big\rangle. \tag{63}$$

We see from the above equations that the usual Born-rule equilibrium state $\rho = |\psi|^2$ is unstable. As a result of quantum-gravitational corrections, nonequilibrium $\rho \neq |\psi|^2$ is created on a timescale τ_{noneq} which can be estimated from the rate of change of the (fine-grained) H-function $H(t) = \int D\phi\, \rho \ln(\rho/|\psi|^2)$. From (59) and (61) we find[15]

$$\frac{dH}{dt} = -\int D\phi\, \frac{\rho}{|\psi|^2} s. \tag{64}$$

Close to equilibrium ($\rho \approx |\psi|^2$) we have

$$\frac{dH}{dt} \approx -\int D\phi\, s = -2\big\langle \hat{H}_2 \big\rangle. \tag{65}$$

[15] When $s = 0$ the exact H is constant but the coarse-grained value decreases (if there is no initial fine-grained structure) [16].

Defining τ_{noneq} as the timescale over which H changes by a factor of order unity, we have the estimate

$$\tau_{\text{noneq}} \approx \frac{1}{2\left|\left\langle \hat{H}_2 \right\rangle\right|}. \tag{66}$$

Note however that, for such effects to build up over time, nonequilibrium must be created faster than relaxation can remove it, which requires conditions where

$$\tau_{\text{relax}} > \tau_{\text{noneq}}. \tag{67}$$

Some quantum-gravity theorists have long been puzzled by the non-Hermitian terms $i\hat{H}_2$ (first found in 1991 and re-derived in more recent papers), which signal a violation of unitarity (the usual norm of ψ is not conserved) [64–68]. Because the non-Hermitian terms are inconsistent with the standard interpretation of $|\psi|^2$ as a probability density, they are often regarded as an artifact to be ignored by fiat. Some authors have advocated formally eliminating these terms by appropriate redefinitions of the wave function [66, 95]. We suggest that such redefinitions may turn out to be an artificial means of disguising genuine physical effects. Our experience with quantum systems on a classical spacetime background teaches us that the Hamiltonian must be Hermitian, but that experience is limited to conditions where quantum-gravitational effects are negligible. As we have seen, non-Hermitian terms are perfectly consistent with pilot-wave theory, according to which they simply generate a gravitational instability of the Born rule: an initial density $\rho = |\psi|^2$ can evolve into a final density $\rho \neq |\psi|^2$.

The derivation of the non-Hermitian terms will now be discussed in more detail.

9 Semiclassical Expansion

Quantum-gravitational corrections to the Schrödinger equation (52) were derived by Kiefer and Singh [64] from a semiclassical expansion

$$\Psi = \exp i\left(\mu S_0 + S_1 + \mu^{-1} S_2 + \cdots\right) \tag{68}$$

of the extended Wheeler–DeWitt equation (10) for $\Psi[g_{ij}, \phi]$, where $\mu = c^2/32\pi G$ (dimensions mass per length). Inserting (68) into the left-hand side of (10), terms of the same order in μ are collected and their sum set to zero. The orders that appear are $\mu^2, \mu, \mu^0, \mu^{-1} \ldots$.

To a first approximation we obtain the usual Schrödinger equation (52) for a field ϕ on a classical background spacetime. We then obtain gravitational corrections to (52), in the form of (very small) Hermitian and non-Hermitian terms in the Hamiltonian. The results found by Kiefer and Singh are summarised below. In pilot-wave theory we must also consider how the semiclassical expansion affects the de Broglie guidance equation (37) for ϕ. As we will see, the guidance equation retains its standard form. Thus, if the semiclassical expansion is to be trusted, it follows from the fundamental equations of quantum gravity that the emergent Born rule is unstable.

9.1 Lowest-Order Schrödinger Approximation

At order μ^2 the expansion (68) yields $\left(\delta S_0/\delta\phi\right)^2 = 0$ so that $S_0 = S_0[g_{ij}]$ depends only on g_{ij}, while at order μ it is found that S_0 satisfies a classical Hamilton-Jacobi equation whose solution defines a classical background spacetime. The trajectories of the classical background can be used to define an effective time parameter t (cf. Eq. (51)).

Order μ^0 yields an equation for S_1, which can be written as an effective time-dependent Schrödinger equation for a zeroth-order (uncorrected) wave functional $\psi^{(0)}[\phi, t]$ on a classical background with metric g_{ij}. Defining

$$\psi^{(0)} = D\exp(iS_1) \tag{69}$$

for an appropriate functional $D[g_{ij}]$, it can be shown that

$$i\frac{\partial\psi^{(0)}}{\partial t} = \int d^3x\, N\hat{\mathcal{H}}_\phi\psi^{(0)} \tag{70}$$

where $\hat{\mathcal{H}}_\phi$ is given by (12). This is the standard Schrödinger equation for a massless (minimally-coupled) real scalar field ϕ with potential $\mathcal{V}(\phi)$ on a classical spacetime background.

As expected, to this order the Wheeler–DeWitt wave functional $\Psi[g_{ij}, \phi]$ takes the WKB form (49), with $\Psi_{\mathrm{WKB}}[g_{ij}] = (1/D)\exp\left(iMS_0\right)$ and $\psi[\phi, g_{ij}] = \psi^{(0)}[\phi, t]$.

In pilot-wave theory we must also consider the de Broglie guidance equation (37) for ϕ. To this order, how is the field velocity $\dot{\phi}$ related to the effective wave functional $\psi^{(0)}[\phi, t]$? We can find out by inserting the expansion (68) into (37) (where $S = \mathrm{Im}\ln\Psi$), yielding

$$\frac{\partial\phi}{\partial t} = \frac{N}{\sqrt{g}}\frac{\delta}{\delta\phi}\left(\mathrm{Re}S_1 + \mu^{-1}\mathrm{Re}S_2 + \cdots\right) . \tag{71}$$

The factor D in (69) can be chosen to be real, so that $\mathrm{Re}S_1$ is equal to the phase of $\psi^{(0)}$. To lowest order we then have a de Broglie velocity,

$$\left(\frac{\partial\phi}{\partial t}\right)^{(0)} = \frac{N}{\sqrt{g}}\frac{\delta S^{(0)}}{\delta\phi} , \tag{72}$$

where $S^{(0)} = \mathrm{Im}\ln\psi^{(0)}$ is the phase of $\psi^{(0)}$.[16] This is the standard de Broglie guidance equation for a field ϕ with wave functional $\psi^{(0)}[\phi, t]$ [96].

[16] This result is of course expected from the WKB form (49) (with $\psi = \psi^{(0)}$).

Thus, in this approximation we recover the usual pilot-wave dynamics of a field on a classical spacetime background, and so we can expect quantum relaxation to the Born rule to occur in the usual way.

9.2 Gravitational Corrections

Following Ref. [64] we now consider higher orders in the semiclassical expansion (68). At order μ^{-1} Kiefer and Singh obtain an equation for S_2. Writing $S_2 = \sigma_2[g_{ij}] + \eta[\phi, g_{ij}]$ for appropriately chosen σ_2, the corrected matter wave functional

$$\psi^{(1)} = \psi^{(0)} \exp(i\eta/\mu) \tag{73}$$

is found to satisfy a corrected Schrödinger equation

$$i\frac{\partial\psi^{(1)}}{\partial t} = \int d^3x\, N\big(\hat{\mathcal{H}}_\phi + \hat{\mathcal{H}}_a + i\hat{\mathcal{H}}_b\big)\psi^{(1)}, \tag{74}$$

where

$$\hat{\mathcal{H}}_a = \frac{1}{8\mu}\frac{1}{\sqrt{g}R}\hat{\mathcal{H}}_\phi^2 \tag{75}$$

and

$$\hat{\mathcal{H}}_b = \frac{1}{8\mu}\frac{\delta}{\delta\tau}\left(\frac{\hat{\mathcal{H}}_\phi}{\sqrt{g}R}\right) \tag{76}$$

are both Hermitian (employing the convenient shorthand $\delta/\delta\tau = \dot{g}_{ij}\delta/\delta g_{ij}$, with $\dot{g}_{ij} = 2NG_{ijkl}\delta S_0/\delta g_{kl}$, to denote a 'many-fingered time derivative' on the background). To this order we have a total effective Hamiltonian of the form (56) with

$$\hat{H}_\phi = \int d^3x\, N\hat{\mathcal{H}}_\phi, \quad \hat{H}_a = \int d^3x\, N\hat{\mathcal{H}}_a, \quad \hat{H}_b = \int d^3x\, N\hat{\mathcal{H}}_b. \tag{77}$$

As noted we have Hermitian and non-Hermitian corrections $\hat{H}_a$ and $i\hat{H}_b$ respectively.[17]

We can now consider the next order in the semiclassical expansion (68) of the de Broglie guidance equation (37). Because the term $\sigma_2[g_{ij}]$ in S_2 is independent of ϕ, the de Broglie velocity (71) takes the form

$$\frac{\partial\phi}{\partial t} = \frac{N}{\sqrt{g}}\frac{\delta}{\delta\phi}\big(\mathrm{Re}S_1 + \mu^{-1}\mathrm{Re}\eta + \cdots\big)\,. \tag{78}$$

The corrected wave functional (73) has a total phase

[17] In an expanding cosmological background the ratio of $i\hat{H}_b$ to $\hat{H}_a$ is roughly of order $\sim H/E$, where $H = \dot{a}/a$ is the Hubble parameter and E is a typical energy for the field [64].

$$\mathrm{Im}\ln\psi^{(1)} = \mathrm{Re}S_1 + \mu^{-1}\mathrm{Re}\eta, \tag{79}$$

and so the corrected de Broglie velocity (78) can once again be written in the standard form,

$$\left(\frac{\partial\phi}{\partial t}\right)^{(1)} = \frac{N}{\sqrt{g}}\frac{\delta S^{(1)}}{\delta\phi}\ , \tag{80}$$

where now $S^{(1)} = \mathrm{Im}\ln\psi^{(1)}$ is the phase of $\psi^{(1)}$.

To conclude, despite the non-Hermitian term in the corrected Schrödinger equation (74), the de Broglie velocity (80) continues to take the standard form (now in terms of $\psi^{(1)}$). In other words, the expression for the velocity remains that associated with the original (uncorrected) Hermitian part $\hat{H}_\phi$ of the Hamiltonian. The non-Hermitian term affects the time evolution of $\psi^{(1)}$—and so indirectly affects the trajectories—but does not change the form of the guidance equation itself. As we have seen, this implies that the Born rule for ϕ is unstable.

More recently, Brizuela et al. [67, 68] have derived similar results for a minisuperspace model of quantum cosmology. The classical background is defined by a scale factor $a(t)$ and a homogeneous field $\phi(t)$. Quantum scalar perturbations (of the background metric combined with the inflaton perturbation) are described by the Mukhanov–Sasaki variable $v_{\mathbf{k}}$ in Fourier space. The wave function $\Psi_{\mathbf{k}}(a, \phi, v_{\mathbf{k}})$ satisfies a Wheeler–DeWitt equation for the mode $\mathbf{k}$, which is solved by means of a semiclassical expansion

$$\Psi_{\mathbf{k}} = \exp\left[i\left(m_{\mathrm{P}}^2 S_0 + m_{\mathrm{P}}^0 S_1 + m_{\mathrm{P}}^{-2} S_2 + \cdots\right)\right] \tag{81}$$

in powers of m_{P}^2. Inserting this into the left-hand side of the Wheeler–DeWitt equation, terms of the same order in m_{P} are collected and their sum set to zero. By this means, Brizuela et al. derive a Schrödinger equation for an effective wave function $\psi_{\mathbf{k}}^{(1)} = \psi_{\mathbf{k}}^{(1)}(v_{\mathbf{k}}, t)$, where the corrections in the effective Hamiltonian have both Hermitian and non-Hermitian parts. The same expansion can again be applied to the de Broglie guidance equation for the perturbations $v_{\mathbf{k}}$ [8]. We again find that the de Broglie velocity takes the standard form proportional to the gradient of the phase $s_{\mathbf{k}}^{(1)} = \mathrm{Im}\ln\psi_{\mathbf{k}}^{(1)}$ of $\psi_{\mathbf{k}}^{(1)}$, and so remains equal to the velocity generated by the (uncorrected) Hermitian part of the Hamiltonian. As in the general case this implies that the Born rule is unstable.

10 Examples of Quantum Instability

The gravitational instability of the Born rule has been studied for several examples. These include a scalar field on de Sitter space, a scalar field close to an evaporating black hole, and an atomic system in the gravitational field of the earth. Here we outline the results obtained so far and some of the potential implications.[18]

[18] For more details see Ref. [8].

10.1 Inflationary Perturbations on De Sitter Space

In Ref. [8], taking the results of Brizuela et al. [67, 68] as a starting point, we derived a simplified model for inflationary perturbations in a far slow-roll limit, on a background with an approximate de Sitter expansion, $a \propto e^{Ht}$, where the Hubble parameter H is almost constant. The resulting equations define a tractable cosmological model of quantum instability in the early inflationary universe.

The perturbations are described by (real) Fourier field components $q_{\mathbf{k}}$. The corrected Schrödinger equation for the effective wave function $\psi_{\mathbf{k}}^{(1)}(q_{\mathbf{k}}, t)$ is found to be

$$i\frac{\partial \psi_{\mathbf{k}}^{(1)}}{\partial t} = \hat{H}_{\mathbf{k}}\psi_{\mathbf{k}}^{(1)} - \frac{\bar{k}^3}{2m_{\mathrm{P}}^2 H^2}\frac{1}{\psi_{\mathbf{k}}^{(0)}}\left[\frac{1}{a^3}(\hat{H}_{\mathbf{k}})^2\psi_{\mathbf{k}}^{(0)} + i\frac{\partial}{\partial t}\left(\frac{1}{a^3}\hat{H}_{\mathbf{k}}\right)\psi_{\mathbf{k}}^{(0)}\right]\psi_{\mathbf{k}}^{(1)}, \quad (82)$$

where

$$\hat{H}_{\mathbf{k}} = -\frac{1}{2a^3}\frac{\partial^2}{\partial q_{\mathbf{k}}^2} + \frac{1}{2}ak^2 q_{\mathbf{k}}^2 \quad (83)$$

is the uncorrected (zeroth-order) Hamiltonian for the field mode, $\psi_{\mathbf{k}}^{(0)}$ is the uncorrected (zeroth-order) wave function, and

$$\bar{k} = \frac{1}{\mathfrak{L}}, \quad (84)$$

where $\mathfrak{L}$ is an arbitrary lengthscale associated with spatial integration in the classical action (to be interpreted as an infrared cutoff) [69]. In the same limit the de Broglie guidance equation for $q_{\mathbf{k}}$ is found to be

$$\frac{dq_{\mathbf{k}}}{dt} = \frac{1}{a^3}\frac{\partial s_{\mathbf{k}}^{(1)}}{\partial q_{\mathbf{k}}}, \quad (85)$$

where $s_{\mathbf{k}}^{(1)} = \mathrm{Im}\ln\psi_{\mathbf{k}}^{(1)}$ is the phase of $\psi_{\mathbf{k}}^{(1)}$. This is the standard de Broglie velocity for Fourier components of a scalar field, with Hamiltonian (83), on a classical expanding background [35].

For a theoretical ensemble with the same wave function $\psi_{\mathbf{k}}^{(1)}(q_{\mathbf{k}}, t)$, the probability density $\rho_{\mathbf{k}}^{(1)}(q_{\mathbf{k}}, t)$ will evolve by the continuity equation

$$\frac{\partial \rho_{\mathbf{k}}^{(1)}}{\partial t} + \frac{\partial}{\partial q_{\mathbf{k}}}\left(\rho_{\mathbf{k}}^{(1)}\dot{q}_{\mathbf{k}}\right) = 0, \quad (86)$$

where $\dot{q}_{\mathbf{k}}$ is the velocity field (85). In contrast, from (82) we find that $\left|\psi_{\mathbf{k}}^{(1)}\right|^2$ satisfies

$$\frac{\partial\left|\psi_{\mathbf{k}}^{(1)}\right|^2}{\partial t} + \frac{\partial}{\partial q_{\mathbf{k}}}\left(\left|\psi_{\mathbf{k}}^{(1)}\right|^2\dot{q}_{\mathbf{k}}\right) = s, \quad (87)$$

where in the notation of Sect. 8 the 'source' s is given by (60) where here

$$\hat{H}_2 = -\frac{\bar{k}^3}{2m_{\rm P}^2 H^2} \frac{1}{\psi_{\mathbf{k}}^{(0)}} \frac{\partial}{\partial t}\left(\frac{1}{a^3}\hat{H}_{\mathbf{k}}\right)\psi_{\mathbf{k}}^{(0)}. \tag{88}$$

These equations can be used to calculate the gravitational production of quantum nonequilibrium during inflation, employing the differential equation (62) for the rate of change of the ratio $f_{\mathbf{k}} = \rho_{\mathbf{k}}/\left|\psi_{\mathbf{k}}\right|^2$ along trajectories. Taking $\psi_{\mathbf{k}}^{(0)}$ to be the Bunch-Davies vacuum wave function, approximate calculations show that the gravitational instability of the Born rule generates a nonequilibrium deficit $\sim 1/k^3$ in the primordial cosmological power spectrum. It has been shown elsewhere that there is no significant relaxation during inflation [37, 97], so the condition (67) will be satisfied and the generated nonequilibrium will persist over time. However, the magnitude of the effect on the power spectrum is far too small to observe in the CMB (for details see Ref. [8]).

By considering only the Hermitian terms in the Hamiltonian, Brizuela et al. [67, 68] show that the gravitationally-corrected wave function induces a similar $\sim 1/k^3$ correction to the power spectrum but of opposite sign (hence a power excess). However, the calculations of Ref. [8] are too approximate to precisely compare the overall magnitudes of these physically-distinct effects.

10.2 Evaporating Black Holes

It is also of interest to consider quantum instability for a field in the background spacetime of an evaporating Schwarzchild black hole. It was argued by Kiefer et al. [98] that in this case the quantum-gravitational corrections to the effective Schrödinger equation will be as in Eqs. (74)–(76) but with the replacement

$$\sqrt{g}R \to -16\pi GM/c^2, \tag{89}$$

where the Schwarzchild radius $r_{\rm S} = 2GM/c^2$ (for a black hole of mass M) provides a natural lengthscale. The non-Hermitian term in (74) reads

$$i\hat{H}_b = i\frac{4\pi\hbar G}{c^4}\int d^3x\, N\frac{\delta}{\delta\tau}\left(\frac{\hat{\mathcal{H}}_\phi}{\sqrt{g}R}\right), \tag{90}$$

where we have inserted $\mu = c^2/32\pi G$ as well as $\hbar$ and c. With the replacement (89), (90) takes the approximate form

$$i\hat{H}_b \simeq -i\frac{\hbar}{4c^2}\frac{d}{dt}\left(\frac{1}{M}\right)\int d^3x\, N\hat{\mathcal{H}}_\phi = i\frac{\hbar}{4c^2}\frac{1}{M^2}\frac{dM}{dt}\hat{H}_\phi, \tag{91}$$

where $\hat{H}_\phi$ is the uncorrected field Hamiltonian (neglecting the rate of change of $\hat{\mathcal{H}}_\phi$ compared with the rate of change of the background geometry). Kiefer et al. suggested that this term might alleviate the problem of black-hole information loss (though such a term is inconsistent with the standard quantum formalism).

Taking the phenomenological time dependence [99, 100]

$$M(t) \simeq M_0 \left(1 - \kappa \frac{m_P^3}{M_0^3} \left(\frac{t}{t_P} \right) \right)^{1/3}, \tag{92}$$

with M_0 the initial mass, κ a numerical factor, and here $m_P = \sqrt{\hbar c/G} \simeq 10^{-5}$ g the standard Planck mass, we have $dM/dt \simeq -\frac{1}{3}\kappa(m_P/t_P)\big(m_P/M\big)^2$. According to (91) the Hamiltonian $\hat{H}_\mathbf{k}$ of a field mode then acquires a non-Hermitian correction $i\hat{H}_2$ (in the notation of Sect. 8) with

$$\hat{H}_2 \simeq -\frac{1}{12}\kappa \left(\frac{m_P}{M} \right)^4 \hat{H}_\mathbf{k} \, . \tag{93}$$

This correction is significant in the final stage of evaporation when M approaches m_P, suggesting that the final burst of Hawking radiation could contain significant departures from the Born rule.[19]

Quantum nonequilibrium is expected to be created on a timescale τ_{noneq} of order (66), which depends inversely on the equilibrium mean energy $E_\mathbf{k} = \langle \hat{H}_\mathbf{k} \rangle$. If we take $E_\mathbf{k} \sim k_B T_H$ where

$$k_B T_H = \frac{\hbar c^3}{G} \frac{1}{8\pi M} = \frac{1}{8\pi} m_P c^2 \frac{m_P}{M} \tag{94}$$

is the Hawking temperature, then from (66) and (93) we have

$$\tau_{\text{noneq}} \sim \frac{48\pi}{\kappa} t_P \left(\frac{M}{m_P} \right)^5 . \tag{95}$$

Corrections to the Born rule will be significant if τ_{noneq} is not too large compared to the evaporation timescale t_{evap}. Taking $1/t_{\text{evap}} \sim (1/M)|dM/dt|$ we have

$$t_{\text{evap}} \sim \frac{3}{\kappa} t_P \left(\frac{M}{m_P} \right)^3 \tag{96}$$

and a ratio

$$\frac{\tau_{\text{noneq}}}{t_{\text{evap}}} \sim \frac{48\pi}{3} \left(\frac{M}{m_P} \right)^2 \tag{97}$$

(the factor κ cancels). Again it seems clear that significant deviations from the Born rule can be generated in the outgoing radiation only in the final stage of evaporation when M approaches m_P.

It is however not known if such deviations could survive quantum relaxation, which may well be significant in the final stage of evaporation when the background spacetime is changing rapidly. Quantum nonequilibrium will build up over time only

[19] The creation of quantum nonequilibrium by evaporating black holes was previously suggested as a possible mechanism for resolving the information-loss puzzle [35, 63, 96, 101], but without a clear theoretical underpinning.

if (67) is satisfied in the regime where M approaches $m_{\rm P}$. Thus we need to know how $\tau_{\rm relax}$ scales with M and to compare this with our estimate $\tau_{\rm noneq} \propto \left(M/m_{\rm P}\right)^5$. This is a matter for future work.

Should nonequilibrium survive in the outgoing radiation, at least in principle the emitted photons could show anomalies in their two-slit interference pattern or in their polarisation probabilities [102]. Realistically, Hawking radiation in the γ-ray region might be detected from exploding primordial black holes (which may form a significant component of dark matter [103]). However, only the very final burst is likely to show significant deviations from the Born rule, making detection difficult.

10.3 An Atom in the Gravitational Field of the Earth

We might ask if the Born rule could be unstable for an atomic system in the gravitational field of the earth. We saw in Sect. 10.2 that the non-Hermitian correction to a field Hamiltonian in the spacetime of a Schwarzchild black hole can plausibly be obtained from (90) by replacing $\sqrt{g}R$ by $-8\pi r_{\rm S}$ where $r_{\rm S} = 2GM/c^2$ is the natural lengthscale of the background. In the gravitational field of the earth we might expect instead to make a replacement of the form

$$\sqrt{g}R \rightarrow -8\pi r_{\rm c}, \tag{98}$$

where $r_{\rm c}$ is the local radius of curvature ($r_{\rm c} \simeq 10^{13}$ cm at the surface of the earth).[20] Inserting this in (90), and writing $\hat{\mathcal{H}}_\phi$ as $\hat{\mathcal{H}}_a$ where $\hat{H}_a = \int d^3x\, N\hat{\mathcal{H}}_a$ is the atomic Hamiltonian, we find an estimated non-Hermitian term

$$i\hat{H}_b \sim -i\frac{\hbar G}{c^4}\frac{1}{r_{\rm c}}\int d^3x\, N\frac{\delta}{\delta\tau}\left(\hat{\mathcal{H}}_a\right) \sim -i\frac{l_{\rm P}}{r_{\rm c}}\frac{\partial \hat{H}_a}{\partial t}t_{\rm P}, \tag{99}$$

where $l_{\rm P}$ and $t_{\rm P}$ are respectively the Planck length and time.

The term (99) is non-zero only if the (uncorrected) atomic Hamiltonian $\hat{H}_a$ is time dependent. The magnitude of (99) is roughly the change in $\hat{H}_a$ over a Planck time suppressed by the ratio $l_{\rm P}/r_{\rm c}$. Needless to say, in ordinary laboratory conditions, this term will be utterly negligible. Furthermore, if $\hat{H}_a$ changes rapidly (to maximise the effect), the atomic wave function will be a superposition of multiple energy eigenstates, and we expect to find quantum relaxation over timescales $\tau_{\rm relax} \ll \tau_{\rm noneq}$. Even if we could probe an atomic ensemble over times $\sim \tau_{\rm noneq}$ (far longer than the age of the universe), any gravitationally-generated nonequilibrium will have long-since relaxed. It then appears that the gravitational creation of quantum nonequilibrium in ordinary laboratory systems—with a dynamical Hamiltonian in a background curved space—is likely to be of theoretical interest only.

[20] A similar suggestion was made by Kiefer and Singh [64], who considered the effect of the Hermitian correction on atomic energy levels.

11 Conclusion

We have argued that, in the deep quantum-gravity regime, with a non-normalisable Wheeler–DeWitt wave functional Ψ, there is no Born rule and the universe is in a perpetual state of quantum nonequilibrium with a probability density $P \neq |\Psi|^2$. Quantum relaxation to the Born rule can occur only when the early universe emerges into a semiclassical or Schrödinger approximation, with a time-dependent and normalisable effective wave functional ψ for a system on a classical space-time background, for which the probability density ρ can evolve towards $|\psi|^2$ (on a coarse-grained level). We conclude that the long-standing hypothesis of primordial quantum nonequilibrium, with relaxation to the Born rule taking place soon after the big bang, follows naturally from the internal logic of quantum gravity (as interpreted in de Broglie–Bohm pilot-wave theory). Furthermore, quantum-gravitational corrections to the Schrödinger approximation, in the form of tiny non-Hermitian terms in the effective Hamiltonian, generate a (very slight) instability of the Born rule, whereby quantum nonequilibrium $\rho \neq |\psi|^2$ can be created from a prior equilibrium ($\rho = |\psi|^2$) state. To observe such effects will be difficult in practice, though possible at least in principle.

When restricted to the Born-rule equilibrium state, the pilot-wave or de Broglie–Bohm formulation of quantum theory is experimentally indistinguishable from textbook quantum mechanics. Wider support for this formulation is likely to be forthcoming should we find experimental evidence for violations of the Born rule—or if the theory allows us to make decisive progress in understanding some vital aspect of fundamental physics. From the results presented here we suggest that this little-used formulation of quantum theory allows us to understand and solve three problems in canonical quantum gravity: (a) to explain why the naive Schrödinger interpretation does not work, (b) to account for the emergence of the Born rule in a semiclassical regime, and (c) to give a consistent meaning to non-Hermitian quantum-gravitational corrections to the effective Schrödinger equation.

The results of this paper also impact on certain philosophical debates concerning the status of the Born rule in de Broglie–Bohm theory. As we have noted, and discussed in detail elsewhere [26], the 'Bohmian mechanics' school employs an essentially circular argument to obtain the Born rule for subsystems by assuming the Born rule for the whole universe at some initial cosmological time.[21] We have seen that, when quantum gravity is taken into account, such an argument has no starting point, since there is no fundamental Born-rule measure for a universe governed by the Wheeler–DeWitt equation (despite attempts by some workers [56, 62] to apply the naive Schrödinger interpretation to pilot-wave gravitation).

It is one hundred years since de Broglie started on the path that, after five years of remarkable developments, brought him in 1927 to pilot-wave theory as we know it today. It is seventy years since the revival and further development of pilot-wave theory in Bohm's papers of 1952. And yet the theory is still not widely known or used, and is often misunderstood. The historical development of pilot-wave theory

[21] In a remarkable reply to Ref. [26], Dürr and Struyve [104] invoke similar circular reasoning in their account of classical coin tossing.

is in certain respects reminiscent of the historical development of the kinetic theory of gases. Beginning with the pioneering work of Bernoulli in the early 18th century, and of Herapath and Waterston in the early 19th century, kinetic theory was more or less ignored until it was taken up by Clausius in an influential paper of 1857 [105]. Another half a century had to pass, with decisive contributions in particular by Maxwell, Boltzmann and Einstein, before theorists were able to interpret Brownian motion as evidence for atoms and kinetic theory. Whether or not a comparable empirical breakthough awaits pilot-wave theory remains to be seen.

Why were physicists in the late nineteenth-century still reluctant to accept the existence of atoms and molecules, long after chemists had already deduced their detailed shapes and compositions? In part there was philosophical opposition from Mach and others, who emphasised the role of sensory perception in physics, while the idea of an objective reality beyond the immediate reach of our senses came to be widely derided as unscientific and metaphysical. Similarly, today there remains widespread opposition to realism in quantum physics. For as long as the details of de Broglie–Bohm trajectories cannot be observed (the uncertainty principle reigns for as long as we are confined to quantum equilibrium) those trajectories will continue to be dismissed as unphysical.

A decisive breakthrough, with an end to seemingly endless philosophical debates, will occur only by extending the boundaries of physics beyond what is currently known and understood. The prospects do not seem entirely remote. As we have argued in this paper, gravitation may hold the key to unlocking the hidden physics of pilot-wave theory.

References

1. de Broglie, L.: La nouvelle dynamique des quanta, in: Électrons et Photons: Rapports et Discussions du Cinquième Conseil de Physique (Gauthier-Villars, Paris, 1928). [English translation in ref. [2]]
2. Bacciagaluppi, G., Valentini, A.: Quantum Theory at the Crossroads: Reconsidering the 1927 Solvay Conference. Cambridge University Press, Cambridge (2009). arXiv:quant-ph/0609184
3. Bohm, D.: A suggested interpretation of the quantum theory in terms of 'hidden' variables. I. Phys. Rev. **85**, 166 (1952)
4. Bohm, D.: A suggested interpretation of the quantum theory in terms of 'hidden' variables. II. Phys. Rev. **85**, 180 (1952)
5. Holland, P.R.: The Quantum Theory of Motion: an Account of the de Broglie–Bohm Causal Interpretation of Quantum Mechanics. Cambridge University Press, Cambridge (1993)
6. Valentini, A., Pearle, P., Saunders, S.: Three Roads to Quantum Reality: Pilot Waves, Dynamical Collapse, Many Worlds. Oxford University Press, Oxford (2023)
7. Valentini, A.: de Broglie–Bohm pilot-wave theory. In: Oxford Research Encyclopedia of Physics. Oxford University Press, Oxford (2023). https://oxfordre.com/physics
8. Valentini, A.: Quantum gravity and quantum probability. arXiv:2104.07966
9. Pauli, W.: in: Louis de Broglie: Physicien et Penseur. Albin Michel, Paris (1953)
10. Keller, J.B.: Bohm's interpretation of the quantum theory in terms of 'hidden' variables. Phys. Rev. **89**, 1040 (1953)
11. Bohm, D.: Proof that probability density approaches $|\psi|^2$ in causal interpretation of the quantum theory. Phys. Rev. **89**, 458 (1953)
12. Bohm, D., Vigier, J.P.: Model of the causal interpretation of quantum theory in terms of a fluid with irregular fluctuations. Phys. Rev. **96**, 208 (1954)
13. Dürr, D., Goldstein, S., Zanghì, N.: Quantum equilibrium and the origin of absolute uncertainty. J. Stat. Phys. **67**, 843 (1992)

14. Dürr, D., Teufel, S.: Bohmian Mechanics: The Physics and Mathematics of Quantum Theory. Springer, Berlin (2009)
15. Tumulka, R.: Bohmian mechanics. In: Knox, E., Wilson, A. (eds.) The Routledge Companion to the Philosophy of Physics. Routledge, New York (2021)
16. Valentini, A.: Signal-locality, uncertainty, and the subquantum H-theorem. I. Phys. Lett. A **156**, 5 (1991)
17. Valentini, A.: Signal-locality, uncertainty, and the subquantum H-theorem. II. Phys. Lett. A **158**, 1 (1991)
18. Valentini, A.: On the pilot-wave theory of classical, quantum and subquantum physics. PhD thesis, International School for Advanced Studies, Trieste, Italy (1992). http://hdl.handle.net/20.500.11767/4334
19. Valentini, A.: Pilot-wave theory of fields, gravitation and cosmology. In: Cushing, J.T., et al. (eds.) Bohmian Mechanics and Quantum Theory: an Appraisal. Kluwer, Dordrecht (1996)
20. Valentini, A.: Hidden variables, statistical mechanics and the early universe. In: Bricmont, J. et al. (eds.) Chance in Physics: Foundations and Perspectives. Springer, Berlin (2001). arXiv:quant-ph/0104067
21. Valentini, A.: Signal-locality in hidden-variables theories. Phys. Lett. A **297**, 273 (2002). arXiv:quant-ph/0106098
22. Valentini, A.: Subquantum information and computation. Pramana-J. Phys. **59**, 269 (2002). arXiv:quant-ph/0203049
23. Valentini, A., Westman, H.: Dynamical origin of quantum probabilities. Proc. R. Soc. A **461**, 253 (2005). arXiv:quant-ph/0403034
24. Pearle, P., Valentini, A.: Quantum mechanics: generalizations. In: Françoise, J.-P. et al. (eds.) Encyclopaedia of Mathematical Physics. Elsevier, North-Holland, Amsterdam (2006). arXiv:quant-ph/0506115
25. Valentini, A.: Beyond the quantum. Phys. World **22N11**, 32 (2009). [arXiv:1001.2758]
26. Valentini, A.: Foundations of statistical mechanics and the status of the Born rule in de Broglie–Bohm pilot-wave theory. In: Allori, V. (ed.) Statistical Mechanics and Scientific Explanation: Determinism, Indeterminism and Laws of Nature. World Scientific, Singapore (2020). arXiv:1906.10761
27. Efthymiopoulos, C., Contopoulos, G.: Chaos in Bohmian quantum mechanics. J. Phys. A Math. Gen. **39**, 1819 (2006)
28. Towler, M.D., Russell, N.J., Valentini, A.: Time scales for dynamical relaxation to the Born rule. Proc. R. Soc. A **468**, 990 (2012). [arXiv:1103.1589]
29. Colin, S.: Relaxation to quantum equilibrium for Dirac fermions in the de Broglie–Bohm pilot-wave theory. Proc. R. Soc. A **468**, 1116 (2012). [arXiv:1108.5496]
30. Abraham, E., Colin, S., Valentini, A.: Long-time relaxation in pilot-wave theory. J. Phys. A Math. Theor. **47**, 395306 (2014). [arXiv:1310.1899]
31. Efthymiopoulos, C., Contopoulos, G., Tzemos, A.C.: Chaos in de Broglie–Bohm quantum mechanics and the dynamics of quantum relaxation, Ann. Fond. Louis de Broglie **42**, 133 (2017). arXiv:1703.09810
32. Drezet, A.: Justifying Born's rule $P_\alpha = |\Psi_\alpha|^2$ using deterministic chaos, decoherence, and the de Broglie–Bohm quantum theory, Entropy **23**, 1371 (2021). arXiv:2109.09353
33. Lustosa, F.B., Colin, S., Perez Bergliaffa, S.E.: Quantum relaxation in a system of harmonic oscillators with time-dependent coupling. Proc. R. Soc. A **477**, 20200606 (2021). arXiv:2007.02939
34. Lustosa, F.B., Pinto-Neto, N., Valentini, A.: Evolution of quantum nonequilibrium for coupled harmonic oscillators. arXiv:2205.13701
35. Valentini, A.: Astrophysical and cosmological tests of quantum theory. J. Phys. A Math. Theor. **40**, 3285 (2007). arXiv:hep-th/0610032
36. Valentini, A.: de Broglie–Bohm prediction of quantum violations for cosmological super-Hubble modes. arXiv:0804.4656
37. Valentini, A.: Inflationary cosmology as a probe of primordial quantum mechanics. Phys. Rev. D **82**, 063513 (2010). arXiv:0805.0163
38. Colin, S., Valentini, A.: Mechanism for the suppression of quantum noise at large scales on expanding space. Phys. Rev. D **88**, 103515 (2013). arXiv:1306.1579
39. Colin, S., Valentini, A.: Primordial quantum nonequilibrium and large-scale cosmic anomalies. Phys. Rev. D **92**, 043520 (2015). arXiv:1407.8262
40. Valentini, A.: Statistical anisotropy and cosmological quantum relaxation. arXiv:1510.02523

41. Colin, S., Valentini, A.: Robust predictions for the large-scale cosmological power deficit from primordial quantum nonequilibrium, Int. J. Mod. Phys. D **25**, 1650068 (2016). arXiv:1510.03508
42. Underwood, N.G., Valentini, A.: Quantum field theory of relic nonequilibrium systems. Phys. Rev. D **92**, 063531 (2015). arXiv:1409.6817
43. Underwood, N. G., Valentini, A.: Anomalous spectral lines and relic quantum nonequilibrium, Phys. Rev. D **101**, 043004 (2020). arXiv:1609.04576
44. Albert, D.Z.: After Physics. Harvard University Press, Cambridge (2015)
45. Goldstein, S.: Bohmian mechanics. In: Zalta, E.N. (ed.) The Stanford Encyclopedia of Philosophy, Fall 2021 edn (2021). https://plato.stanford.edu/archives/fall2021/entries/qm-bohm/
46. Rovelli, C.: Quantum Gravity. Cambridge University Press, Cambridge (2004)
47. DeWitt, B.S.: Quantum theory of gravity. I. The canonical theory. Phys. Rev. **160**, 1113 (1967)
48. Kiefer, C.: Quantum Gravity. Oxford University Press, Oxford (2012)
49. Unruh, W.G., Wald, R.M.: Time and the interpretation of canonical quantum gravity. Phys. Rev. D **40**, 2598 (1989)
50. Isham, C.J.: Conceptual and geometrical problems in quantum gravity. In: Mitter, H., Gausterer, H. (eds.) Recent Aspects of Quantum Fields. Springer, Berlin (1991)
51. Kuchař, K.V.: Time and interpretations of quantum gravity. In: Kunstatter, G., Vincent, D., Williams, J. (eds.) Proceedings of the 4th Canadian Conference on General Relativity and Relativistic Astrophysics. World Scientific, Singapore (1992) [Reprinted: Kuchař, K.V.: Int. J. Mod. Phys. D **20**, 3 (2011)]
52. Isham, C.J.: Canonical quantum gravity and the problem of time. In: Ibort, L. A., Rodriguez, M. A. (eds.) Integrable Systems, Quantum Groups, and Quantum Field Theories. Kluwer, London (1993). arXiv:gr-qc/9210011
53. Kuchař, K.V.: The problem of time in quantum geometrodynamics. In: Butterfield, J. (ed.) The Arguments of Time. Oxford University Press, Oxford (1999)
54. Anderson, E.: The Problem of Time: Quantum Mechanics Versus General Relativity. Springer, Cham (2017)
55. Kiefer, C., Peter, P.: Time in quantum cosmology. Universe **8**, 36 (2022). arXiv:2112.05788
56. Vink, J.C.: Quantum potential interpretation of the wave function of the universe. Nucl. Phys. B **369**, 707 (1992)
57. Horiguchi, T.: Quantum potential interpretation of the Wheeler–DeWitt equation. Mod. Phys. Lett. A **9**, 1429 (1994)
58. Shtanov, Yu.V.: Pilot wave quantum cosmology. Phys. Rev. D **54**, 2564 (1996). arXiv:gr-qc/9503005
59. Pinto-Neto, N.: The Bohm interpretation of quantum cosmology. Found. Phys. **35**, 577 (2005). arXiv:gr-qc/0410117
60. Pinto-Neto, N., Fabris, J.C.: Quantum cosmology from the de Broglie–Bohm perspective. Class. Quantum Grav. **30**, 143001 (2013). arXiv:1306.0820
61. Pinto-Neto, N.: The de Broglie–Bohm quantum theory and its application to quantum cosmology. Universe **7**, 134 (2021). arXiv:2111.03057
62. Dürr, D., Struyve, W.: Quantum Einstein equations. Class. Quantum Grav. **37**, 135002 (2020). arXiv:2003.03839
63. Valentini, A.: Trans-Planckian fluctuations and the stability of quantum mechanics. arXiv:1409.7467
64. Kiefer, C., Singh, T.P.: Quantum gravitational corrections to the functional Schrödinger equation. Phys. Rev. D **44**, 1067 (1991)
65. Kiefer, C., Krämer, M.: Quantum gravitational contributions to the cosmic microwave background anisotropy spectrum. Phys. Rev. Lett. **108**, 021301 (2012). arXiv:1103.4967
66. Bini, D., Esposito, G., Kiefer, C., Krämer, M., Pessina, F.: On the modification of the cosmic microwave background anisotropy spectrum from canonical quantum gravity. Phys. Rev. D **87**, 104008 (2013). arXiv:1303.0531
67. Brizuela, D., Kiefer, C., Krämer, M.: Quantum-gravitational effects on gauge-invariant scalar and tensor perturbations during inflation: the de Sitter case, Phys. Rev. D **93**, 104035 (2016). arXiv:1511.05545
68. Brizuela, D., Kiefer, C., Krämer, M.: Quantum-gravitational effects on gauge-invariant scalar and tensor perturbations during inflation: the slow-roll approximation. Phys. Rev. D **94**, 123527 (2016). arXiv:1611.02932

69. Kamenshchik, A.Y., Tronconi, A., Venturi, G.: Signatures of quantum gravity in a Born–Oppenheimer context. Phys. Lett. B **734**, 72 (2014). arXiv:1403.2961
70. Hartle, J.B., Hawking, S.W.: Wave function of the universe. Phys. Rev. D **28**, 2960 (1983)
71. Hawking, S.W.: The quantum state of the universe. Nucl. Phys. B **239**, 257 (1984)
72. Hawking, S.W., Page, D.: Operator ordering and the flatness of the universe. Nucl. Phys. B **264**, 185 (1986)
73. Hawking, S.W., Page, D.: How probable is inflation? Nucl. Phys. B **298**, 789 (1988)
74. Wheeler, J.A.: Superpsace and the nature of quantum geometrodynamics. In: DeWitt, C., Wheeler, J.A. (eds.) Battelle Rencontres: 1967 Lectures in Mathematics and Physics. Benjamin, New York (1968)
75. Rovelli, C.: Quantum mechanics without time: a model. Phys. Rev. D **42**, 2638 (1990)
76. Rovelli, C.: Time in quantum gravity: an hypothesis. Phys. Rev. D **43**, 442 (1991)
77. Rovelli, C.: Forget time, FQXi Essay on the Nature of Time (2009). arXiv:0903.3832
78. Barbour, J.B.: The timelessness of quantum gravity. I. The evidence from the classical theory. Class. Quantum Grav. **11**, 2853 (1994)
79. Barbour, J.B.: The timelessness of quantum gravity. II. The appearance of dynamics in static configurations. Class. Quantum Grav. **11**, 2875 (1994)
80. Halliwell, J.J.: Trajectories for the wave function of the universe from a simple detector model. Phys. Rev. D **64**, 044008 (2001). [arXiv:gr-qc/0008046]
81. Halliwell, J.J.: Probabilities in quantum cosmological models: a decoherent histories analysis using a complex potential. Phys. Rev. D **80**, 124032 (2009). arXiv:0909.2597
82. Halliwell, J.J.: Decoherent histories analysis of minisuperspace quantum cosmology, J. Phys.: Conf. Ser. **306**, 012023 (2011). arXiv:1108.5991
83. Hellmann, F., Mondragon, M., Perez, A., Rovelli, C.: Multiple-event probability in general-relativistic quantum mechanics. Phys. Rev. D **75**, 084033 (2007). arXiv:gr-qc/0610140
84. Mondragon, M., Perez, A., Rovelli, C.: Multiple-event probability in general-relativistic quantum mechanics: a discrete model. Phys. Rev. D **76**, 064005 (2007). arXiv:0705.0006
85. Struyve, W., Valentini, A.: de Broglie–Bohm guidance equations for arbitrary Hamiltonians. J. Phys. A Math. Theor. **42**, 035301 (2009). arXiv:0808.0290
86. Liddle, A.R., Lyth, D.H.: Cosmological Inflation and Large-Scale Structure. Cambridge University Press, Cambridge (2000)
87. Mukhanov, V.: Physical Foundations of Cosmology. Cambridge University Press, Cambridge (2005)
88. Peter, P., Uzan, J.-P.: Primordial Cosmology. Oxford University Press, Oxford (2009)
89. Aghanim, N., et al. (Planck Collaboration): Planck 2015 results. XI. CMB power spectra, likelihoods, and robustness of parameters, Astron. Astrophys. **594**, A11 (2016)
90. Vitenti, S., Peter, P., Valentini, A.: Modeling the large-scale power deficit with smooth and discontinuous primordial spectra, Phys. Rev. D **100**, 043506 (2019). arXiv:1901.08885
91. Valentini, A.: Hidden variables and the large-scale structure of space–time. In: Craig, W.L., Smith, Q. (eds.) Einstein, Relativity and Absolute Simultaneity. Routledge, London (2008). arXiv:quant-ph/0504011
92. Pinto-Neto, N., Sergio Santini, E.: The consistency of causal quantum geometrodynamics and quantum field theory. Gen. Rel. Grav. **34**, 505 (2002). arXiv:gr-qc/0009080
93. Valentini, A.: On Galilean and Lorentz invariance in pilot-wave dynamics. Phys. Lett. A **228**, 215 (1997). arXiv:0812.4941
94. Zeh, H.D.: Time in quantum gravity. Phys. Lett. A **126**, 311 (1988)
95. Kiefer, C., Wichmann, D.: Semiclassical approximation of the Wheeler–DeWitt equation: arbitrary orders and the question of unitarity. Gen. Relativ. Gravit. **50**, 66 (2018). arXiv:1802.01422
96. Valentini, A.: Black holes, information loss, and hidden variables. arXiv:hep-th/0407032
97. Kandhadai, A., Valentini, A.: Perturbations and quantum relaxation. Found. Phys. **49**, 1 (2019). [arXiv:1609.04485]
98. Kiefer, C., Müller, R., Singh, T.P.: Quantum gravity and non-unitarity in black hole evaporation. Mod. Phys. Lett. A **9**, 2661 (1994). arXiv:gr-qc/9308024
99. DeWitt, B.S.: Quantum field theory in curved spacetime. Phys. Rep. **19**, 295 (1975)
100. Wald, R.M.: General Relativity. University of Chicago Press, Chicago (1984)
101. Kandhadai, A., Valentini, A.: Mechanism for nonlocal information flow from black holes, Int. J. Mod. Phys. A **35**, 2050031 (2020). arXiv:191205374v1

102. Valentini, A.: Universal signature of non-quantum systems. Phys. Lett. A **332**, 187 (2004). arXiv:quant-ph/0309107
103. Carr, B., Kuhnel, F., Sandstad, M.: Primordial black holes as dark matter, Phys. Rev. D **94**, 083504 (2016). arXiv:1607.06077
104. Dürr, D., Struyve, W.: Typicality in the foundations of statistical physics and Born's rule. In: Allori, V. et al. (eds.) Do Wave Functions Jump? Springer, Cham (2021). arXiv:1910.08049
105. Brush, S.G.: John James Waterston and the kinetic theory of gases. Am. Sci. **49**, 202 (1961)