Foundations of Physics (2022) 52:114
https://doi.org/10.1007/s10701-022-00634-1

Understanding Time Reversal in Quantum Mechanics: A New Derivation

Shan Gao[1]

Received: 4 August 2022 / Accepted: 28 September 2022 / Published online: 6 October 2022

Abstract
Why does time reversal involve two operations, a temporal reflection and the operation of complex conjugation? Why is it that time reversal preserves position and reverses momentum and spin? This puzzle of time reversal in quantum mechanics has been with us since Wigner's first presentation. In this paper, I propose a new solution to this puzzle. First, it is shown that the standard account of time reversal can be derived based on the assumption that the probability current is reversed by the time reversal transformation. Next, this assumption is justified and the meaning of time reversal is clarified by analyzing the relationship between the rates of change and the instantaneous quantities which determine them. Finally, I explain how the new analysis help solve the puzzle of time reversal in quantum mechanics.

Keywords Quantum mechanics · Time reversal · Continuity equation · Probability current

1 Introduction

Why does time reversal involve two operations, a temporal reflection and the operation of complex conjugation in quantum mechanics? Why is it that time reversal preserves position and reverses momentum and spin? This puzzle of time reversal has been with us since Wigner's [1] first presentation, although some progress has been made to solve it recently (see, e.g. [2–4]). According to some authors, time reversal

✉ Shan Gao
gaoshan2017@sxu.edu.cn

1 Research Center for Philosophy of Science and Technology, Shanxi University, Taiyuan 030006, People's Republic of China

"can involve nothing whatsoever other than reversing the velocities of the particles" ([5], p. 20), and "It does not make sense to time-reverse a truly instantaneous state of a system" [6].[1] While according to others [9, 10, 3], this is not the case. In this paper, I will try to solve this puzzle of time reversal. I will first give a new derivation of the standard account of time reversal in quantum mechanics based on the assumption that the probability current is reversed by the time reversal transformation. Then, I will argue that this assumption can be justified by analyzing the relationship between the rates of change and the instantaneous quantities which determine them. Finally, I will explain how the new analysis help solve the puzzle of time reversal in quantum mechanics.

2 A New Derivation

Consider the Schrödinger equation for a spin-0 quantum system in an external scalar potential:

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r},t)\right]\psi(\mathbf{r},t), \tag{1}$$

where $\hbar$ is Planck's constant divided by 2π, $\psi(\mathbf{r},t)$ is the wave function of the system, m is the mass of the system, and $V(\mathbf{r},t)$ is an external scalar potential. From this equation we can derive the continuity equation:

$$\frac{\partial\rho(\mathbf{r},t)}{\partial t} + \nabla\cdot\mathbf{j}(\mathbf{r},t) = 0, \tag{2}$$

where $\rho(\mathbf{r},t) = |\psi(\mathbf{r},t)|^2$ and $\mathbf{j}(\mathbf{r},t) = \frac{\hbar}{2mi}[\psi^*(\mathbf{r},t)\nabla\psi(\mathbf{r},t) - \psi(\mathbf{r},t)\nabla\psi^*(\mathbf{r},t)]$ are probability density and probability current density, respectively.

Now I will show how the standard account of time reversal in quantum mechanics can be derived based on the assumption that the probability current is reversed by the time reversal transformation. First, according to this assumption we have $T\mathbf{j}(\mathbf{r},t) = -\mathbf{j}(\mathbf{r},-t)$, where T is the time reversal operator. Next, it can be argued that time reversal does not change the probability density. From a physical point of view, the probability density of finding a particle in certain position in space does not depend on the direction of time. Moreover, from a mathematical point of view, it can be proved that any transformation of $\rho(\mathbf{r},t)$, $F(\rho(\mathbf{r},t))$, which satisfies the nomalized condition $\int F(\rho(\mathbf{r},t))d\mathbf{r} = 1$ for any $\rho(\mathbf{r},t)$, must be an identity transformation.[2] Then, we have $T\rho(\mathbf{r},t) = \rho(\mathbf{r},-t)$. These two transformation rules ensure the time reversal invariance of the continuity equation.

[1] It has also been argued that the transformation referred to as 'time reversal' in quantum mechanics does not deserve the name, and it should be more appropriately described as motion reversal ([7], p.377; [8], p.266).

[2] I thank Phil Pearle and Rodi Tumulka for showing me a proof of this result.

By writing the wave function in the polar form $\psi = Re^{iS/\hbar}$, where R and S are real functions, we can obtain the following relation:

$$\mathbf{j}(\mathbf{r},t) = \frac{1}{m}\rho(\mathbf{r},t)\nabla S(\mathbf{r},t). \tag{3}$$

By using the transformation rules for $\rho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$, we have $TS(\mathbf{r},t) = -S(\mathbf{r},-t) + C_0$, where C_0 is a real constant. Then we can obtain the standard antiunitary transformation rule for the wave function: $T\psi(\mathbf{r},t) = \psi^*(\mathbf{r},-t)$ when ignoring an overall constant phase. Based on this transformation rule for the wave function, we can derive the transformation rule for every observable from its definition (or its operation on the wave function). For example, for position $\mathbf{r}$, we have $T\mathbf{r}T^{-1} = \mathbf{r}$, and for momentum $\mathbf{p} = -i\hbar\nabla$, we have $T\mathbf{p}T^{-1} = -\mathbf{p}$, and for angular momentum $\mathbf{L} = \mathbf{r}\times\mathbf{p}$, we have $T\mathbf{L}T^{-1} = -\mathbf{L}$.

In addition, by analyzing the probability current acceleration:

$$\frac{\partial \mathbf{v}(\mathbf{r},t)}{\partial t} = \frac{1}{m}[\nabla Q(\mathbf{r},t) - \nabla V(\mathbf{r},t)], \tag{4}$$

where $\mathbf{v}(\mathbf{r},t) = \frac{\mathbf{j}(\mathbf{r},t)}{\rho(\mathbf{r},t)}$ is the local velocity for the probability current, and $Q(\mathbf{r},t) = \frac{\hbar^2}{2m}\frac{\nabla^2 R(\mathbf{r},t)}{R(\mathbf{r},t)}$, we can obtain the transformation rule for the scalar potential: $TV(\mathbf{r},t) = V(\mathbf{r},-t)$. Notably this transformation rule applies to the electric scalar potential $T\phi(\mathbf{r},t) = \phi(\mathbf{r},-t)$. Using the definition $\mathbf{E} = -\nabla\boldsymbol{\phi}$, we can obtain the transformation rule for the electric field $T\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},-t)$. Furthermore, by analyzing the continous equation for a charged system in an electromagnetic field, we can also obtain the transformation rules for the magnetic potentials and fields. The probability current density for a spin-0 system with mass m and charge Q in an external electromagnetic field is

$$\mathbf{j}(\mathbf{r},t) = \frac{1}{m}\rho(\mathbf{r},t)[\nabla S(\mathbf{r},t) - Q\mathbf{A}(\mathbf{r},t)], \tag{5}$$

where $\mathbf{A}(\mathbf{r},t)$ is the magnetic vector potential. Then $T\mathbf{j}(\mathbf{r},t) = -\mathbf{j}(\mathbf{r},-t)$ requires $T\mathbf{A}(\mathbf{r},t) = -\mathbf{A}(\mathbf{r},-t)$. Using the definition $\mathbf{B} = \nabla\times\mathbf{A}$, we can obtain the transformation rule for the magnetic field $T\mathbf{B}(\mathbf{r},t) = -\mathbf{B}(\mathbf{r},-t)$.

Lastly, we can also obtain the time reversal transformation rule for spin in a similar way. The probability current for a spin-s system with mass m and charge Q and magnetic moment μ_s in an external electromagnetic field is

$$\begin{aligned}\mathbf{j}(\mathbf{r},t) =& \frac{1}{2m}[(\psi^*(\mathbf{r},t)\mathbf{p}\psi(\mathbf{r},t) - \psi(\mathbf{r},t)\mathbf{p}\psi^*(\mathbf{r},t)) - 2Q\mathbf{A}(\mathbf{r},t)\rho(\mathbf{r},t)]\\ &+ \frac{\mu_s}{s}\nabla\times(\psi^*(\mathbf{r},t)\mathbf{S}\psi(\mathbf{r},t)),\end{aligned} \tag{6}$$

where $\mathbf{S}$ is the spin operator. Then $T\mathbf{j}(\mathbf{r},t) = -\mathbf{j}(\mathbf{r},-t)$ requires $T\mathbf{S}(\mathbf{r},t) = -\mathbf{S}(\mathbf{r},-t)$. Based on the transformation rules for spin and the wave function, we can also derive the result $T^2 = -I$ for spin-1/2 systems.

The above analysis provides a new derivation of the standard time reversal transformation rules in quantum mechanics, which ensures that the Schrödinger equation is time reversal invariant. The analysis can be extended to relativistic quantum mechanics.

3 Understanding Time Reversal in Quantum Mechanics

In the following, I will argue that the assumption that the probability current is reversed by the time reversal transformation can be justified. I will also explain how the new analysis help solve the puzzle of time reversal in quantum mechanics.

First of all, this assumption is in accordance with our intuition that time reversal reverses the direction of a current. However, it should be pointed out that unlike the standard velocity in Newtonian mechanics, the probability current is not defined as the rate of change of an instantaneous configurational quantity; rather, it is also an instantaneous quantity, though not configurational. This means that one cannot directly determine the transformation rule for the probability current by its definition, and this is different from the situation of standard velocity in Newtonian mechanics.[3]

Next, as noted before, it has been debated whether an instantaneous quantity should be changed by time reversal. According to some authors, it does not make sense to time-reverse a truly instantaneous quantity [6], and time reversal can involve nothing other than reversing the rates of change of instantaneous quantities such as velocities of particles [5]. This is a nonstandard view of time reversal. On this view, time reversal will keep the probability current density, as well as the probability density, unchanged. Then, in the time-reversed world, when the net probability current flows into a volume, the probability in the volume does not increase but decrease. In other words, this nonstandard view violates the continuity equation.

Note that different from the Schrödinger equation for the wave function, the continuity equation for the probability density and current density has a direct physical meaning. It is a local and stronger form of the probability conservation law. A weak version of the probability conservation law says that the total probability of obtaining all possible results is one. The continuity equation says that when the probability density changes continuously or the probability current is continuous, the increase/decrease of the probability in a volume is equal to the net probability that flows into/out the volume. The probability conservation law is comprehensible, and its validity is justified by its physical meaning. The total probability of obtaining all possible results can only be one, not be any other value such as one third. And when the net probability current flows into a volume, the probability in the volume must increase

[3] Note that even though in the de Broglie-Bohm theory we can determine the transformation rule for the velocity of a Bohmian particle by its definition, which is assumed to be equal to the current velocity, we still need to resort to the time reversal invariance of the guiding equation to derive the standard transformation rule for the wave function. Then, why not directly assume the time reversal invariance of the Schrödinger equation? In my view, the de Broglie-Bohm theory does not help much in solving the puzzle of time reversal in quantum mechanics (cf. [11]).

and cannot decrease. This is a requirement of logic and definition. Thus it is arguable that the continuity equation should also be valid in the time-reversed world. In other words, the continuity equation should be time reversal invariant.

Then, what is wrong with the nonstandard view of time reversal? and why should an instantaneous quantity such as probability current be reversed by the time reversal transformation? In my view, when the rate of change (of a time reversal invariant quantity) is determined by an instantaneous quantity, this instantaneous quantity should also be reversed by the time reversal transformation as the rate of change. If this is not true, then the reversed rate of change cannot be explained in the time-reversed world.[4] The nonstandard view's violation of the continuity equation is just such a case. By the continuity equation, the change of the probability density over time is produced and determined by the probability current. If the rate of change of the probability density is time-reversed but the probability current is not time-reversed, then the change of the probability density over time cannot be explained in the time-reversed world, and the probability conservation law will also be violated. For example, in the time-reversed world, when the net probability current flows into a volume, the probability in the volume does not increase but decrease. Then the decrease of the probability in the volume cannot be explained, and the continuity equation is also violated. Note, however, that if an instantaneous quantity does not determine the rate of change of something invariant by time reversal, then it is arguable that this instantaneous quantity should not be reversed by time reversal, as the nonstandard view rightly holds.

The above analysis provides a possible way to solve the puzzle of time reversal in quantum mechanics. First, the probability density is arguably not changed by the time reversal transformation, and thus the rate of change of the probability density is reversed by the time reversal transformation. Next, since the rate of change of the probability density is determined by the probability current according to the continuity equation, it is arguable that the probability current should be reversed by the time reversal transformation. Then why time reversal involves complex conjugation is because the phase of the wave function is an integral of the probability current density (divided by the probability density) and time reversal reversing the probability current (and keeping the probability density unchanged) amounts to taking the complex conjugation of the wave function.[5] Moreover, why time reversal reverses momentum, spin, and magnetic fields is because these quantities appear in the probability current density, and reversing the probability current requires reversing them.

[4] This argument can also be used in Newtonian mechanics and Maxwell's theory of electromagnetism. For example, in Newtonian mechanics, not only the standard velocity, which is defined as the derivative of position with respect to time, but also the intrinsic velocity [12], which determines the standard velocity, should be reversed by the time reversal transformation.

[5] Several authors have given a similar account [2, 9, 13].

Finally, two points need to be emphasized. First, the above solution to the puzzle of time reversal is independent of the measurability of the probability current.[6] Moreover, the solution is also independent of the physical meaning of the wave function or the ontology of quantum mechanics (cf. [14]). Next, the density and current density in the continuity equation can be measured by protective measurements, and the continuity equation may thus have a deeper ontological meaning. According to the principle of protective measurement [15–18], when the wave function of a single quantum system is known, one can measure both ρ and $\mathbf{j}$ by a series of protective measurements on the system. Let the explicit form of the measured wave function at a given instant t be $\psi(x)$, and the measured observable A be (normalized) projection operators on small spatial regions V_n having volume v_n:

$$A = \begin{cases} \frac{1}{v_n}, \text{ if } & x \in V_n, \\ 0, \text{ if } & x \notin V_n. \end{cases} \tag{7}$$

A protective measurement of A then yields

$$\langle A \rangle = \frac{1}{v_n} \int_{V_n} |\psi(x)|^2 dv, \tag{8}$$

which is the average of the density $\rho(x) = |\psi(x)|^2$ over the small region V_n. Similarly, we can measure another observable $B = \frac{\hbar}{2mi}(A\nabla + \nabla A)$. The measurement yields

$$\langle B \rangle = \frac{1}{v_n} \int_{V_n} \frac{\hbar}{2mi}(\psi^* \nabla \psi - \psi \nabla \psi^*) dv = \frac{1}{v_n} \int_{V_n} j(x) dv. \tag{9}$$

This is the average value of the current density $j(x)$ in the region V_n. Then when $v_n \to 0$ and after performing measurements in sufficiently many regions V_n we can measure $\rho(x)$ and $j(x)$ everywhere in space. When assuming the psi-ontic view [19], $\rho(x)$ and $\mathbf{j}(x)$, when multiplied by the mass and charge of the system, can be explained as the mass and charge density and current density [20],[7] and the continuity equation can also be explained as the local form of the conservation law for mass and charge. This may provide further support for the validity of the continuity equation.

[6] There have been worries about the measurability of the probability current in the continuity equation. For example, Sakurai wrote, "we would like to caution the reader against a too literal interpretation of j as ρ times the velocity defined at every point in space, because a simultaneous precision measurement of position and velocity would necessarily violate the uncertainty principle." (Sakurai, 1996, pp.102, 103).

[7] Take $\rho(x)$ as an example. When assuming the psi-ontic view, the density $\rho(x) = |\psi(x)|^2$ will be a physical property of the measured system. Then, what density is the density $\rho(x)$? Since a measurement must always be realized by a certain physical interaction between the measured system and the measuring device, the density must be, in the first place, the density of a certain interacting charge. For instance, if the measured system is charged and the measurement is realized by an electrostatic interaction between the measured system and the measuring device, then the density multiplied by the charge of the measured system Q, namely $Q\rho(x)$, will be the charge density of the measured system in position x. Similarly, $Q\mathbf{j}$ will be the charge current density of the measured system in position x. A detailed analysis of the origin of the charge density and charge current density can be found in [20].

4 Conclusion

In this paper, I have argued that the standard account of time reversal in quantum mechanics can be derived based on the assumption that the probability current is reversed by the time reversal transformation. Moreover, this assumption is justified and the meaning of time reversal is clarified by analyzing the relationship between the rates of change and the instantaneous quantities which determine them. This analysis provides a new solution to the puzzle of time reversal in quantum mechanics. It remains to be seen whether this solution is fully satisfying and whether there are other better and complete solutions.

References

1. Wigner, E. P. (1931).: Group Theory and its Application to the Quantum Mechanics of Atomic Spectra. New York: Academic Press (1959)
2. Callender, C.: Quantum mechanics: keeping it real? Br. J. Philos. Sci. (2021). https://doi.org/10.1086/715032
3. Roberts, B.: Three myths about time reversal in quantum theory. Philos. Sci. **84**, 315–334 (2017)
4. Roberts, B. (2020). Time Reversal. The Routledge Companion to the Philosophy of Physics, Eleanor Knox and Alistair Wilson (eds)
5. Albert, D.Z.: Time and Chance. Harvard University Press, Cambridge, MA (2000)
6. Callender, C.: Is time 'Handed' in a quantum world? Proceedings of the Aristotelian Society **100**(3), 247–269 (2000)
7. Ballentine, L.E.: Quantum Mechanics: A Modern Development. World Scientific Publishing Company, Singapore (1998)
8. Sakurai, J.J.: Modern Quantum Mechanics, Revised Addison Wesley, Reading, MA (1994)
9. Earman, J.: What time reversal is and why it matters. Int. Stud. Philos. Sci. **16**(3), 245–264 (2002)
10. Malament, D.B.: On the time reversal invariance of classical electromagnetic theory. Stud. Hist. Philos. Mod. Phys. **35**, 295–315 (2004)
11. Allori, V., Goldstein, S., Tumulka, R., Zanghì, N.: On the common structure of Bohmian mechanics and the Ghirardi-Rimini-Weber theory, British Journal for the Philosophy of Science 59 (3), 353. Section **4**, 2 (2008)
12. Tooley, M.: In defence of the existence of states of motion. Philos. Top. **16**, 225–254 (1988)
13. Sebens, C.: Quantum mechanics as classical physics. Philos. Sci. **82**(2), 266–291 (2015)
14. Struyve, W. (2020). Time-reversal invariance and ontology. https://philsci-archive.pitt.edu/17682/
15. Aharonov, Y., Anandan, J., Vaidman, L.: Meaning of the wave function. Phys. Rev. A **47**, 4616 (1993)
16. Aharonov, Y., Vaidman, L.: Measurement of the Schrödinger wave of a single particle. Phys. Lett. A **178**, 38 (1993)
17. Gao, S. (ed.): Protective Measurement and Quantum Reality: Toward a New Understanding of Quantum Mechanics. Cambridge University Press, Cambridge (2015)
18. Piacentini, F., et al.: Determining the quantum expectation value by measuring a single photon. Nat. Phys. **13**, 1191 (2017)
19. Pusey, M., Barrett, J., Rudolph, T.: On the reality of the quantum state. Nat. Phys. **8**, 475–478 (2012)
20. Gao, S.: The Meaning of the Wave Function. In Search of the Ontology of Quantum Mechanics. Cambridge University Press, Cambridge (2017)