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Quantum Statistics of Identical Particles

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Abstract
The empirical rule that systems of identical particles always obey either Bose or Fermi statistics is customarily imposed on the theory by adding it to the axioms of nonrelativistic quantum mechanics, with the result that other statistical behaviors are excluded a priori. A more general approach is to ask what other many-particle statistics are consistent with the indistinguishability of identical particles. This strategy offers a way to discuss possible violations of the Pauli Exclusion Principle, and it leads to some interesting issues related to preparation of states and a superselection rule arising from invariance under the permutation group.

Keywords Many-body statistics · Nonrelativistic quantum theory · Pauli Exclusion Principle · Quantum state preparation · Superselection rule

Mathematics Subject Classification 20C35 · 20B30 · 20C30

1 Introduction

The strong empirical evidence that all systems of identical particles obey either Bose or Fermi statistics is one of the most striking features of quantum physics. An associated theoretical puzzle of long standing is that these quantum-statistical properties of many-particle systems cannot be derived from the axioms of quantum mechanics; instead, they must be imposed by an additional assumption. In the seminal paper of Messiah and Greenberg [1] this is expressed as follows:

Symmetrization Postulate (SP): State vectors describing several identical particles are either symmetric (bosons) or antisymmetric (fermions) under permutations of the particle labels.

The familiar version of quantum mechanics—called SPQM below—is defined by adding the Symmetrization Postulate to the axioms of nonrelativistic quantum mechanics, [2, Chap. IIIb]. For N identical particles this means that the Hilbert

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space of states is either $\mathfrak{H}_B^{(N)}$(symmetric state vectors for bosons) or $\mathfrak{H}_F^{(N)}$ (antisymmetric state vectors for fermions).

The SPQM version of quantum theory has been phenomenally successful, but even successful theories should be subjected to periodic experimental tests. Any rule, e.g. the Pauli Exclusion Principle (PEP)–that imposes strict conditions on predictions of experimental results should be retested as experimental techniques improve. This empirical motivation, combined with the lack of any convincing theoretical explanation of the Symmetrization Postulate, has stimulated a number of experimental searches for possible SP-violations [3–10]. In view of the continuing interest in these issues, a review of a more general theoretical approach may lead to useful insights. In order to limit the theoretical possibilities this generalization might allow, the following discussion will be confined to a minimal extension of SPQM that combines the idea of indistinguishability of identical particles with the axioms of quantum mechanics [1].

In what follows, a hermitian operator, A, that represents a measurable physical quantity is called an *observable*, and the set of its eigenvalues is denoted by $ev(A)$. In general there can be hermitian operators that do not represent any physical quantity; therefore, a condition picking out those hermitian operators that may be observables must be given for each physical system. An *outcome*, $\left(a\,, \mathcal{E}_a(A)\right)$, of a measurement of A consists of the measured value, $a \in ev(A)$, and the associated eigenspace, $\mathcal{E}_a(A)$, with the basis set,

$$\begin{aligned}\mathfrak{B}\left[\mathcal{E}_a(A)\right] &:= \{|a : \mu\rangle \mid A|a : \mu\rangle \\ &= a|a : \mu\rangle,\ 1 \leq \mu \leq d_a \text{ (degeneracy of } a)\big\}.\end{aligned} \tag{1}$$

For a set, $\boldsymbol{A} := \left\{A_1, \dots, A_J\right\}$, of compatible (commuting) observables the outcome, $\left(\boldsymbol{a}\,, \mathcal{E}_{\boldsymbol{a}}(\boldsymbol{A})\right)$, of a joint measurement consists of the joint measured values, $\boldsymbol{a} = \left(a_1, \dots, a_J\right)$, and the joint eigenspace,

$$\mathcal{E}_{\boldsymbol{a}}(\boldsymbol{A}) := \bigcap_{j=1}^{J} \mathcal{E}_{a_j}\left(A_j\right), \quad \dim \mathcal{E}_{\boldsymbol{a}}(\boldsymbol{A}) = d_{\boldsymbol{a}} \text{ (degeneracy of } \boldsymbol{a}). \tag{2}$$

If the purpose of a measurement is to leave the system in a unique pure state corresponding to a specific eigenvalue, it is called a preparation of the state. Since this procedure involves rejecting outcomes with other eigenvalues, it is also called filtering.

In Sect. 2 the notion of identical particles is reviewed, together with a description of systems of identical particles. A definition of the indistinguishability of identical particles is presented in Sect. 3 and used to formulate an extension of SPQM that permits violations of the Symmetrization Postulate. The special quantum features of systems of indistinguishable particles are presented in Sect. 4. Experiments seeking PEP-violations are briefly discussed in Sect. 5, together with a toy model that illustrates which features of IPQM would be involved in their analysis. A general discussion is presented in Sect. 6.

2 Identical Particles

An 'elementary particle' is currently understood to be one of the objects listed in the Standard Model. Some bound states of the elementary particles, e.g. nucleons, nuclei, atoms, and molecules, can also be regarded as particles as long as they do not experience interactions strong enough to excite their internal degrees of freedom. Each of these particles is identified by a finite set of *intrinsic properties* that are the values of measurements on a single particle, e.g. mass, charge and spin. This identification is assumed to be complete in the strong sense that there are no additional single-particle measurements that could further define the particle or distinguish between two particles with the same intrinsic properties. Two particles are *identical* if they have the same intrinsic properties, and identical particles belong to the same *species* [11]. Each of these particles—with appropriate caveats for quarks and gluons—can exist in the vacuum isolated from all other particles. This property distinguishes them from those excited states of many-particle systems that are called quasiparticles. Systems of quasiparticles can display exotic forms of statistics [12, 13] , but the present paper is intended to apply only to statistics arising from the indistinguishability of identical particles.

The axioms of quantum theory do not provide any special rules for systems of identical particles, but they do provide a framework for studying their special properties. For this purpose, it is sufficient to consider a system composed of only one particle species. The single-particle state space, $\mathfrak{H}^{(1)}_{spc}$, is defined by a basis set, $\mathfrak{B}\left[\mathfrak{H}^{(1)}_{spc}\right] = \left\{|\theta_1\rangle, |\theta_2\rangle, \dots\right\}$, where each state identifier, θ, is a set of quantum numbers appropriate to the particle species in question, e.g. $\theta = (\boldsymbol{k}, s)$ where $\boldsymbol{k}$ is a wave number and $s\hbar$ is an eigenvalue of the z-component of the spin. With the assignment of distinct *particle labels*, $n = 1, 2, \dots, N$, to the individual particles, the single-particle state space, $\mathfrak{H}^{(1)}_n$, for particle n is a copy of $\mathfrak{H}^{(1)}_{spc}$, and single-particle state vectors in $\mathfrak{H}^{(1)}_n$ are written as $|\psi\rangle_n$. Since no assumption about the behavior of state vectors under permutations of the particle labels has been made, the N-particle state space, $\mathfrak{H}^{(N)}$, is the full tensor product of the single-particle state spaces, $\mathfrak{H}^{(N)} = \mathfrak{H}^{(1)}_1 \otimes \cdots \otimes \mathfrak{H}^{(1)}_N$, with the basis set,

$$\begin{aligned} &\mathfrak{B}\left[\mathfrak{H}^{(N)}\right] = \left\{|\boldsymbol{\theta}\rangle := |\theta_{1'}\rangle_1 \otimes \cdots \otimes |\theta_{N'}\rangle_N \mid \boldsymbol{\theta} := \left(\theta_{1'}, \dots, \theta_{N'}\right)\right\}, \\ &\langle \boldsymbol{\kappa}|\boldsymbol{\theta}\rangle := \left\langle \kappa_{1'}|\theta_{1'}\right\rangle_1 \cdots \left\langle \kappa_{N'}|\theta_{N'}\right\rangle_N = \delta_{\boldsymbol{\kappa}\boldsymbol{\theta}} := \delta_{\kappa_{1'}\theta_{1'}} \cdots \delta_{\kappa_{N'}\theta_{N'}}. \end{aligned} \tag{3}$$

The ordering convention is that the tensor products are always written in increasing order of the particle labels. The primed subscripts $\left\{1', \dots, N'\right\}$ in $\boldsymbol{\theta}$ are called *state labels*, since n' in $|\theta_{n'}\rangle_n$ identifies the single-particle state that is assigned to particle n.

The $N!$ different assignments of labels to particles are related to each other by the Symmetric Group, $\mathcal{S}_N$, composed of permutations, $n \to \mathcal{P}(n)$, of the labels $\{1, \dots, N\}$. For any pair of permutations $\mathcal{K}$ and $\mathcal{P}$ the product rule for the group $\mathcal{S}_N$ is $(\mathcal{K}\mathcal{P})(n) = \mathcal{K}(\mathcal{P}(n))$. Since the particle labels have no physical significance, a permutation of the particle labels, with the state labels held fixed, is a passive transformation (*passive permutation*) analogous to a rotation of the coordinate axes with the physical system held fixed. A passive permutation, $\mathcal{P}$, acts on a basis vector $|\boldsymbol{\theta}\rangle$ in (3) by first permuting the particle labels–$|\theta_{n'}\rangle_n \to |\theta_{n'}\rangle_{\mathcal{P}(n)}$—and then arranging the tensor product of the $|\theta_{n'}\rangle_{\mathcal{P}(n)}$'s in increasing order of $\mathcal{P}(n)$. The combination of these two operations is equivalent to leaving the particle labels unchanged while performing the inverse permutation of the state labels:

$$\mathcal{P} : |\theta_{1'}\rangle_1 \otimes \cdots \otimes |\theta_{N'}\rangle_N \to \left|\theta_{\mathcal{P}^{-1}(1')}\right\rangle_1 \otimes \cdots \otimes \left|\theta_{\mathcal{P}^{-1}(N')}\right\rangle_N. \tag{4}$$

A permutation of the state labels in $\boldsymbol{\theta} = \left(\theta_{1'}, \ldots, \theta_{N'}\right)$, with the particle labels held fixed, is an *active permutation*, analogous to a rotation of the physical system with the coordinate axes held fixed. Each active permutation $\mathcal{P}$ acts on $\mathfrak{H}^{(N)}$ by an operator, $D(\mathcal{P})$, defined by:

$$D(\mathcal{P})|\boldsymbol{\theta}\rangle := \left|\widetilde{\mathcal{P}}(\boldsymbol{\theta})\right\rangle \text{ for all } \boldsymbol{\theta}, \quad \text{where } \widetilde{\mathcal{P}}(\boldsymbol{\theta}) := \left(\theta_{\mathcal{P}(1')}, \ldots, \theta_{\mathcal{P}(N')}\right). \tag{5}$$

A straightforward argument shows that $D(\mathcal{P})$ is a unitary operator that satisfies,

$$D(\mathcal{P})D(\mathcal{K}) = D(\mathcal{P}\mathcal{K}), \tag{6}$$

so that $\mathcal{P} \to D(\mathcal{P})$ is a unitary representation of $\mathcal{S}_N$ with carrier space $\mathfrak{H}^{(N)}$. Comparing (5) to (4) shows that the effect of a passive permutation can be expressed as

$$\mathcal{P} : |\boldsymbol{\theta}\rangle \to D\left(\mathcal{P}^{-1}\right)|\boldsymbol{\theta}\rangle, \tag{7}$$

i.e. the passive permutation $\mathcal{P}$ of the particle labels has the same effect as the active permutation $\mathcal{P}^{-1}$ of the state labels.

3 Indistinguishability of Identical Particles

The assumption that there are no additional single-particle measurements that can distinguish between two identical particles is the source of the fundamental intuition that merely relabelling identical particles cannot change any measurement outcomes. This means that two N-particle state vectors related by a passive permutation of the particle labels must yield the same probability distributions for the measured values of all observables. Combining this with the relation (7) between passive and active permutations suggests that the idea of the indistinguishability of identical particles is captured by the following:

Indistinguishability Postulate: Two state vectors of several identical particles that differ only by an active permutation of state labels yield the same probability distributions for measurements of all observables.

This statement applies to all measurements on the entire system, not to single-particle measurements on isolated particles. This is the essential difference between indistinguishability and identity of particles. The Indistinguishability Postulate and the Symmetrization Postulate are both consistent with the axioms of quantum theory, but neither is derivable from them. The version of quantum mechanics defined by adding the Indistinguishability Postulate to the axioms will be called IPQM. By virtue of its use of the word 'observables,' the Indistinguishability Postulate itself implies the rule that determines which hermitian operators may be counted as observables.

For any state $|\Psi\rangle$ the probability of the measurement outcome, $\left(a\,,\mathcal{E}_a(A)\right)$, for A is,

$$\begin{aligned} prob(a|\Psi) &= \sum_{\mu=1}^{d_a} |\langle a\,:\,\mu|\Psi\rangle|^2 \\ &= \sum_{\mu=1}^{d_a} \langle\Psi|a\,:\,\mu\rangle\langle a\,:\,\mu|\Psi\rangle \\ &= \left\langle\Psi\middle|\Pi\left(\mathcal{E}_a(A)\right)\middle|\Psi\right\rangle, \end{aligned} \tag{8}$$

where $\Pi\left(\mathcal{E}_a(A)\right)$ is the projection operator onto $\mathcal{E}_a(A)$. For any observable A; any eigenvalue $a \in ev(A)$; any pure state $|\Psi\rangle$; and any active permutation;

$$|\Psi\rangle \rightarrow \left|\Psi_{\mathcal{P}}\right\rangle = D(\mathcal{P})|\Psi\rangle; \tag{9}$$

the Indistinguishability Postulate requires that the probability of finding a in a measurement of A is the same for the original and the permuted state,

$$prob(a|\Psi) = prob\left(a|\Psi_{\mathcal{P}}\right). \tag{10}$$

Applying (8) and (9) to both sides of this equation leads to,

$$\begin{aligned} &\left\langle\Psi\middle|\Pi\left(\mathcal{E}_a(A)\right)\middle|\Psi\right\rangle \\ &\quad= \left\langle\Psi_{\mathcal{P}}\middle|\Pi\left(\mathcal{E}_a(A)\right)\middle|\Psi_{\mathcal{P}}\right\rangle \\ &\quad= \left\langle\Psi\middle|D(\mathcal{P})^{\dagger}\Pi\left(\mathcal{E}_a(A)\right)D(\mathcal{P})\middle|\Psi\right\rangle, \end{aligned} \tag{11}$$

for all $|\Psi\rangle \in \mathfrak{H}^{(N)}$ and all $\mathcal{P} \in \mathcal{S}_N$. Two operators with the same expectation value in all states are equal; therefore, (11) imposes a condition on the projection operator $\Pi\left(\mathcal{E}_a(A)\right)$ itself,

$$\begin{aligned} &\Pi\left(\mathcal{E}_a(A)\right) \\ &\quad= D(\mathcal{P})^{\dagger}\Pi\left(\mathcal{E}_a(A)\right)D(\mathcal{P}) \\ &\quad= \sum_{\mu=1}^{d_a} D(\mathcal{P})^{\dagger}|a\,:\,\mu\rangle\langle a\,:\,\mu|D(\mathcal{P}), \end{aligned} \tag{12}$$

for all $\mathcal{P} \in \mathcal{S}_N$. Since $D(\mathcal{P})$ is unitary, the transformed operator, $A_{\mathcal{P}} := D(\mathcal{P})^{\dagger}AD(\mathcal{P})$, has the same eigenvalues as A, and the corresponding transformed eigenvectors are $|\mathcal{P};a\,:\,\mu\rangle := D(\mathcal{P})^{\dagger}|a\,:\,\mu\rangle$. Substituting this into (12) yields

$$\Pi\left(\mathcal{E}_a(A)\right) = \sum_{\mu=1}^{d_a} |\mathcal{P};a\,:\,\mu\rangle\langle\mathcal{P};a\,:\,\mu| = \Pi\left(\mathcal{E}_a\left(A_{\mathcal{P}}\right)\right). \tag{13}$$

The operators A and $A_{\mathcal{P}}$ have the same eigenvalues and the same eigenspaces; therefore, they are equal. Thus every observable A must satisfy, $A = A_{\mathcal{P}} = D(\mathcal{P})^{\dagger} A D(\mathcal{P})$, which is equivalent to,

$$[A, D(\mathcal{P})] = 0, \quad \text{for all } \mathcal{P} \in \mathcal{S}_N. \tag{14}$$

An operator A that satisfies this condition is said to be *permutation-invariant*. Thus the Indistinguishability Postulate yields the,

Permutation-Invariance Rule: Observables for a system of identical particles must be permutation-invariant.

This is a necessary–but not sufficient–condition for a hermitian operator to be an observable. An equivalent statement is: A hermitian operator that does not satisfy (14) cannot be an observable. It is important to remember that there can be hermitian operators that satisfy (14) but are not observables. These cases depend on particular properties of the physical system under consideration. An important general consequence of (14) is that for systems of identical particles there are no observables that act on a single particle or any proper subset of particles. Every observable must act on the entire system.

The behavior of state vectors under permutations is an equally important feature of quantum theory for identical particles. A subspace $\mathfrak{N}$ of $\mathfrak{H}^{(N)}$ is called permutation-invariant if $D(\mathcal{P}) : \mathfrak{N} \to \mathfrak{N}$ for all $\mathcal{P} \in \mathcal{S}_N$, or equivalently if $\Pi(\mathfrak{N})$ is a permutation-invariant operator. Consequently, the representation $\mathcal{P} \to D(\mathcal{P})$ on $\mathfrak{H}^{(N)}$ induces a representation of S_N on every permutation-invariant subspace of $\mathfrak{H}^{(N)}$. Since observables are permutation-invariant operators, each eigenspace of an observable is a permutation-invariant subspace and thus a carrier space for a representation of S_N. The expression (2) shows that a joint eigenspace of compatible observables is also a carrier space for a representation of $\mathcal{S}_N$. A permutation-invariant subspace is called irreducible if it has no proper permutation-invariant subspaces,[1] and reducible otherwise. A reducible subspace can be expressed as the direct sum of irreducible subspaces [14, Chap. 3-13]. An irreducible (reducible) subspace carries an irreducible (reducible) representation of $\mathcal{S}_N$ [14, Chap. 7]. The irreducible representations of $\mathcal{S}_N$ are denoted by $\mathcal{P} \to \mathfrak{D}^{(\gamma)}(\mathcal{P})$—or simply $\mathfrak{D}^{(\gamma)}$–where $\mathfrak{D}^{(\gamma)}$ is a $d_\gamma \times d_\gamma$ unitary matrix and d_γ is the dimension of the representation. The index γ runs over the finite set Γ_N defined in "Appendix 1".

4 Quantum Mechanics for Indistinguishable Particles

In the SPQM version of quantum mechanics the state space, $\mathfrak{H}_{SP}^{(N)}$, for N identical particles is either $\mathfrak{H}_B^{(N)}$ or $\mathfrak{H}_F^{(N)}$; consequently, every $|\Psi\rangle \in \mathfrak{H}_{SP}^{(N)}$ satisfies $D(\mathcal{P})|\Psi\rangle = \sigma_{\mathcal{P}}|\Psi\rangle$, where $\sigma_{\mathcal{P}} = 1$ for bosons and $\sigma_{\mathcal{P}} = (+1, -1)$ as $\mathcal{P}$ is (even, odd) for fermions. Since the operators in quantum theory act on the Hilbert space of states—i.e. they send the Hilbert space into itself–every operator A acting on $\mathfrak{H}_{SP}^{(N)}$ satisfies, $D(\mathcal{P})A|\Psi\rangle = \sigma_{\mathcal{P}} A|\Psi\rangle$, which in turn yields,

$$[D(\mathcal{P}), A]|\Psi\rangle = D(\mathcal{P})A|\Psi\rangle - AD(\mathcal{P})|\Psi\rangle = 0. \tag{15}$$

[1] A subspace is proper if it is neither the whole space nor the null subspace consisting of the zero vector alone.

Thus in SPQM all hermitian operators acting on $\mathfrak{H}_{SP}^{(N)}$ automatically satisfy the permutation-invariance condition () required for observables.

By contrast, in the IPQM version of quantum mechanics the state space is the complete tensor product $\mathfrak{H}^{(N)}$ of the single-particle spaces; no symmetry conditions are imposed on the state vectors. Instead, the permutation-invariance condition (14) is used to pick out the hermitian operators that may be observables. Thus in IPQM there are always hermitian operators acting on $\mathfrak{H}^{(N)}$ that do not satisfy (14) and cannot be observables.

4.1 Complete Sets of Compatible Observables

In quantum mechanics for distinguishable particles a *complete set of commuting operators* (CSCOP) is a finite set, $\boldsymbol{C} = \left\{C_1, \ldots, C_K\right\}$, of mutually commutative hermitian operators for which each joint eigenvalue is nondegenerate, so that each joint eigenspace is one-dimensional [2, Chap. IID3b]. In specific applications a CSCOP is usually constructed from the relevant observables, e.g. $\boldsymbol{C} = \left\{H, L^2, L_z\right\}$ for a scalar particle moving in a central potential. Sets of hermitian operators satisfying this definition can be constructed in many ways, but it is always implicitly assumed that a CSCOP can be formed with observables. This is an important issue for applications to state preparation.

In IPQM any attempt to apply the idea of a CSCOP to a system of indistinguishable particles encounters the serious difficulty that there are no CSCOPs. To see why, assume that a set, $\boldsymbol{A}$, of observables is a CSCOP, then every eigenspace of $\boldsymbol{A}$ must be one-dimensional. On the other hand, each eigenspace of $\boldsymbol{A}$ is a carrier space for a representation of $\mathcal{S}_N$, and only the symmetric and antisymmetric representations are one-dimensional, c.f. "Appendix 1". Thus the unique basis vector, $|\boldsymbol{a}\rangle$, for each eigenspace would have to belong either to $\mathfrak{H}_B^{(N)}$ or to $\mathfrak{H}_F^{(N)}$. The eigenstates defined by a CSCOP are supposed to form a basis set for $\mathfrak{H}^{(N)}$; consequently, every vector $|\Psi\rangle$ in $\mathfrak{H}^{(N)}$ would be of the form,

$$|\Psi\rangle = |\Psi : B\rangle + |\Psi : F\rangle, \tag{16}$$

where $|\Psi : B\rangle \in \mathfrak{H}_B^{(N)}$ and $|\Psi : F\rangle \in \mathfrak{H}_F^{(N)}$. This is true for $N = 2$, but false for $N \geq 3$. If $\mathcal{P}$ is an even permutation, then $D(\mathcal{P})$ would leave all vectors satisfying (16) invariant; consequently, any $|\Psi\rangle \in \mathfrak{H}^{(N)}$ that is not invariant under even permutations is a counter example to (16). For $N \geq 3$ suppose the components of $\boldsymbol{\theta} = \left(\theta_{1'}, \ldots, \theta_{N'}\right)$ are all distinct, then the basis vector $|\boldsymbol{\theta}\rangle$ is not invariant under even permutations, or indeed any permutations at all; therefore, for $N \geq 3$ there are no CSCOPs for systems of indistinguishable particles described by IPQM. On the other hand, this argument does not apply to SPQM, since all state vectors are in either $\mathfrak{H}_B^{(N)}$or $\mathfrak{H}_F^{(N)}$ to begin with. Thus CSCOPs are not forbidden in SPQM.

The physical necessity of preparing states by measurements of observables means that any replacement for the CSCOP idea will still involve joint measurements of some finite set, $\boldsymbol{A} = \left\{A_1, \ldots, A_K\right\}$, of compatible observables. The definition of the joint eigenspace, $\mathcal{E}_{\boldsymbol{a}}(\boldsymbol{A})$, in (2) guarantees that

$$\dim\left[\mathcal{E}_{\boldsymbol{a}}(\boldsymbol{A})\right] \leq \dim\left[\mathcal{E}_{a_k}\left(A_k\right)\right] \quad k=1,\ldots,K. \tag{17}$$

With this in mind, it is useful to restrict the observables in $\boldsymbol{A}$ by requiring that at least one of them, say A_1, has only finite degeneracies, i.e. every eigenspace of A_1 is finite-dimensional. When this requirement is satisfied, the joint eigenspaces $\mathcal{E}_{\boldsymbol{a}}(\boldsymbol{A})$ are all finite dimensional carrier spaces for reducible or irreducible representations of $\mathcal{S}_N$.

Irreducibility is automatic for nondegenerate eigenvalues, since the eigenspaces are one-dimensional. If $\boldsymbol{a}$ is degenerate and $\mathcal{E}_{\boldsymbol{a}}(\boldsymbol{A})$ is irreducible, a natural question is whether the $d_{\boldsymbol{a}}$-fold degeneracy can be reduced by enlarging $\boldsymbol{A}$ to $\left\{A_1,\ldots,A_K,Z\right\}$, where Z is a permutation-invariant hermitian operator that commutes with all members of $\boldsymbol{A}$. Since $\mathcal{E}_{\boldsymbol{a}}(\boldsymbol{A})$ is invariant under Z, there will be permutation-invariant eigenspaces, $\mathcal{E}_z(Z)$, that are subspaces of $\mathcal{E}_{\boldsymbol{a}}(\boldsymbol{A})$, but the irreducible space $\mathcal{E}_{\boldsymbol{a}}(\boldsymbol{A})$ does not contain any proper, permutation-invariant subspaces. This leaves two possibilities: either $\mathcal{E}_z(Z)=\{\mathbf{0}\}$, or $\mathcal{E}_z(Z)=\mathcal{E}_{\boldsymbol{a}}(\boldsymbol{A})$. The first is trivial and the second means that every vector in $\mathcal{E}_{\boldsymbol{a}}(\boldsymbol{A})$ is an eigenvector of Z with a common eigenvalue z. Thus for any permutation-invariant, hermitian operator Z that is compatible with $\boldsymbol{A}$,

$$\left\langle \boldsymbol{a} : \mu|Z|\boldsymbol{a} : \mu'\right\rangle = z\,\delta_{\mu\mu'} \quad \left(\mu,\mu'=1,\ldots,d_{\boldsymbol{a}}\right). \tag{18}$$

Therefore, no measurement of an observable compatible with all the observables in $\boldsymbol{A}=\left\{A_1,\ldots,A_K\right\}$ can distinguish between the pure states in an irreducible eigenspace of $\boldsymbol{A}$. This is the physical significance of irreducibility with respect to the Symmetric Group.

If $\boldsymbol{a}$ is degenerate and $\mathcal{E}_{\boldsymbol{a}}(\boldsymbol{A})$ is reducible, the $d_{\boldsymbol{a}}$-fold degeneracy can be partially resolved by expressing $\mathcal{E}_{\boldsymbol{a}}(\boldsymbol{A})$ as the direct sum of irreducible subspaces,

$$\mathcal{E}_{\boldsymbol{a}}(\boldsymbol{A}) = \bigoplus_{i=1}^{n_a}\mathcal{E}_{\boldsymbol{a}}^{(i)}(\boldsymbol{A}), \quad d_{\boldsymbol{a}} = \sum_{i=1}^{n_a} d_{\boldsymbol{a}i}, \tag{19}$$

where $\mathcal{E}_{\boldsymbol{a}}^{(i)}(\boldsymbol{A})$ is a carrier space for an an irreducible representation $\mathfrak{D}^{(\gamma_i)}$. According to (18) it is not possible to resolve the irreducible eigenspaces $\mathcal{E}_{\boldsymbol{a}}^{(i)}(\boldsymbol{A})$ by measuring any compatible observable, but it is possible to label them by using the permutation-invariant, hermitian operator,

$$Z := \sum_{\boldsymbol{a}\in ev(\boldsymbol{A})}\sum_{i=1}^{n_a} z_{\boldsymbol{a}i}\Pi\left(\mathcal{E}_{\boldsymbol{a}}^{(i)}(\boldsymbol{A})\right), \tag{20}$$

where the $z_{\boldsymbol{a}i}$'s are real and distinct. Every vector $|\psi\rangle$ in each irreducible subspace $\mathcal{E}_{\boldsymbol{a}}^{(i)}(\boldsymbol{A})$ satisfies $Z|\psi\rangle = z_{\boldsymbol{a}i}|\psi\rangle$; therefore, every eigenspace of the extended set, $\left\{A_1,\ldots,A_K,Z\right\}$, is irreducible.

This argument establishes the existence of a permutation-invariant, hermitian operator that ensures irreducibility for every eigenspace of the extended set, but there is no guarantee that this operator represents a measurable quantity. Just as in the case of CSCOPs, the existence of an observable that has the same effect as Z must be assumed. When this is true, the extended set is an example of a

Complete Set of Compatible Observables (CSCOB), which is defined as a collection, $\boldsymbol{A} = \{A_1, \ldots, A_J\}$, of compatible observables for which the eigenspace in each outcome, $(\boldsymbol{a}, \mathcal{E}_{\boldsymbol{a}}(\boldsymbol{A}))$, of a joint measurement of $\boldsymbol{A}$ carries an irreducible representation $\mathfrak{D}^{(\gamma_{\boldsymbol{a}})}$ of $\mathcal{S}_N$. The irreducible eigenspaces obtained by measuring a CSCOB play the role of the pure states obtained by measurement of a CSCOP for distinguishable particles.

The eigenvalue set $ev(\boldsymbol{A})$ for a CSCOB naturally decomposes into subsets labelled by $\gamma \in \Gamma_N$,

$$ev(\boldsymbol{A}, \gamma) := \{\boldsymbol{a} \in ev(\boldsymbol{A}) \mid \gamma_{\boldsymbol{a}} = \gamma\}, \tag{21}$$

i.e. for $\boldsymbol{a} \in ev(\boldsymbol{A}, \gamma)$ the eigenspace $\mathcal{E}_{\boldsymbol{a}}(\boldsymbol{A})$ is a carrier space for the irreducible representation $\mathfrak{D}^{(\gamma)}$, with the basis set

$$\mathfrak{B}\left[\mathcal{E}_{\boldsymbol{a}}(\boldsymbol{A})\right] = \{|\gamma, \boldsymbol{a} : g\rangle \mid g = 1, \ldots, d_\gamma\}, \tag{22}$$

where the basis vectors transform by,

$$D(\mathcal{P})|\gamma, \boldsymbol{a} : g\rangle = \sum_{g'=1}^{d_\gamma} |\gamma, \boldsymbol{a} : g'\rangle \mathfrak{D}^{(\gamma)}_{g'g}(\mathcal{P}). \tag{23}$$

A superposition (mixture) of basis vectors $|\gamma, \boldsymbol{a} : g\rangle$ with a common value of γ is said to be a pure (mixed) state of *symmetry type* γ. Superpositions (mixtures) of states of several distinct symmetry types are called *hybrid-symmetry* pure (mixed) states. Every $|\Psi\rangle \in \mathfrak{H}^{(N)}$ can be expressed as a superposition of states of different symmetry types,

$$|\Psi\rangle = \sum_{\gamma \in \Gamma_N} C_\gamma |\Psi^\gamma\rangle; \quad \text{where } |\Psi^\gamma\rangle := \sum_{\boldsymbol{a} \in ev(\boldsymbol{A},\gamma)} \sum_{g=1}^{d_\gamma} \Psi^{\gamma \boldsymbol{a} g} |\gamma, \boldsymbol{a} : g\rangle, \tag{24}$$

$$\langle \Psi^\gamma | \Psi^\gamma \rangle = \sum_{\boldsymbol{a} g} |\Psi^{\gamma \boldsymbol{a} g}|^2 = 1, \quad \text{and} \quad \langle \Psi | \Psi \rangle = \sum_\gamma \left| C_\gamma \right|^2 = 1. \tag{25}$$

4.2 State Preparation

In SPQM a pure state $|\Psi\rangle$ that is a unique eigenvector of a set of compatible observables is said to be a *preparable pure state*. Preparation of states is, therefore, an important application of CSCOPs that are composed of observables. If a physical system has such a CSCOP and $|\Psi\rangle$ is one of unique eigenvectors of $\boldsymbol{C}$, then $|\Psi\rangle$ is prepared by measuring $\boldsymbol{C}$ and accepting the outcome with the joint eigenvalue corresponding to $|\Psi\rangle$. Since systems of indistinguishable particles under IPQM do not support CSCOPs, this description of state preparation has to be worked out anew.

A measurement of a set, $\boldsymbol{A}$, of compatible observables leaves the system in one of the eigenspaces of $\boldsymbol{A}$. If this eigenspace happens to be one-dimensional, then the unique basis vector $|\boldsymbol{a}\rangle$ is a preparable pure state. For indistinguishable particles, the one-dimensional eigenspaces of $\boldsymbol{A}$ carry the symmetric or antisymmetric representation of $\mathcal{S}_N$; therefore, all preparable pure states are either in $\mathfrak{H}_B^{(N)}$ or in $\mathfrak{H}_F^{(N)}$. This severe restriction of the set of preparable pure states suggests that in IPQM the idea of state preparation should be extended to mixed states described by density operators. Since observables are permutation-invariant, every observable Y satisfies,

$$Tr[\varrho Y] = Tr\left[\varrho D^{\dagger}(\mathcal{P}) Y D(\mathcal{P})\right] = Tr\left[D(\mathcal{P})\varrho D^{\dagger}(\mathcal{P}) Y\right], \tag{26}$$

for any density operator ϱ and all $\mathcal{P} \in \mathcal{S}_N$. Summing both sides of this equation over $\mathcal{P} \in \mathcal{S}_N$ yields $Tr[\varrho Y] = Tr[\bar{\rho} Y]$, where,

$$\bar{\rho} := \frac{1}{N!} \sum_{\mathcal{P}\in\mathcal{S}_N} D(\mathcal{P})\rho D^{\dagger}(\mathcal{P}), \tag{27}$$

is a permutation-invariant density operator. Thus no generality is lost by requiring physical density operators to be permutation-invariant.

Let $\boldsymbol{A}$ be a CSCOB, then a measurement with outcome $\left(\boldsymbol{a}, \mathcal{E}_{\boldsymbol{a}}(\boldsymbol{A})\right)$ leaves the system in a state described by a density operator of the form,

$$\rho = \sum_{g=1}^{d_\gamma} \sum_{g'=1}^{d_\gamma} |\gamma, \boldsymbol{a} : g\rangle \rho_{gg'} \left\langle \gamma, \boldsymbol{a} : g' \right|. \tag{28}$$

Since the basis vectors $|\gamma, \boldsymbol{a} : g\rangle$ are joint eigenvectors of $\boldsymbol{A}$, the operator ρ commutes with $\boldsymbol{A}$; consequently, (18) implies that $\rho_{gg'} \propto \delta_{gg'}$. Combining this with the unit trace condition yields,

$$\varrho = \frac{1}{d_\gamma} \sum_{g=1}^{d_\gamma} |\gamma, \boldsymbol{a} : g\rangle\langle\gamma, \boldsymbol{a} : g| = \frac{1}{d_\gamma} \Pi\left(\mathcal{E}_{\boldsymbol{a}}(\boldsymbol{A})\right). \tag{29}$$

Thus a measurement of the CSCOB $\boldsymbol{A}$ leaves the system in a state described by a unique density operator. This state is called a *preparable mixed state*. The result (29) is also called a maximal state preparation or a maximal filtering [1] . This is the best that can be done for indistinguishable particles described by IPQM.

4.3 Superselection Rule

The combination of the permutation-invariance rule with the first two of Schur's Lemmas, c.f. "Appendix 1", is the basis for a proof of the following result: *A permutation-invariant operator, X, cannot connect states of different symmetry types* [1]. Consequently, for any permutation-invariant operator X,

$$\left\langle \gamma', \boldsymbol{a}' : g' | X | \gamma, \boldsymbol{a} : g \right\rangle = \delta_{\gamma'\gamma} M^{\gamma}_{g'g}(X), \tag{30}$$

where,

$$M^{\gamma}_{g'g}(X) := \left\langle \gamma, \boldsymbol{a}' : g' | X | \gamma, \boldsymbol{a} : g \right\rangle; \quad \left(g',\ g = 1, \dots, d_{\gamma}\right). \tag{31}$$

The permutation-invariance condition for X leads to,

$$M^{\gamma}(X) = M^{\gamma}\left(D(\mathcal{P})^{\dagger} X D(\mathcal{P})\right) = \mathfrak{D}^{(\gamma)}(\mathcal{P})^{\dagger} M^{\gamma}(X) \mathfrak{D}^{(\gamma)}(\mathcal{P}), \tag{32}$$

which, by the unitarity of $\mathfrak{D}^{(\gamma)}(\mathcal{P})$, yields,

$$\mathfrak{D}^{(\gamma)}(\mathcal{P}) M^{\gamma}(X) = M^{\gamma}(X) \mathfrak{D}^{(\gamma)}(\mathcal{P}), \tag{33}$$

for all $\mathcal{P} \in \mathcal{S}_N$. For any permutation-invariant operator X and any $\boldsymbol{a}$, $\boldsymbol{a}' \in ev(\boldsymbol{A}, \gamma)$, an application of the third Schur's Lemma to the identity (33) shows that the matrix $M^{\gamma}(X)$ is proportional to the identity matrix;[2] therefore,

$$\left\langle \gamma, \boldsymbol{a}' : g' | X | \gamma, \boldsymbol{a} : g \right\rangle = \left\langle \gamma, \boldsymbol{a}' || X || \gamma, \boldsymbol{a} \right\rangle \delta_{g'g}. \tag{34}$$

Applying (30) and (34) to the case $X = 1$ yields the orthonormality conditions,

$$\left\langle \gamma', \boldsymbol{a}' : g' \middle| \gamma, \boldsymbol{a} : g \right\rangle = \delta_{\gamma\gamma'} \delta_{\boldsymbol{a}\boldsymbol{a}'} \delta_{gg'}. \tag{35}$$

The state vectors of symmetry type γ form a subspace, $\mathfrak{H}^{(N)}_{\gamma}$, with basis set

$$\mathfrak{B}\left[\mathfrak{H}^{(N)}_{\gamma}\right] = \left\{ |\gamma, \boldsymbol{a} : g\rangle \mid \boldsymbol{a} \in ev(\boldsymbol{A}, \gamma),\ g = 1, \dots, d_{\gamma} \right\}. \tag{36}$$

As γ ranges through Γ_N the combined basis vectors of the $\mathfrak{H}^{(N)}_{\gamma}$'s provide a basis set for $\mathfrak{H}^{(N)}$. By virtue of (35) the $\mathfrak{H}^{(N)}_{\gamma}$'s are mutually orthogonal and thus linearly independent. Furthermore, (30) shows that permutation-invariant operators cannot connect $\mathfrak{H}^{(N)}_{\gamma}$ and $\mathfrak{H}^{(N)}_{\gamma'}$ for $\gamma \neq \gamma'$.These properties combine to yield the,

Superselection Rule: For any permutation-invariant operator A,

$$A : \mathfrak{H}^{(N)}_{\gamma} \to \mathfrak{H}^{(N)}_{\gamma} \text{ and } \mathfrak{H}^{(N)} = \bigoplus_{\gamma \in \Gamma_N} \mathfrak{H}^{(N)}_{\gamma}. \tag{37}$$

The subspaces $\mathfrak{H}^{(N)}_{\gamma}$ are called superselection sectors [15, Chap III.1]. Since observables and physical density operators are permutation-invariant, they do not connect different superselection sectors.

The superselection rule has significant consequences for time evolution. The unitary time evolution operator, $U(t) := \exp(-itH/\hbar)$, is permutation-invariant by virtue of the permutation-invariance of the Hamiltonian; therefore, it satisfies (). Thus $\langle \Phi | U(t) | \Psi \rangle = 0$ if $|\Phi\rangle$ and $|\Psi\rangle$ belong to different superselection sectors. In particular, this means that $U(t)|\Psi^{\gamma}(0)\rangle$ remains in $\mathfrak{H}^{(N)}_{\gamma}$ at all later times,

$$|\Psi^{\gamma}(t)\rangle := U(t)|\Psi^{\gamma}(0)\rangle \in \mathfrak{H}^{(N)}_{\gamma} \quad \text{for all } t > 0, \tag{38}$$

so that the terms of different symmetry type in (24) evolve independently,

[2] The notation, $\langle \gamma, \boldsymbol{a}' || X || \gamma, \boldsymbol{a} \rangle$, for the proportionality coefficient is chosen by analogy to the reduced matrix element used in connection with the rotation group

$$|\Psi(t)\rangle = U(t)|\Psi(0)\rangle = \sum_{\gamma} C_{\gamma} U(t)|\Psi^{\gamma}(0)\rangle = \sum_{\gamma} C_{\gamma}|\Psi^{\gamma}(t)\rangle. \tag{39}$$

Since SP-violations are expected to be small, it is safe to assume that there is a dominant term, $C_{\gamma_{FB}}$, in (24) and (), where either $\gamma_{FB} = F$ (Fermi representation) or $\gamma_{FB} = B$ (Bose representation). Thus the strength of possible SP-violations is determined by the initial amplitudes, C_{γ}, for $\gamma \neq \gamma_{FB}$. Note that combining (15)—which shows that every operator in the SPQM version satisfies the necessary condition () for observables—with (38)—which shows that time evolution preserves symmetry type—implies that the SPQM version can be completely recovered from the IPQM version by simply setting $C_{\gamma} = 0$ for all $\gamma \neq \gamma_{FB}$. In other words, the Symmetrization Postulate is equivalent to the equally mysterious condition that the only superselection sector present in any initial state is either $\gamma = F$ or $\gamma = B$. A less stringent assumption is that $\left|C_{\gamma}\right| \ll \left|C_{\gamma_{FB}}\right|$for all $\gamma \neq \gamma_{FB}$. Combining this with the normalization condition () yields

$$\left|C_{\gamma}\right| \ll 1 \quad \text{for all } \gamma \neq \gamma_{FB}, \tag{40}$$

which again says that violations of the Symmetrization Postulate are rare.

The superselection rules commonly encountered in quantum theory are associated with continuous symmetries, e.g. rotations and gauge transformations. Both of these examples are often said to impose restrictions on the superposition principle. For example, superpositions of states with integer and half-integer total angular momentum, or superpositions of states with different net charges are both said to be forbidden. As explained in [15, Chap III.1] another way to understand this situation is that these superpositions are not forbidden; rather, they act as mixed states for all observables. The superselection rule () has the same effect for the discrete symmetry group $\mathcal{S}_N$. The expectation value of any permutation-invariant operator X for a general pure state $|\Psi\rangle$ is,

$$\langle\Psi|X|\Psi\rangle = \sum_{\gamma'} \sum_{\gamma} C^{*}_{\gamma'} C_{\gamma} \left\langle \Psi^{\gamma'}|X|\Psi^{\gamma} \right\rangle = \sum_{\gamma} \left|C_{\gamma}\right|^2 \langle\Psi^{\gamma}|X|\Psi^{\gamma}\rangle, \tag{41}$$

where the final form is a consequence of the superselection rule. Since () holds for all observables and only involves the magnitudes $\left|C_{\gamma}\right|$, there are no interference terms between states of different symmetry type and no information about the phases of the coefficients, C_{γ}, can be obtained from any measurement of observables, Even though $|\Psi\rangle$ is a hybrid-symmetry pure state, (41) shows that the expectation value of any observable is an average over a statistical mixture of the definite-symmetry pure states included in $|\Psi\rangle$. In other words, for evaluating averages of observables the pure state $|\Psi\rangle$ acts as a mixed state described by the permutation-invariant density operator,

$$\rho_{\Psi} := \sum_{\gamma} |\Psi^{\gamma}\rangle \left|C_{\gamma}\right|^2 \langle\Psi^{\gamma}|. \tag{42}$$

5 Search for SP-Violations

The strong conditions imposed on state vectors by the Symmetrization Postulate form the basis of the conventional (SPQM) description of all systems of identical particles, ranging from a small number of particles involved in a scattering event to interacting many-body systems, e.g. Bose–Einstein condensates, superfluids, superconductors, etc.. This broad range of influence implies an equally broad range of possible experiments to search for SP-violations. The formalism developed in the previous sections could, for example, be used to construct an extended version of quantum statistical mechanics that is not limited to Bose or Fermi statistics. However, experiments on many-body systems may not be the most useful approach. The difficulty is that the small size of SP-violations would very likely produce subtle effects that would be extremely difficult to detect. This may be the reason that experiments for both Fermi systems [3–7] and Bose systems [8–10] typically involve interactions of single electrons or photons with atoms or molecules. While detection of SP-violating events for these systems will also be extremely difficult, these experiments can take advantage of selection rules imposed by Bose or Fermi statistics. For example, in SPQM the initial and final states for an electron (photon) scattering event cannot include a symmetric (antisymmetric) state of two electrons (photons)

In order to obtain an observable signal from a weak violation of an SP-imposed selection rule, it is essential to have a large flux of incident particles. One way to achieve this with electrons is to induce a strong flow of current through a conductor. Experiments using this arrangement to test for PEP-violations have been conducted and carefully analyzed [6]. The idea of these experiments is that a radiation cascade would occur when a conduction electron is captured by an atom in the crystal lattice. Captures producing X-rays would, however, be PEP-forbidden since the relevant lower energy levels are fully occupied. Thus an emission of X-rays in the appropriate energy range would be a signal of a PEP-violation. A model for this event must evidently go beyond SPQM, and the IPQM version of quantum mechanics provides a minimal extension suited to this purpose.

In order to illustrate how the unfamiliar aspects of IPQM are involved in an analysis of such experiments, it is useful to consider a simplified model. The relevant issues can not arise for single particles and are essentially trivial for two-particle systems, which only support the symmetric and antisymmetric representations. Thus the simplest model systems that are informative are those with three indistinguishable particles. These considerations suggest the following toy model. The incident electron and the electrons in the target atom are modeled by three identical, non-interacting, spin-1/2 particles confined to one space dimension. Unperturbed particle dynamics are described by a spin-independent single-particle Hamiltonian H_0, and the coupling to the radiation field is given by H_{rad}. The single-particle quantum numbers are $\theta = (\epsilon, s)$, where ϵ and $s\hbar$ are respectively eigenvalues of H_0 and the z-component of the spin. In the initial state, the atom is modeled by two particles in the ground state of H_0, with total energy $2\epsilon_0$, and the incident electron is modeled by one particle in an excited state, with energy

$\epsilon_1 > \epsilon_0$. The final state has all three particles in the ground state. For distinguishable particles this situation could be described by assigning $|\boldsymbol{\theta}_{int}\rangle = \left|(\epsilon_0, 1/2), (\epsilon_0, -1/2), (\epsilon_1, 1/2)\right\rangle$, and $\left|\boldsymbol{\theta}_{fin}\right\rangle = \left|(\epsilon_0, 1/2), (\epsilon_0, -1/2), (\epsilon_0, 1/2)\right\rangle$ as nominal initial and final state vectors respectively.

In the SPQM version for fermions, the state vectors $|\boldsymbol{\theta}_{int}\rangle$ and $\left|\boldsymbol{\theta}_{fin}\right\rangle$ would be replaced by the antisymmetrized states,

$$\begin{aligned} \left|F, \boldsymbol{\theta}_{int}\right\rangle &:= \frac{1}{\sqrt{3!}} \sum_{\mathcal{P}\in\mathcal{S}_3} s_{\mathcal{P}} D(\mathcal{P}) \left|\boldsymbol{\theta}_{int}\right\rangle, \\ \left|F, \boldsymbol{\theta}_{fin}\right\rangle &:= \frac{1}{\sqrt{3!}} \sum_{\mathcal{P}\in\mathcal{S}_3} s_{\mathcal{P}} D(\mathcal{P}) \left|\boldsymbol{\theta}_{fin}\right\rangle = 0, \end{aligned} \tag{43}$$

where $s_{\mathcal{P}} = (+1, -1)$ for (even, odd) $\mathcal{P}$. The second equation is an example of the rule that two identical fermions cannot occupy the same single-particle state.

In the IPQM version all three irreducible representations of $\mathcal{S}_3$, *cf.* "Appendix 2", must be considered: $\gamma = F$, $\gamma = B$, and $\gamma = I$. The irreducible representations that overlap an unsymmetrized three-particle state $|\boldsymbol{\kappa}\rangle$ are determined by calculating the normalized projection of $|\boldsymbol{\kappa}\rangle$ onto a basis vector in a carrier space for the irreducible representation $\mathfrak{D}^{(\gamma)}$, i.e.,

$$|\gamma; \boldsymbol{\kappa} : g\rangle := \frac{\Pi_{\gamma g}|\boldsymbol{\kappa}\rangle}{\sqrt{\left\langle \boldsymbol{\kappa} \middle| \Pi_{\gamma g} \middle| \boldsymbol{\kappa} \right\rangle}}, \tag{44}$$

where $\Pi_{\gamma g}$ is one of the projection operators, Π_B,Π_F, and Π_{Ig}, defined in (51)–(53). Evaluating (44) for $|\boldsymbol{\theta}_{int}\rangle$ yields,

$$\begin{aligned} \left|B, \boldsymbol{\theta}_{int} : 1\right\rangle &\propto \Pi_B \left|\boldsymbol{\theta}_{int}\right\rangle \neq 0, \\ \left|F, \boldsymbol{\theta}_{int} : 1\right\rangle &\propto \Pi_F \left|\boldsymbol{\theta}_{int}\right\rangle \neq 0, \\ \left|I, \boldsymbol{\theta}_{int} : g\right\rangle &\propto \Pi_{Ig} \left|\boldsymbol{\theta}_{int}\right\rangle \neq 0, \end{aligned} \tag{45}$$

which means that all three representations overlap with $|\boldsymbol{\theta}_{int}\rangle$. For $\left|\boldsymbol{\theta}_{fin}\right\rangle$,

$$\begin{aligned} \left|B, \boldsymbol{\theta}_{fin} : 1\right\rangle &\propto \Pi_B \left|\boldsymbol{\theta}_{fin}\right\rangle \neq 0, \\ \left|I, \boldsymbol{\theta}_{fin} : g\right\rangle &\propto \Pi_{Ig} \left|\boldsymbol{\theta}_{fin}\right\rangle \neq 0, \\ \left|F, \boldsymbol{\theta}_{fin} : 1\right\rangle &\propto \Pi_F \left|\boldsymbol{\theta}_{fin}\right\rangle = 0, \end{aligned} \tag{46}$$

i.e. only the representations B and I overlap with $\left|\boldsymbol{\theta}_{fin}\right\rangle$. The general normalized states that can describe this experiment are an initial state

$$\left|\Psi_{int}\right\rangle = C_{int}^{F}\left|F, \theta_{int} : 1\right\rangle + C_{int}^{B}\left|B, \theta_{int} : 1\right\rangle + \sum_{g=1}^{2} C_{int}^{Ig}\left|I, \theta_{int} : g\right\rangle, \tag{47}$$

and a final state,

$$\left|\Psi_{fin}\right\rangle = C_{fin}^{B}\left|B, \theta_{fin} : 1\right\rangle + \sum_{g=1}^{2} C_{fin}^{Ig}\left|I, \theta_{fin} : g\right\rangle. \tag{48}$$

A detectable PEP-violating event for this model would be the emission of radiation at the resonance frequency ω given by $\hbar\omega = \left(2\epsilon_0 + \epsilon_1\right) - \left(3\epsilon_0\right) = \epsilon_1 - \epsilon_0$. Since H_{rad} is permutation invariant, the general result (34) and the superselection rule (37) combine to yield the transition matrix element,

$$\begin{aligned}\left\langle\Psi_{fin}\middle|H_{rad}\middle|\Psi_{int}\right\rangle =& C_{fin}^{B*}C_{int}^{B}\left\langle B, \theta_{fin}\middle|H_{rad}\middle|B, \theta_{int}\right\rangle \\ &+ \left\{\sum_{g=1}^{2} C_{fin}^{Ig*}\, C_{int}^{Ig}\right\}\left\langle I, \theta_{fin}\middle\|H_{rad}\middle\|I, \theta_{int}\right\rangle,\end{aligned} \tag{49}$$

which shows that the probability of detecting a PEP-violation is determined by the initial amplitudes C_{int}^{B} and C_{int}^{Ig} for the Bose and mixed representations of $\mathcal{S}_3$. This is an explicit example of the general conclusion following from (38). When the initial state vector satisfies (40), the expansion coefficients satisfy $\left|C_{int}^{B}\right| \ll 1$ and $\left|C_{int}^{Ig}\right| \ll 1$, so that the transition rate is small.

6 Discussion

The IPQM version of quantum mechanics is complicated by the necessity of considering all irreducible representations of $\mathcal{S}_N$. As shown in Sect. 4.2 this requires changes in the usual description of state preparation. The most significant new feature of IPQM is the presence of the superselection rule associated with the discrete symmetry group $\mathcal{S}_N$. As shown in Sect. 4.3 this has important effects on time evolution and the interpretation of superpositions of states of different symmetry type. These features of IPQM are consequences of three assumptions: (A1) The axioms of quantum theory. (A2) The definition of identical particles. (A3) The Indistinguishability Postulate. These assumptions lead to the following properties: (1) Permutation symmetries hold without regard to interactions. (2) Permutation symmetries hold for all interparticle separations. (3) The symmetry type of a state is preserved by time evolution. The first two properties are purely kinematical, and the third follows from (38).

The analysis of the toy model of a PEP-violating experiment in Sect. is focussed on the implications of properties (1)–(3) of IPQM, but an alternative description has been proposed for this class of experiments. In this approach the target and incident electrons are treated as separate systems, and the incident electrons are said to be *fresh* [6]. The meaning of the term ‘fresh electrons’ is implicit in the following assumptions: (i) The fresh electrons have never interacted with the target electrons. (ii) No symmetry conditions have been established between the fresh electrons and the target electrons. (iii) Establishment of any symmetry conditions between the fresh electrons and the target electrons depends on interactions between them. The assumptions (i)–(iii) of the fresh-electron model are clearly inconsistent with properties (1)–(3) of IPQM and thus with one or all of the assumptions (A1)–(A3). This does not mean that the fresh electron assumptions are unphysical, but it does imply that they require an extension of SPQM that is substantially more radical than IPQM. This is not necessarily a bad feature; changing something as fundamental as the PEP may well require a radically new theory.

The decision to formulate IPQM using nonrelativistic quantum theory was not made for the sake of simplicity; indeed, it cannot be avoided. It is not possible to formulate a relativistic quantum theory with a fixed number of particles [16, Chap. 1.1]; consequently, any theory that assumes a fixed number of particles—in particular IPQM—is necessarily nonrelativistic. The fact that IPQM does not impose any connection between spin and statistics [1], is also be related to the use of nonrelativistic quantum theory. The spin–statistics connection has only been established by using an argument based on relativistic quantum field theory [15, 17] to prove the **spin–statistics theorem**: *integer spin particles are bosons and half-integer spin particles are fermions*. The statement of this theorem seems to suggest that only Bose and Fermi statistics are possible in a relativistic theory, but this is misleading. The various proofs of the spin–statistics theorem depend on the following assumption: : *quantum fields evaluated at spacelike separated points either commute or anti-commute* (CorAC). In some proofs this assumption is included in the hypothesis of the theorem and in others it is found in the associated axioms of field theory. In either case CorAC is used to impose the empirical restriction to Bose or Fermi statistics on relativistic quantum field theory, just as the Symmetrization Postulate imposes it on nonrelativistic quantum mechanics. If it could be demonstrated that CorAC follows from the axioms of any consistent relativistic quantum field theory, then the conclusion would be that only Bose or Fermi statistics are possible. If, instead, there is a consistent field theory for which CorAC could be false, then it might be possible to introduce other kinds of statistics into field theory.

Appendix: The Symmetric Group

The Symmetric Group $\mathcal{S}_N$ is of order $N!$ and each permutation in $\mathcal{S}_N$ can be written as the product of transpositions. A permutation is even (odd) if it is the product of an even (odd) number of transpositions. The parity of a permutation $\mathcal{P}$ is $s_{\mathcal{P}} = (+1, -1)$ for (even, odd) $\mathcal{P}$.

Each irreducible representation , $\mathcal{P} \to \mathfrak{D}^{(\gamma)}(\mathcal{P})$, of $\mathcal{S}_N$ is finite dimensional and the labels, $\gamma := \left(\lambda_1, \dots, \lambda_N\right)$, range over the finite set, Γ_N, of solutions of $\lambda_1 + \cdots + \lambda_N = N$, subject to $\lambda_n \geq \lambda_{n+1} \geq 0$. For each N there are only two one-dimensional representations: The symmetric Bose representation, $\gamma = B = (N, 0, \dots, 0)$, and the antisymmetric Fermi representation, $\gamma = F = (1, 1, \dots, 1)$. Projection operators, $\Pi_{\gamma g}$, satisfying,

$$\Pi_{\gamma' g'}\Pi_{\gamma g} = \delta_{\gamma'\gamma}\delta_{g' g}\Pi_{\gamma g} \quad \left(\gamma, \gamma' \in \Gamma_N, \quad g = 1, \dots, d_\gamma\right), \tag{50}$$

are used to define basis vectors for a space $\mathfrak{R}$ carrying an irreducible representation of $\mathcal{S}_N$ [14, Chap. 7].

Schur's Lemmas

The following statements hold for any finite group G [14, Chap 3.14].

Lemma 1 *Let $\mathfrak{D}(g)$ and $\mathfrak{D}'(g)$ be matrices of the irreducible representations $\mathfrak{D}$ and $\mathfrak{D}'$ of G with different dimensions, then a matrix M satisfying $\mathfrak{D}(g)M = M\mathfrak{D}'(g)$ for all $g \in G$ necessarily vanishes.*

Lemma 2 *Let $\mathfrak{D}(g)$ and $\mathfrak{D}'(g)$ be matrices of the irreducible representations $\mathfrak{D}$ and $\mathfrak{D}'$ with the same dimension. If a matrix M satisfies $\mathfrak{D}(g)M = M\mathfrak{D}'(g)$ for all $g \in G$, then either $\mathfrak{D}(g)$ and $\mathfrak{D}'(g)$ are equivalent or $M = 0$.*

Lemma 3 *If the matrices $\mathfrak{D}(g)$ provide an irreducible representation of G and $M\mathfrak{D}(g) = \mathfrak{D}(g)M$ for all $g \in G$, then M is a multiple of the identity matrix.*

Symmetric Group of Order 3

Group elements: The 3! permutations in $\mathcal{S}_3$ are $\left\{e, \mathcal{P}_{12}, \mathcal{P}_{13}, \mathcal{P}_{23}, \mathcal{P}_{132}, \mathcal{P}_{123}\right\}$, where,
$e(n) = n$; $\mathcal{P}_{ij}(j) = i$; $\mathcal{P}_{ij}(i) = j$; $\mathcal{P}_{ij}(n) = n$ for $n \neq i, j$;
$\mathcal{P}_{132}(1) = 3, \mathcal{P}_{132}(3) = 2\,, \mathcal{P}_{132}(2) = 1;$
$\mathcal{P}_{123}(1) = 2, \mathcal{P}_{123}(2) = 3\,, \mathcal{P}_{123}(3) = 1.$

Irreducible representations

The labels $\gamma = \left(\lambda_1, \lambda_2, \lambda_3\right)$ in Γ_3 satisfy $\lambda_1 + \lambda_2 + \lambda_3 = 3$ with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$. There are three solutions: $\gamma = (3, 0, 0) = B$, $\gamma = (1, 1, 1) = F$, and $\gamma = (2, 1, 0) = I$. The first two respectively label the one-dimensional Bose and Fermi representations, and the third labels a two-dimensional representation of intermediate symmetry. The projection operators are given by,

$$\text{Bose:}\Pi_{(3,0,0)} = \Pi_B \,:=\frac{1}{3!}\sum_{\mathcal{P}\in\mathcal{S}_3} D(\mathcal{P}), \tag{51}$$

$$\text{Fermi:}\Pi_{(1,1,1)} = \Pi_F \,:= \frac{1}{3!}\sum_{\mathcal{P}\in\mathcal{S}_3} s_{\mathcal{P}} D(\mathcal{P}), \tag{52}$$

$$
\begin{aligned}
&\text{Intermediate:}\Pi_{Ig} = \Pi_{(2,1,0)g} \; (g = 1, 2), \\
&\Pi_{I1} := \frac{1}{3}\big(\hat{e} + D(\mathcal{P}_{12}) \nwarrow D(\mathcal{P}_{13}) \nwarrow D(\mathcal{P}_{123})\big), \\
&\Pi_{I2} := \frac{1}{3}\big(\hat{e} - D(\mathcal{P}_{12}) + D(\mathcal{P}_{13}) - D(\mathcal{P}_{132})\big).
\end{aligned}
\tag{53}
$$

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