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Quantum Behavior of a Classical Particle Subject to a Random Force

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Abstract
We give a partial answer to the question whether the Schrödinger equation can be derived from the Newtonian mechanics of a particle in a potential subject to a random force. We show that the fluctuations around the classical motion of a one dimensional harmonic oscillator subject to a random force can be described by the Schrödinger equation for a period of time depending on the frequency and the energy of the oscillator. We achieve this by deriving the postulates of Nelson's stochastic formulation of quantum mechanics for a random force depending on a small parameter. We show that the same result applies to small potential perturbations around the harmonic oscillator. We also show that the noise spectrum can be chosen to obtain the result for all oscillator frequencies for fixed mass. We discuss heuristics to generalize the result for a particle in one dimension in a potential where the motion can be described using action-angle variables. The main motivation of this paper is to provide a step for constructing a Newtonian theory which would approximately reproduce quantum mechanics both in unitary evolution and measurement regimes.

Keywords Quantum mechanics · Stochastic mechanics · Brownian motion · Method of stochastic averaging

1 Introduction

Despite the successes of quantum theory there remains the solution of the measurement problem and its unification with general relativity. Much effort has been spent assuming quantum mechanics is fundamental and applies to smallest and largest possible scales. At the smallest scales where quantum effects in gravity should take place, although we can form mathematically consistent quantum gravity theories, we have no experimental guidance yet and have extreme conceptual difficulties making

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sense of a quantum description of space-time. At large scales we observe that nature behaves classically, which is not possible to understand within the standard postulates of quantum mechanics since those do no pretense to explain the measurement processes and the quantum to classical transition in a fundamental fashion.

Regardless of the successes and failures of quantum mechanics, in this paper we would like to retain the Newtonian–Einsteinian notions. We try to answer the simplest possible question: Can a single non-relativistic quantum particle in a potential in one dimension be described by Newtonian mechanics? A lot of effort has been put in deriving hidden variable theories but the answer to this question is still missing: there is no proof that it is impossible and there is no proof that all quantum effects described by a general solution of the Schrödinger equation can be accounted classically. Indeed the latter seems almost impossible since it is very hard to imagine how a classical particle would exhibit quantum interference.

The main motivation for pursuing an answer to this question is to ultimately find a theory which would approximately reproduce quantum mechanics both in unitary evolution and measurement regimes. For the construction of this theory, since we give up on quantum mechanics as fundamental, it is natural to keep the second most fundamental principle: relativity. In the non-relativistic limit, relativity reduces to Newtonian mechanics. Therefore we ask whether there exist a Newtonian formulation of non-relativistic quantum mechanics at least approximately. A Newtonian formulation requires a particle description. There exist a stochastic particle description of quantum mechanics due to Nelson [1]. In this paper, we derive Nelson's stochastic particle theory in one dimension as an approximation to the Newtonian motion of a particle perturbed by a random force for small fluctuations around the classical motion. We show that when a classical harmonic oscillator is subject to a specific random force, its fluctuating motion around the classical trajectory can be described by the Schrödinger equation for a range of values of frequency, energy and time. We show that this generalizes to potential perturbations around the harmonic oscillator. We also show that the noise spectrum can be chosen to obtain the result for all oscillator frequencies for fixed mass. We further discuss a way to generalize the result to potentials admitting action-angle variables. The mathematical tools that we use to achieve this are the method of stochastic averaging and Nelson's formulation of Schrödinger equation in terms of stochastic particle trajectories. In an upcoming paper, we generalize the results to all quantum states (arbitrary superposition states) for the harmonic oscillator [2].

Here, we set aside the question of many particles which would involve Bell's theorem. It is widely believed that it is impossible to have a local Newtonian–Enstheinian explanation of entangled states. Although the mostly forgotten rigorous analysis of Bell inequalities by Nelson [3–5] distinguishes between passive and active locality and makes it possible for stochastic theories to be in principle able to explain entangled states. We will treat the entangled states elsewhere.

Before the significant discovery of Nelson [1] that it is possible to give a stochastic account of Schrödinger equation, it had been widely believed this was impossible because diffusions are dissipative but there is a notion of conserved energy in the quantum mechanical evolution. Nelson showed that it is possible to construct conservative diffusions which are equivalent to the Schrödinger

equation. However Nelson's formulation is not Newtonian: the particle is subject to random motion in its position space contrary to that the random effects should appear as forces in a Newtonian theory. Here we attempt to answer whether Nelson's formulation can be derived from a phase space stochastic process where the random term appears as a force. Indeed this is the first and perhaps the most important of the open problems stated in his book [6]. This question was most openly investigated by Smolin [7] who gave sufficient conditions for a cosmological theory to reduce to Nelson's theory.

The same type of questions have been asked and were tried to be answered mostly by the stochastic electrodynamics community [8]. There one assumes that an electrically charged particle is coupled to a background stochastic electric field with a specific spectrum and is also subject to electromagnetic radiation reaction. One is able to show that in equilibrium one can choose the spectrum to match with all the energy eigenstates of a harmonic oscillator. However there lacks a universal spectrum working for all energy eigenstates and the superposition states seem to be elusive. There are two main lines of attempted derivations of Schrödinger equation both running into difficulties. In the first approach by integrating out the velocity evolution one tries to reduce to a position space process. Schrödinger equation holds if one can neglect certain radiative terms in the equations but there is no justification for how the system reaches a state such that those terms can be neglected and how long the system stays in that state such that the approximation is valid. In the second approach it is shown that if one assumes that there are multiple ergodic energy states then stochastic variables can be described by matrix variables and one obtains Heisenberg's theory. However it seems difficult to construct a stochastic system exhibiting classical multiple ergodic energy states which matches with the quantum energy eigenvalues and to describe the transition between energy eigenstates in such a framework. Perhaps the most important objection against stochastic electrodynamics is that it only applies to charged particles and lacks universality. However we think the questions asked and attempted to be answered in this model are valuable and give insights for further developments.

The paper is organized as follows. In Sect. 2 we briefly review Ito calculus and stochastic differential equations-the mathematical framework that we use in the rest of the paper. In Sect. 3 we give an account of Nelson's stochastic formulation of the Schrödinger equation for a non-relativistic particle in one dimension. We introduce the two postulates of Nelson which are equivalent to the Schrödinger equation in Madelung form. In Sect. 4 we show that the Newton–Nelson law is satisfied by a particle subject to a random force proportional to white noise. In Sect. 5 we introduce the method of stochastic averaging to be used in the following section to derive Nelson's first postulate. In Sect. 6 we show that Nelson's first postulate is approximately satisfied for a time interval depending on the energy and the frequency of the oscillator by choosing a suitable spectrum for random force. We further show how a colored spectrum yields Nelson's two postulates for oscillators of all frequencies with fixed mass. We discuss how this result generalizes to small potential perturbations around the harmonic oscillator.

In Sect. 7 we give heuristics to generalize the result to arbitrary potentials admitting action angle variables. In Sect. 8 we discuss the results.

2 Review of Stochastic Differential Equations

We give a brief review of Ito stochastic calculus and stochastic differential equations. We will only state results formally which are relevant for our purposes and refer the reader to standard textbooks on the subject (e.g. [9, 10]). Let $\xi(t)$ be the Gaussian process with zero mean and unit variance (also known as white noise), i.e.

$$\langle \xi(t) \rangle = 0, \quad \forall t \tag{1}$$

and for times $(t_1, t_2, \ldots, t_n)$, $(\xi(t_1), \xi(t_2), \ldots, \xi(t_n))$ are Gaussian correlated random variables with co-variance

$$\langle \xi(t_1)\xi(t_2) \rangle = \delta(t_1 - t_2). \tag{2}$$

Note that for $t_1 \neq t_2$, $\xi(t_1)$ and $\xi(t_2)$ are independent. We define the Wiener process $W(t)$ as the formal time integral of $\xi(t)$:

$$W(t) = \int_0^t \xi(s) ds, \tag{3}$$

where we set the initial time to $t = 0$ without loss of generality. We can also write this as $dW(t) = \xi(t)dt$. The Wiener process is again Gaussian since it is a linear combination of independent Gaussian random variables. Its mean is zero as can be directly seen from the definition. Its co-variance is calculated as

$$\langle dW(t_1) dW(t_2) \rangle = \int_0^{t_1} \int_0^{t_2} \langle \xi(s_1)\xi(s_2) \rangle ds_1 ds_2 = \min(t_1, t_2). \tag{4}$$

From this we see that formally $dW(t)$ is of order $\sqrt{dt}$. We will be dealing with stochastic differential equations in the rest of the paper. Suppose we would like to make sense of the following initial value problem for the scalar variable $x(t)$:

$$\frac{dx(t)}{dt} = f(x(t)) + g(x(t))\xi(t) \tag{5}$$

with $p(x, t = 0)) = p_0(x)$ for some initial probability distribution $p_0(x)$. An ambiguity arises when we would like to make sense of the product $g(x(t))\xi(t)$. We know that since $\xi(t)$ is independent of $\xi(s)$ for $s < t$, it is independent of $g(x(s))$ for $s < t$. But the product concerns the same times. In order to remedy this difficulty we will write the equation in differential form:

$$dx(t) = f(x(t))dt + g(x(t))dW(t) \tag{6}$$

which is a formal way to write the integral equation:

$$x(t) = x_0 + \int_0^t f(x(s))ds + \int_0^t g(x(s))dW(s). \tag{7}$$

Now if we can make sense of the integral that includes $dW(s)$ term we can define the stochastic differential equation in terms of the integral equation. There are more than one ways to define a stochastic integral. In this paper we will operate with the Ito definition. For the other famous (Stratonovich) definition see [9, 10]. We adopt the following definition:

$$\int_0^t g(x(s))dW(s) = \lim_{\Delta s \to 0} \sum_i g(x(s_i))(dW(s_{i+1}) - dW(s_i)) \tag{8}$$

where $\Delta s = s_{i+1} - s_i, \quad \forall i$. Therefore the increment $dW(s_{i+1}) - dW(s_i)$ is independent of $g(x(s_i))$. However with this definition we need to update the chain rule of calculus. Suppose that we would like to calculate the equation that is obeyed by a function of x, say $y = f(x)$. Remember that $dW(t)$ is of order $\sqrt{dt}$. Thus in order to correctly calculate dy we should expand it up to second order. Without proof we state the Ito's lemma:

$$dy = \frac{df}{dx}dx + \frac{1}{2}\frac{d^2f}{dx^2}(dx)^2 = \frac{df}{dx}dx + \frac{1}{2}\frac{d^2f}{dx^2}g^2(x)dt. \tag{9}$$

Note that in the expansion of $(dx)^2$ we omitted terms of order $dt^{3/2}$ and only kept those of order dt and $\sqrt{dt}$. We will also need the two dimensional version of this. Suppose we have two processes x_1 and x_2 with independent Wiener processes $dW_1(t)$ and $dW_2(t)$:

$$dx_i(t) = f_i(x_1, x_2)dt + g_i(x_1, x_2)dW_i(t). \tag{10}$$

If $y = f(x_1, x_2)$ then we can write the differential dy as

$$dy = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x_1^2}g_1^2 + \frac{\partial^2 f}{\partial x_2^2}g_2^2\right)dt. \tag{11}$$

We will frequently invoke these results in the following sections.

3 Review of Nelson's Stochastic Mechanics

We give a review of Nelson's stochastic formulation of non-relativistic quantum mechanics in one dimension. For more details see Nelson's original paper [1], his two books [6, 11] and Guerra's review [12]. Consider the Schrödinger equation:

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(x)\right)\psi(x,t). \tag{12}$$

Putting $\psi(x,t) = \sqrt{\rho(x,t)}e^{\frac{i}{\hbar}S(x,t)}$ we get the Madelung equations:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}\left(\rho \frac{1}{m}\frac{\partial S}{\partial x}\right) \tag{13}$$

$$\frac{\partial S}{\partial t} = -\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 - U(x) + \frac{\hbar^2}{2m}\frac{1}{\sqrt{\rho}}\frac{\partial^2}{\partial x^2}\sqrt{\rho} \tag{14}$$

where $\rho(x, t)$ is the probability of finding the particle at (x, t) and $S(x, t)$ is the phase of the wave function. We recognize first of the equations as the continuity equation with velocity $\frac{1}{m}\frac{\partial S}{\partial x}$. The second of the equations apart from the last term (quantum potential) on the right hand side is the Hamilton–Jacobi equation. Thus if $\hbar = 0$, we have the classical ensemble of particles. The Newton's equations of motion are then the equations that characteristic curves obey corresponding to this set of Madelung partial differential equations. Since the quantum potential term depends on the probability $\rho(x, t)$, giving deterministic characteristics seems not possible. However as Nelson proved [1, 6, 11], it is possible to give a Markovian stochastic process associated to the solution of Madelung equations in position space. We start by assuming that a particle obeys the following stochastic differential equation:

$$dx(t) = b(x(t), t)dt + \sqrt{\frac{\hbar}{m}}dW(t) \tag{15}$$

where $b(x(t), t)$ is a general function and dW is the Wiener process. We call this as Nelson's first postulate. The diffusion equation associated to this is [9, 10]

$$\frac{\partial \rho(x, t)}{\partial t} = -\frac{\partial}{\partial x}(b(x, t)\rho(x, t)) + \frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\rho(x, t) \tag{16}$$

where $\rho(x, t)$ is the probability of finding the particle at x at time t. In order to match with the continuity equation we define

$$\frac{\partial}{\partial x}S(x, t) = m(b(x, t) - \frac{\hbar}{2m}\frac{\partial}{\partial x}\log \rho(x, t)) \tag{17}$$

where we assumed that $\rho(x, t)$ is nowhere zero. For a discussion of what happens at zeros see [6]. Indeed, the probability of the particle arriving at places where $\rho(x, t) = 0$ is zero since $b(x, t)$ is singular at zeros and acts repulsively on the stochastic particle trajectory pushing the particle away from the zeros. We want $S(x, t)$ just defined in this way to satisfy the quantum Hamilton–Jacobi equation. We could postulate it as a partial differential equation but Nelson found a way to write this solely in terms of the stochastic particle trajectory. The quantum Hamilton–Jacobi equation can be shown to be equivalent to the following equation:

$$\frac{1}{2}(D_+D_- + D_-D_+)x(t) = -\frac{1}{m}\frac{\partial U(x)}{\partial x}|_{x(t)} \tag{18}$$

where D_+ and D_- are forward and backward derivatives which will be defined below, the right hand side is the classical acceleration of the particle evaluated on the stochastic trajectory and the left hand side is the time-symmetric stochastic

acceleration. This is the stochastic analogue of Newton's second law. Thus we call this as Newton–Nelson law or Nelson's second postulate. The forward and backward derivatives are defined to be

$$D_+x(t) = \lim_{\Delta t\to 0^+} E[\frac{x(t+\Delta t)-x(t)}{\Delta t}|x(t)] \tag{19}$$

$$D_-x(t) = \lim_{\Delta t\to 0^+} E[\frac{x(t)-x(t-\Delta t)}{\Delta t}|x(t)] \tag{20}$$

where $E[f|x(t)\ ]$ denotes the expectation of f conditioned on $x(t)$. For any function $F(x, t)$ we can write its forward and backward derivatives explicitly as follows

$$(D_+F)(x,t) = \frac{\partial}{\partial t}F(x,t) + b(x,t)\frac{\partial}{\partial x}F(x,t) + \frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}F(x,t) \tag{21}$$

$$\begin{aligned}(D_-F)(x,t) = &\frac{\partial}{\partial t}F(x,t) + (b(x,t) - \frac{\hbar}{m}\frac{\partial}{\partial x}\log\rho(x,t))\frac{\partial}{\partial x}F(x,t)\\ &- \frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}F(x,t).\end{aligned} \tag{22}$$

The derivation of the formula for D_+ is straightforward but the calculation of D_- is subtler [6, 11, 12]. Using these formulas it is straightforward to show that the Newton–Nelson law is equivalent to the x derivative of the second Madelung equation (3.3). It has been shown that for each solution of the Schrödinger equation there is an associated stochastic process satisfying Nelson's postulates and if Nelson's postulates are satisfied that one can construct a wave function which satisfies the Schrödinger equation with its absolute square the probability density of the position of particle. The stochastic formulation can be generalized to particles propagating in higher dimensions, multiple particles, fields and particles with spin [6, 12].

4 Newton–Nelson Law

In this section we will show that the Newton–Nelson law is satisfied by a particle in a potential in one dimension subject to a random force. Consider a particle of mass m in a potential $U(x)$ subject to a random force:

$$\begin{aligned}dx(t) &= v(t)dt\\ dv(t) &= a(x(t))dt + \sigma dW(t)\end{aligned} \tag{23}$$

where (x, v) denotes the position and velocity variables, dW is the Wiener process, σ is a positive constant and

$$a(x) = -\frac{U'(x)}{m} = -\frac{1}{m}\frac{dU(x)}{dx}. \tag{24}$$

We will make use of the following formulas for forward and backward derivatives conditioned on fixed ($x(t)$, $v(t)$) of a function $G(x, v, t)$ which can be found in section 5 of Guerra's review [12]:

$$\begin{aligned}(D_+G)(x,v,t)|_{(x(t),v(t))} &= \lim_{\Delta t\to 0^+} E[\frac{G(x(t+\Delta t),v(t+\Delta t),t+\Delta t)-G(x(t),v(t),t)}{\Delta t}|x(t),v(t)] \\ &= \frac{\partial G}{\partial t}+v\frac{\partial G}{\partial x}+a(x)\frac{\partial G}{\partial v}+\frac{\sigma^2}{2}\frac{\partial^2 G}{\partial v^2}\end{aligned} \tag{25}$$

$$\begin{aligned}(D_-G)(x,v,t)|_{(x(t),v(t))} &= \lim_{\Delta t\to 0^+} E[\frac{G(x(t),v(t),t)-G(x(t-\Delta t),v(t-\Delta t),t-\Delta t)}{\Delta t}|x(t),v(t)] \\ &= \frac{\partial G}{\partial t}+v\frac{\partial G}{\partial x}+(a(x)-\sigma^2\frac{\partial}{\partial v}\log\rho(x,v,t))\frac{\partial G}{\partial v}-\frac{\sigma^2}{2}\frac{\partial^2 G}{\partial v^2}\end{aligned} \tag{26}$$

where $\rho(x, v, t)$ is the probability of finding the particle at x with velocity v at time t. We also need the following result on conditional expectations for a set of random variables (x, y, z):

$$E[F(z)|x] = \int E[F(z)|x,v]p(v|x)dv \tag{27}$$

for any function $F(z)$. To see this we write the conditional probability of $p(z|x)$ in terms of $p(z|x, v)$ and $p(v|x)$ as

$$\begin{aligned}p(z|x) &= \frac{p(z,x)}{p(x)} = \frac{\int p(z,x,v)dv}{p(x)} = \frac{\int p(z|x,v)p(x,v)dv}{p(x)} \\ &= \int p(z|x,v)p(v|x)dv.\end{aligned} \tag{28}$$

Using this we compute

$$\begin{aligned}E[F(z)|x] &= \int F(z)p(z|x)dx = \int\int F(z)p(z|x,v)p(v|x)dxdv \\ &= \int E[F(z)|x,v]p(v|x)dv.\end{aligned} \tag{29}$$

To derive Newton–Nelson law we will calculate the stochastic acceleration $\frac{1}{2}(D_+D_- + D_-D_+)x$. From Eqs. 21 and 22 we see that

$$D_+x(t) = D_-x(t) = v(t), \tag{30}$$

where conditioning on $v(t)$ does not matter. Next we calculate $D_+D_-x(t)$ and $D_-D_+x(t)$ conditioned on $(x(t), v(t))$ using Eqs. 25 and 26:

$$D_+D_-x(t) = D_+|_{(x(t),v(t))}v(t) = a(x(t)) \tag{31}$$

$$D_-D_+x(t) = D_-|_{(x(t),v(t))}v(t) = a(x(t)) - \sigma^2\frac{\partial}{\partial v}\log\rho(x,v,t). \tag{32}$$

Hence

$$\frac{1}{2}(D_+D_- + D_-D_+)x(t)|_{(x(t),v(t))} = a(x(t)) - \frac{\sigma^2}{2}\frac{\partial}{\partial v}\log\rho(x,v,t). \tag{33}$$

In order the calculate the stochastic acceleration, which is conditioned only on $x(t)$, we use equation4.5:

$$\begin{aligned}\frac{1}{2}(D_+D_- + D_-D_+)x(t)|_{x(t)} &= \int \frac{1}{2}(D_+D_- + D_-D_+)x(t)|_{(x(t),v(t))}p_t(v|x)dv \\ &= a(x(t)) - \frac{\sigma^2}{2}\int_{-\infty}^{\infty}\frac{\partial p_t(v|x)}{\partial v}dv = a(x(t)).\end{aligned} \tag{34}$$

where the total derivative term $\frac{\sigma^2}{2}\int_{-\infty}^{\infty}\frac{\partial p_t(v|x)}{\partial v}dv$ vanishes for bounded energy E since $p_t(v|x) = 0$ for $|v| \geq \sqrt{\frac{2E}{m}}$. Thus we have shown that the Newton–Nelson law is satisfied by the process given by Eq. 23. This result was stated without proof in [11] for the particle in a potential subject to linear friction in equilibrium.

5 Method of Stochastic Averaging

In this section we introduce the method of averaging of stochastic differential equations. There are several formulations of stochastic averaging though we will only consider the theorem due to Khas'minskii [13–17] applied to two dimensional systems in Ito form. Consider the process (x, y):

$$dx(t) = \epsilon^2 f_1(x(t), y(t), t)dt + \epsilon g_1(x(t), y(t), t)dW(t) \tag{35}$$

$$dy(t) = \epsilon^2 f_2(x(t), y(t), t)dt + \epsilon g_2(x(t), y(t), t)dW(t) \tag{36}$$

where dW is the Wiener process and $0 < \epsilon \ll 1$ which means that (x, y) are slowly varying in time as compared to f_i and g_i. We assume that f_i and g_i are sufficiently continuously differentiable and bounded. Then for times of order $O(1/\epsilon^2)$ the dynamics can be uniformly approximated (over the time interval) by the following averaged system[1]:

$$dx(t) = \epsilon^2\bar{f}_1(x(t), y(t))dt + \epsilon\sigma_{11}(x(t), y(t))dW_1(t) + \epsilon\sigma_{12}(x(t), y(t))dW_2(t) \tag{37}$$

$$dy(t) = \epsilon^2\bar{f}_2(x(t), y(t))dt + \epsilon\sigma_{21}(x(t), y(t))dW_1(t) + \epsilon\sigma_{22}(x(t), y(t))dW_2(t) \tag{38}$$

[1] More precisely the original process converges weakly to the averaged process as $\epsilon \to 0$.

where dW_1 and dW_2 are independent Wiener processes and the averaged functions are given by

$$\bar{f}_i(x,y) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f_i(x,y,t)dt \tag{39}$$

$$(\sigma\sigma^T)_{ij}(x,y) = \lim_{T\to\infty} \frac{1}{T}\int_0^T g_i(x,y,t)g_j(x,y,t)dt \tag{40}$$

where σ^T denotes the matrix transpose of σ. Note that σ is unique up to a time dependent orthogonal transformation $R(t)$ as $\sigma R(t)(\sigma R(t))^T = \sigma\sigma^T$. It can be shown that $R(t)[dW_1(t)\, dW_2(t)]^T$ is again a Wiener process therefore replacing σ with $\sigma R(t)$ does not change the diffusion process (x, y). For applications below we need the periodic version of averaging. For periodic systems we can write

$$\bar{f}_i(x,y) = \frac{1}{T}\int_0^T f_i(x,y,t)dt \tag{41}$$

$$\sigma\sigma^T)_{ij}(x,y) = \frac{1}{T}\int_0^T g_i(x,y,t)g_j(x,y,t)dt \tag{42}$$

where T is the period of oscillations. The stochastic averaging principle is a generalization of its deterministic version which can found in [18]. For deterministic averaging of a one dimensional system in action-angle variables see [19]. For more on stochastic averaging see the review [16] and the books [15, 17].

6 Nelson's First Postulate for a Harmonic Oscillator

Consider the harmonic oscillator with frequency ω and mass m subject to a random force with position and velocity variables (x, v):

$$\begin{aligned} dx(t) &= v(t)dt \\ dv(t) &= -\omega^2 x(t)dt + \epsilon\omega dW(t). \end{aligned} \tag{43}$$

Assume that the initial energy of the oscillator is E_0 is probability 1. We will show that we can make the choice $\epsilon = \sqrt{\frac{2\hbar}{m}} \ll 1$ such that the position process $x(t)$ is approximately Markovian and satisfies Nelson's first postulate for a time interval depending on $\hbar$, E_0 and ω. We show this by approximating the dynamics given by Eq. 43 by an averaged process using the method of stochastic averaging. We will see that this will induce a noise term for the x variable which is necessary to satisfy Nelson's first postulate. Note that the noise term appearing in Eq. 43 is not necessarily small as it is proportional to $\epsilon\omega$ as ω can be large.

To proceed note that the dynamics in phase space is not in standard form for averaging. Therefore apply the coordinate transformation

$$\begin{aligned} x &= r\cos(\omega t + \phi) \\ v &= -\omega r \sin(\omega t + \phi) \end{aligned} \tag{44}$$

or

$$\begin{aligned} r &= \sqrt{x^2 + \frac{v^2}{\omega^2}} \\ \phi &= -\arctan\left(\frac{v}{\omega x}\right) - \omega t. \end{aligned} \tag{45}$$

To calculate the differential of r and ϕ we use Ito's lemma and obtain

$$\begin{aligned} dr &= \frac{x}{r}dx + \frac{v}{\omega^2 r}dv + \frac{1}{2}\frac{x^2}{\omega^2 r^3}(dv)^2 = \frac{(\epsilon\omega)^2}{2}\frac{x^2}{\omega^2 r^3}dt + \epsilon\omega\frac{v}{\omega^2 r}dW \\ d\phi &= \frac{v}{\omega r^2}dx - \frac{x}{\omega r^2}dv + \frac{1}{2}\frac{2xv}{\omega^3 r^4}(dv)^2 - \omega dt = (\epsilon\omega)^2\frac{xv}{\omega^3 r^4}dt - \epsilon\omega\frac{x}{\omega r^2}dW. \end{aligned} \tag{46}$$

We see that both r and ϕ are slowly varying. Therefore we apply the method of averaging over one period $T = \frac{2\pi}{\omega}$ of the harmonic oscillator. This amounts to fixing r and averaging over the angle variable. Denote the time average of a function $f(x, v)$ by

$$\langle f(x,v)\rangle_T = \frac{1}{T}\int_0^T f(x(t), v(t))dt. \tag{47}$$

The evolution equations can be approximated by the following averaged equations over time intervals of order $O(1/\epsilon^2)$ noting that the noise terms are of order ϵ and the dt terms are of order ϵ^2:

$$\begin{aligned} dr &= \frac{(\epsilon\omega)^2}{2}\frac{1}{\omega^2 r^3}\langle x^2\rangle_T dt + \epsilon\omega\frac{1}{\omega^2 r}\sqrt{\langle v^2\rangle_T}dW_1 \\ d\phi &= (\epsilon\omega)^2\frac{1}{\omega^3 r^4}\langle xv\rangle_T dt + \epsilon\omega\frac{1}{\omega r^2}\sqrt{\langle x^2\rangle_T}dW_2 \end{aligned} \tag{48}$$

where dW_1 and dW_2 are independent Wiener processes. The averaged quantities are calculated to be

$$\begin{aligned} \langle x^2\rangle_T &= \frac{1}{2\pi}\int_0^{2\pi} r^2\cos^2\theta d\theta = \frac{r^2}{2} \\ \langle v^2\rangle_T &= \frac{\omega^2}{2\pi}\int_0^{2\pi} r^2\sin^2\theta d\theta = \frac{\omega^2 r^2}{2} \\ \langle xv\rangle_T &= -\frac{\omega}{2\pi}\int_0^{2\pi} r^2\sin\theta\cos\theta d\theta = 0 \end{aligned} \tag{49}$$

where we have chosen σ in Eq. 42 as diagonal as $\langle xv \rangle_T = 0$. Substitute the averaged quantities in the averaged equations to get:

$$\begin{aligned} dr &= \frac{\epsilon^2}{4}\frac{1}{r}dt + \epsilon\frac{1}{\sqrt{2}}dW_1 \\ d\phi &= \epsilon\frac{1}{\sqrt{2}r}dW_2. \end{aligned} \tag{50}$$

The averaged evolution of the amplitude of oscillations r is independent of ϕ and the evolution of ϕ is determined by the evolution of r. Using averaged equations we can derive the averaged evolution of the position variable x using Eqs. 44 and 50:

$$\begin{aligned} dx &= dr\cos(\omega t + \phi) - r\sin(\omega t + \phi)(\omega dt + d\phi) - \frac{1}{2}r\cos(\omega t + \phi)(d\phi)^2 \\ &= vdt + \epsilon\frac{1}{\sqrt{2}r}(xdW_1 + \frac{v}{\omega}dW_2). \end{aligned} \tag{51}$$

Note that the method of stochastic averaging induced a stochastic term for the x variable where in the original dynamics defined by Eq. 43 the stochastic term only appears in the dynamics of the velocity variable. We can simplify the stochastic term noting that given $x(t)$ and $v(t)$, $dW_1(t)$ and $dW_2(t)$ are independent Gaussian processes. A linear combination

$$a(x(t), v(t))dW_1(t) + b(x(t), v(t))dW_2(t) \tag{52}$$

of independent zero mean Gaussian processes is again a zero mean Gaussian process with variance $a^2(x(t), v(t)) + b^2(x(t), v(t))$. Therefore the equation for the position variable can be written as:

$$dx = vdt + \frac{\epsilon}{\sqrt{2}}dW \tag{53}$$

where dW is the Wiener process. In general this is not a Markov process since v itself is fluctuating and is dependent on x. However if somehow we can assume that the amplitude r is constant then we can express v in terms of x as

$$v = \pm\omega\sqrt{r^2 - x^2} \tag{54}$$

obtaining the Markov process

$$dx = \pm\omega\sqrt{r^2 - x^2}dt + \frac{\epsilon}{\sqrt{2}}dW. \tag{55}$$

Now to match with Nelson's first postulate (Eq. 15) we must choose

$$\epsilon = \sqrt{\frac{2\hbar}{m}}. \tag{56}$$

With this choice of ϵ we can justify the assumption that r remains approximately constant as follows. Assume that initially $r = r_0$ with probability 1. For sufficiently small times we can assume that r is well approximated by r_0. To see this introduce the energy variable

$$E = \frac{1}{2}m\omega^2 r^2. \tag{57}$$

Using Ito's lemma its dynamics is calculated to be:

$$dE = (\epsilon\omega)^2 \frac{m}{2}dt + \epsilon\omega\sqrt{mE}dW_1 \tag{58}$$

with $E_0 = \frac{1}{2}m\omega^2 r_0^2$. Set $E(t) = E_0 + \delta E(t)$. Roughly $\delta E(t)$ grows as

$$\max(\epsilon^2\omega^2 mt, \epsilon\omega\sqrt{mE_0 t}) \tag{59}$$

where $\sqrt{t}$ dependence arises from the Wiener term. The method of stochastic averaging is accurate for times $O(1/\epsilon^2)$. Therefore if $\epsilon^2\omega^2 mt \ll E_0$ and $t \ll 1/\epsilon^2$, we can assume that $\delta E(t)$ is small therefore $E(t) \approx E_0$ and $r(t) \approx r_0$. Putting $\epsilon = \sqrt{2\hbar/m}$, $E(t)$ can be treated as constant in the time interval $O(\min(E_0/\hbar\omega^2, 1/\epsilon^2))$. Hence the approximation holds for longer times t as E_0 gets larger as long as $t < m/\hbar$. We can write the dynamics in the following suggestive form

$$dx = v(x, E_0)dt + \sqrt{\frac{\hbar}{m}}dW \tag{60}$$

where $v(x, E_0) = \pm\sqrt{\frac{2}{m}(E_0 - \frac{1}{2}m\omega^2 x^2)}$ is the classical velocity of the particle with energy E_0. If the stochastic term is absent then this equation would be the classical equation of motion for the particle. Hence the phase space diffusion process gives rise to a position space Markov process as a small random fluctuation around the classical trajectory.

We initially made the assumption that the random force depends on the frequency of the oscillator. Then it is natural to ask whether we can choose a universal random force term which would give the same result for an arbitrary frequency. Such a choice is indeed possible. So instead of the Markovian model start from

$$\begin{aligned} \dot{x}(t) &= v(t) \\ \dot{v}(t) &= -\omega^2 x(t) + \xi(t). \end{aligned} \tag{61}$$

Let $\xi(t)$ be a zero mean Gaussian process with covariance

$$\langle \xi(t)\xi(t+\tau)\rangle = c(\tau) \tag{62}$$

with its Fourier transform, the power spectrum

$$S(\Omega) = \int_{-\infty}^{\infty} c(\tau)e^{-i\Omega\tau}d\tau. \tag{63}$$

It can be shown that upon averaging the (r, ϕ) evolution over one period of the oscillator, only the resonant term corresponding to $\Omega = \omega$ contributes to the averaged equations [16]:

$$\begin{aligned} dr &= \frac{S(\omega)}{8\pi}\frac{1}{\omega^2 r}dt + \sqrt{\frac{S(\omega)}{4\pi}}\frac{1}{\omega}dW_1 \\ d\phi &= \sqrt{\frac{S(\omega)}{4\pi}}\frac{1}{\omega r}dW_2. \end{aligned} \tag{64}$$

Thus if we choose $S(\Omega) = \frac{4\pi\hbar}{m}\Omega^2$, we recover the previous results. Note that the averaged equations obtained for the colored noise are the same as the ones obtained before for white noise with spectrum dependent on frequency. Hence the processes defined by Eqs. 43 and 61 are good approximations to each other. Therefore Newton–Nelson law is approximately satisfied for the colored noise case as it is satisfied for equation 6.1.

Now we consider small potential perturbations around the harmonic oscillator:

$$\begin{aligned} dx(t) &= v(t)dt \\ dv(t) &= -\omega^2 x(t)dt - \eta\frac{1}{m}\frac{dU(x)}{dx}|_{x=x(t)}dt + \epsilon\omega dW(t) \end{aligned} \tag{65}$$

where $\frac{dU(x)}{dx}$ is $O(1)$ and $\eta \ll 1$. We implement the change of variables (eq.6.2). The resulting equations for (r, ϕ) is then given by

$$\begin{aligned} dr &= -\eta\frac{v}{\omega^2 r}\frac{1}{m}\frac{dU(x)}{dx}dt + \frac{(\epsilon\omega)^2}{2}\frac{x^2}{\omega^2 r^3}dt + \epsilon\omega\frac{v}{\omega^2 r}dW \\ d\phi &= \eta\frac{x}{\omega r^2}\frac{1}{m}\frac{dU(x)}{dx}dt + (\epsilon\omega)^2\frac{xv}{\omega^3 r^4}dt - \epsilon\omega\frac{x}{\omega r^2}dW. \end{aligned} \tag{66}$$

We assume that the terms involving η is small. Therefore for times $O(\max(1/\eta, 1/\epsilon^2))$, we can apply the method of stochastic averaging to all terms in the (r, ϕ) equation. However since we would like to retain the terms corresponding to the potential perturbation, we only average over the ϵ terms but not over the η terms. After averaging, we go back to (x, v) variables to obtain

$$dx = v(x, E_0)dt + \frac{\epsilon}{\sqrt{2}}dW \tag{67}$$

where this time $v(q, E_0)$ is the velocity associated to the perturbed potential:

$$v(x, E_0) = \pm\sqrt{\frac{2}{m}(E_0 - \frac{1}{2}m\omega^2 x^2 - \eta U(x))}. \tag{68}$$

The argument for that E remains approximately constant is similar to the one obtained for the pure harmonic oscillator.

7 Nelson's First Postulate for a General Potential

In this section we would like to generalize the results for the harmonic oscillator to a general potential. However we will be able to show less. Due to the difficulty in calculating averages explicitly for general potentials we will be only able to show that we can choose the random force dependent on energy (unlike the harmonic oscillator case where the random force is independent of coordinates) such that Nelson's first law is satisfied using a heuristic averaging procedure. We will restrict to potentials for which the motion can be described using action-angle variables. Therefore consider a particle of mass m in one dimension in a potential $U(x)$ subject to a random force:

$$\begin{aligned} dx(t) &= \frac{p(t)}{m}dt \\ dp(t) &= -U'(x(t))dt + \epsilon dW(t) \end{aligned} \tag{69}$$

where $(x, p = mv)$ denotes the position and momentum variables and $U'(x) = \frac{dU(x)}{dx}$. We first perform the coordinate transformation from (x, p) to (x, E) where E is the energy of the particle defined by

$$E(x,p) = \frac{p^2}{2m} + U(x). \tag{70}$$

Using Ito's lemma we calculate dE as

$$dE = U'(x)dx + \frac{p}{m}dp + \frac{1}{2m}(dp)^2 = \frac{\epsilon^2}{2m}dt + \epsilon\frac{p(x,E)}{m}dW. \tag{71}$$

Note that if the stochastic term is absent then the energy would be conserved. We can also express dx in terms of (x, E) by solving for p in terms of (x, E) in the definition of energy:

$$dx = \pm\frac{1}{m}\sqrt{2m(E - U(x))}dt. \tag{72}$$

We now assume that the classical motion can be described by action-angle variables (ϕ, I) [19]. In terms of the action-angle variables the classical deterministic equations of motion can be written as

$$\begin{aligned} d\phi &= \omega(I)dt \\ dI &= 0 \end{aligned} \tag{73}$$

where the frequency is

$$w(I) = \frac{dE(I)}{dI} \tag{74}$$

and the energy is a function of the action variable alone. Instead of the action variable we will use the energy variable since the energy is a function of the action but

not the angle variable. We further assume that this mapping is one-to-one. Define the action function (not the action variable) as

$$S(I,x) = \int_{x_0}^{x} p(x',E)dx' \tag{75}$$

for an arbitrary initial point x_0. Then the action variable is defined to be proportional to the action function S over one period of motion:

$$I = \frac{1}{2\pi}\oint pdx' \tag{76}$$

and the angle variable is

$$\phi = \frac{\partial S(I,x)}{\partial I} = \int_{x_0}^{x} \frac{\partial p(x',E)}{\partial I}dx'. \tag{77}$$

Taking the derivative inside the integral we have

$$\phi = \omega(I)\int_{x_0}^{x} \frac{\partial p(x',E)}{\partial E}dx' = m\omega(I)\int_{x_0}^{x} \frac{1}{p}dx'. \tag{78}$$

We define

$$f(x,E) = \int_{x_0}^{x} \frac{1}{p}dx' \tag{79}$$

so that

$$\phi(x,E) = m\omega(I(E))f(x,E). \tag{80}$$

Having defined the angle variable we are ready to perform the change of coordinates from (x, E) to (ϕ, E). Using Ito's lemma we calculate $d\phi$ as

$$\begin{aligned} d\phi &= m\frac{\partial}{\partial E}(\omega f)dE + \frac{m}{2}\frac{\partial^2}{\partial E^2}(\omega f)(dE)^2 + m\omega\frac{\partial f}{\partial x}dx \\ &= \omega dt + \frac{\epsilon^2}{2}\left(\frac{\partial}{\partial E}(\omega f) + \frac{p^2}{m}\frac{\partial^2}{\partial E^2}(\omega f)\right)dt + \epsilon\frac{\partial}{\partial E}(\omega f)pdW. \end{aligned} \tag{81}$$

We see that ϕ is slowly varying except the ωdt term. In order to have all the right hand side terms small we further introduce the new angle variable θ as

$$\theta = \phi - \omega t \tag{82}$$

and compute its differential as

$$
\begin{aligned}
d\theta &= d\phi - \frac{d\omega}{dE} t dE - \frac{1}{2}\frac{d^2\omega}{dE^2} t (dE)^2 - \omega dt \\
&= \frac{\epsilon^2}{2}\left(\frac{\partial}{\partial E}(\omega f) + \frac{p^2}{m}\frac{\partial^2}{\partial E^2}(\omega f) - t\frac{d^2\omega}{dE^2}\frac{p^2}{m^2}\right)dt - \frac{\epsilon^2}{2m}\frac{d\omega}{dE} t dt \\
&\quad + \epsilon\frac{p}{m}\left(m\frac{\partial}{\partial E}(\omega f) - t\frac{d\omega}{dE}\right)dW.
\end{aligned} \tag{83}
$$

We have finalized the set of coordinate transformations which yielded slowly varying (E, θ) coordinates. Next we average the dynamics over a period $T = \frac{2\pi}{\omega(E)}$ fixing E in $(x(t), p(t))$ to obtain the approximate averaged equations. The averaged equations for (E, θ) are

$$
\begin{aligned}
dE &= \frac{\epsilon^2}{2m} dt + \epsilon\sigma_{11}(E) dW_1 + \epsilon\sigma_{12}(E) dW_2 \\
d\theta &= \frac{\epsilon^2}{2} F(E) dt + \epsilon\sigma_{21}(E) dW_1 + \epsilon\sigma_{22}(E) dW_2,
\end{aligned} \tag{84}
$$

where dW_1 and dW_2 are independent Wiener processes, $F(E)$ is given by

$$
F(E) = \left\langle \omega\frac{\partial f}{\partial E} + \frac{p^2}{m}\left(2\frac{d\omega}{dE}\frac{\partial f}{\partial E} + \omega\frac{\partial^2 f}{\partial E^2}\right)\right\rangle_T \tag{85}
$$

and the matrix σ is determined from

$$
(\sigma\sigma^T)_{ij} = D_{ij} \tag{86}
$$

The components of D are the averages

$$
\begin{aligned}
D_{11} &= \left\langle \frac{p^2}{m^2}\right\rangle_T \\
D_{12} = D_{21} &= \left\langle \frac{p^2}{m}\omega\frac{\partial f}{\partial E}\right\rangle_T \\
D_{22} &= \left\langle p^2\omega^2\left(\frac{\partial f}{\partial E}\right)^2\right\rangle_T
\end{aligned} \tag{87}
$$

where we have simplified $F(E)$ and D_{ij} noting that since when we take averages over the classical trajectory we can set $dx = \frac{p}{m}dt$ so that $t = mf$. Now we will go back to the dynamics of x to check if Nelson's first postulate is satisfied. First calculating the dynamics of (E, ϕ) then calculating the dynamics of x we obtain the equation of the form

$$
dx = \frac{p}{m}dt + \epsilon^2 K(x, p) + \epsilon(G_1(x, p) dW_1 + G_2(x, p) dW_2) \tag{88}
$$

for some functions $K(x,\ p)$, $G_1(x, p)$ and $G_2(x, p)$. The stochastic term is dependent on coordinates. We invoke without rigorous justification a heuristic averaging

procedure although dx is not in the standard form. We fix E and average the small terms over the angles.

$$dx = \frac{p}{m}dt + \epsilon^2 \bar{K}(E) + \epsilon(\bar{G}_1(E)dW_1 + \bar{G}_2(E)dW_2) \tag{89}$$

where $\bar{K}(E)$, $\bar{G}_1(E)$ and $\bar{G}_2(E)$ are the averages of $K(x,p)$, $G_1(x,p)$ and $G_2(x,p)$. Since $dW_1(t)$ and $dW_2(t)$ are independent zero mean Gaussian processes we can write

$$dx = \frac{p}{m}dt + \epsilon^2 \bar{K}(E) + \epsilon G(E)dW \tag{90}$$

where dW is the Wiener process and $G^2(E) = \bar{G}_1(E)^2 + \bar{G}_2(E)^2$. Now as in the harmonic oscillator case for sufficiently small times E is almost constant. We are not able to calculate $G(E)$ explicitly in terms of E(or the action variable I) and $\omega(E)$ and its derivatives since the averages appearing in D_{ij} seem to be difficult to evaluate analytically. Therefore unlike the harmonic oscillator case we have the weaker result: one can choose

$$\epsilon G(E) = \sqrt{\frac{\hbar}{m}} \tag{91}$$

to satisfy Nelson's first postulate. However in this case we need to choose ϵ dependent on E and this is unfavorable regarding universality: for every value of energy we need to choose a different correlation coefficient for the noise. We saw in the harmonic oscillator case that choosing the colored random force with spectrum proportional to ω^2 we can satisfy Nelson's first postulate for all oscillator frequencies. If one could evaluate $G(E)$ explicitly and if one can verify the heuristic method of averaging, one can check whether the same random force spectrum gives rise to Nelson's first law for arbitrary potentials admitting action-angle description.

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We have shown that it is possible to choose a random force such that fluctuations around the classical trajectory can be described by the Schrödinger equation for the harmonic oscillator. We achieved this by showing that the Newton–Nelson law is satisfied for general potentials and Nelson's first postulate is approximately satisfied for states described by constant energy upon averaging the dynamics. We further generalized this result to the case of small perturbations around the harmonic oscillator. We also showed that there exists a colored Gaussian noise such that Nelson's two postulates are approximately satisfied for an oscillator with any frequency with fixed mass. We attempted to generalize the results for the harmonic oscillator to a general potential admitting action-angle variables. However due to the difficulty in evaluating averages and the need for an heuristic averaging principle we are only able to show that there is a random force depending on energy, mass and frequency of the system such that we can obtain Nelson's first postulate.

In this paper we did not touch the issue of superposition states. Although Nelson's formulation is equivalent to the Schrödinger equation and describes superposition states there is no guarantee that the Nelson's process associated to a superposition state could be derived from a phase space process. In an upcoming paper, we show that all the superposition states of a harmonic oscillator can be derived from the phase space process presented in this paper [2].

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