



# The Hole Argument in Homotopy Type Theory

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## Abstract

The Hole Argument is primarily about the meaning of general covariance in general relativity. As such it raises many deep issues about identity in mathematics and physics, the ontology of space–time, and how scientific representation works. This paper is about the application of a new foundational programme in mathematics, namely homotopy type theory (HoTT), to the Hole Argument. It is argued that the framework of HoTT provides a natural resolution of the Hole Argument. The role of the Univalence Axiom in the treatment of the Hole Argument in HoTT is clarified.

**Keywords** Hole argument · Homotopy type theory · General covariance · Univalence

## 1 Introduction

The Hole Argument is primarily about the meaning of general covariance in general relativity (GR). As such it raises many deep issues about identity in mathematics and physics, the ontology of space–time, and how scientific representation works. This paper is about the application of a new foundational programme in mathematics, namely homotopy type theory (HoTT), to the Hole Argument. It is argued that the framework of HoTT provides a natural resolution of the Hole Argument. The role of the Univalence Axiom in the treatment of the Hole Argument in HoTT is clarified.

Einstein said that “[t]he general laws of nature are to be expressed by equations which hold good for all systems of co-ordinates [...] (generally co-variant)” [4]. This does not mean that the laws of physics take the same form in all coordinate systems (although that remains a common and not unreasonable rough formulation (see, for example, Dainton [2], [372]), because, for example, classical electromagnetism can be formulated in a generally covariant way (very briefly,  $dF = 0$  and  $d^*F = j$ ,

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where  $F$  is the Maxwell–Faraday tensor representing the electromagnetic field, and  $j$  is the ‘four-current’ that represents charge density and current density), but the familiar Maxwell equations are only recovered in inertial coordinate systems. Rather, general covariance means the equations of the theory are formulated in a way that is independent of co-ordinate systems as is the language of the tensor calculus.<sup>1</sup> GR was the first theory to be formulated in a generally covariant way. The theory is defined by the Einstein Field Equation as follows (in units chosen so that the gravitational constant and the speed of light are both 1 and without the cosmological constant).

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}$$

Models of GR are Lorenzian manifolds with metric and stress-energy fields defined on them that together satisfy this equation, and so (dropping the indices) they are of the form  $\langle M, g, T \rangle$ . If there are electromagnetic fields or other fields then they contribute to  $T$ , but models of GR often involve only gravitating matter, or may have no matter at all.

The maps that transform between coordinate systems defined on differential manifolds are diffeomorphisms. A **diffeomorphism** is a smooth (i.e. infinitely differentiable) transformation between manifolds that has a smooth inverse. Diffeomorphisms on manifolds induce mappings on the geometrical objects defined on the manifold—the metric and the matter fields.

A diffeomorphism  $h$  from  $M$  to  $M'$  lifts to give an **isometry**  $(M, g) \rightarrow (M', h^*g)$ . ( $h^*$  is also automatically defined for any other fields on  $M$ ).<sup>2</sup>

Any two models of GR that are related by diffeomorphism are empirically equivalent, in the sense that all questions about measurable physical quantities have the same answers in any two models that are related by diffeomorphism. In the case where  $M' = M$ , the diffeomorphism group partitions the set of solutions into equivalence classes, where two equivalent models have the same observable properties. The set of solutions is closed under diffeomorphisms meaning that, for any  $h$  and any  $\langle M, g, T \rangle$  that is a model,  $\langle M, h^*g, h^*T \rangle$  is also a model.<sup>3</sup>

While theories other than GR can be given a generally covariant formulation, their sets of solutions are not closed under the action of the diffeomorphism group, but under proper subgroups of it. So, for example, the set of solutions of SR is closed under the Lorentz transformations, and the set of solutions of Newtonian gravitation is closed under the Galilean group. What makes GR special is that the invariance group of the fields that satisfy the Einstein equation is the diffeomorphism group, so the solutions of GR are closed under arbitrary coordinate transformations. This is why general covariance seems essential to GR, but not to Newtonian gravity, special relativistic mechanics, or classical electrodynamics.

<sup>1</sup> Thanks to Sam Fletcher and Jim Wetherall for discussion of this and other issues, and also to Dan Christensen, John Dougherty, Michael Shulman, and two anonymous referees. Kretschmann [6] argued that Einstein’s prescription was methodologically inert because any theory can be written in a generally covariant way in the sense above by using the tensor calculus or some other mathematical ingenuity.

<sup>2</sup> Strictly the ‘pullback’  $h^*g$  should be written  $h^{-1*}g$ .

<sup>3</sup> The notation  $hM$  is incorrect because the  $h$  maps the points of  $M$ .

Being generally covariant is perhaps necessary, but is definitely not sufficient for this ‘general invariance’. There must also be no absolute or prior geometric elements independent of matter sources. If this is somehow necessary and sufficient for general invariance, and hence to capture what is special about GR, it is difficult to make rigorous (see [5]).

Note in the philosophical literature it is standardly assumed that two models  $\langle M, g, T \rangle$  and  $\langle M', g', T' \rangle$  are **isomorphic** iff their underlying manifolds are isomorphic, and for some diffeomorphism  $h$ ,  $g' = h^*g$  and  $T' = h^*T$ . In the case where  $M = M'$ , the diffeomorphism  $h$  may be thought of as only being applied to  $g$  and  $T$  and not to the points of  $M$ . This is important in what follows. The next section explains the Hole Argument as it is usually formulated.

## 2 The Hole Argument

Given some model of GR,  $\langle M, g, T \rangle$ , any diffeomorphism  $h$  that maps  $M$  to itself can be extended to a transformation on the model, giving  $\langle M, h^*g, h^*T \rangle$  which is also an empirically equivalent model. In particular, for any  $A^\alpha, B^\beta$  and  $A^{\bar{\alpha}}, B^{\bar{\beta}}$  representing two particular spacetime events in the two models:

$$g^{\alpha\beta}(A^\alpha, B^\beta) = h^*g^{\bar{\alpha}\bar{\beta}}(h^*A^{\bar{\alpha}}, h^*B^{\bar{\beta}})$$

This had better be so because these expressions compute in their respective coordinate systems to give the spatiotemporal distance between events which is an observable, and should be the same for all observers if the local lightcone structure is to be fully invariant.

Consider some model  $\langle M, g, T \rangle$ , and apply a diffeomorphism that is the identity everywhere outside some region, called ‘the Hole’, and not the identity inside the Hole. The result must also be a model of the field equations. Consider a point  $p \in M$  inside the Hole. In the new model in general  $p$  has different values of the fields than it does in the old model. Hence, for example, the worldline of some galaxy going through the Hole may pass through  $p$  in one model but not in the other.

If we regard the points of  $M$  as representing the points of spacetime, so that the two models represent observationally equivalent but different possible worlds, then we have a failure of determinism if some of the past state of the galaxy lies outside of the Hole, because this past state does not determine whether the galaxy passes through  $p$  or not. Einstein initially took this to mean we could not have a generally covariant theory [10].

$\langle M, g, t \rangle$  and  $\langle M, g', t' \rangle$  are standardly (see, for example, [10]) defined to be **Leibniz equivalent** iff  $g' = h^*g$  and  $T' = h^*T$  for some diffeomorphism  $h$  (so they are Leibniz equivalent iff they are isomorphic in the sense of the definition at the end of the previous section). The **relationalist** takes Leibniz equivalent models to represent the same world so the Hole Argument is not a problem for them. According to Earman and Norton [3], however, the substantialist must think that Leibniz equivalent models can in general represent different worlds. This is because they take the substantialist to

be committed to **manifold substantivalism** according to which the points of the manifold  $M$  represent spacetime points, and the particular spacetime points represented by particular points of the manifold do not vary between the models above.

In particular Earman and Norton assume that if  $p \in M$  represents some spacetime point  $e$  in the context of one model, the same point must represent the same spacetime point in any other model built on  $M$ . However, it is possible to deny that the same  $p$  represents different spacetime points in different models while still maintaining substantialism in the sense that space-time is taken to exist independently of matter. There are various versions of this **sophisticated substantivalism** (see [11]). For example, according to **metric field substantivalism** spacetime points are represented by combinations of  $M$  and  $g$ , and the points of the bare manifold do not represent spacetime points at all.

Keeping track of which elements of the model represent which space-time points in different models is easier if the representation relation is thought of as a function from mathematics to physical events or spacetime points. The domain of the function could be the manifold  $M$  (as with manifold substantivalism). Then one form of sophisticated substantivalism is just the denial that the representation function is the same in different models. Alternatively, the domain could be defined to be the collection of all tuples  $\langle M, g \rangle$ , which gives metric field substantivalism according to which only the manifold and the metric together represent spacetime. What the Hole Argument shows depends on what is assumed about how spacetime is represented by models of GR. The putative indeterminism that is supposed to follow from substantivalism is only an issue if points of the manifold represent spacetime points independently of the rest of the model.

Philosophical debate about the Hole Argument aside, in scientific practice it is standardly supposed that Leibniz equivalent models of GR all represent the same physical world. The gravitational field can then be defined as an equivalence class of models ([12, Sect. 1.3], quoted by [13, p. 2]).

Diffeomorphisms are the isomorphisms in the category of differentiable manifolds and arguably, manifolds are only given up to diffeomorphism, and hence we might regard being related by a diffeomorphism as an identity criterion for manifolds. However, it does not follow that being related by diffeomorphism is an identity criterion for models of a given theory, because the assumption that diffeomorphism is the equivalence relation on models is a feature of GR not pure mathematics.

In the next section, the idea that HoTT helps with the Hole Argument is introduced.

### 3 Homotopy Type Theory to the Rescue?

HoTT, as explained below, includes the Univalence Axiom which says very roughly that equivalence is equivalent to identity. If diffeomorphism is taken to be the equivalence relation on models, under Univalence all models related by diffeomorphism are identified. Hence, it might be argued, the models in the Hole Argument related by the hole diffeomorphism necessarily represent the same possible world. The rest of this paper considers how this kind of argument might be made rigorous, and more generally how the difference between intensional type theory and set theory relates to the conceptual and philosophical issues concerning how models in GR represent spacetime.

Michael Shulman [13] argues that what is wrong with the Hole Argument is that the diffeomorphism  $h$  that witnesses the isometry of the two models is not the one that is used to compare the points in the two models (the latter being just the identity map between the manifold and itself). According to him, the apparent indeterminism and paradox of the Hole Argument only arise because the hole diffeomorphism  $h$  is used to generate a map  $h^*$  that is applied to the metric, while  $h$  is not applied to the manifold itself to generate the new model. Thinking in terms of set theory, which is extensional, it is not relevant whether  $h$  is applied to both because the model is just the collection of the manifold and the fields each of which is thought of as an independent entity and  $M$  is mapped to itself by  $h$ . It is explained in what follows why HoTT does not facilitate this way of constructing a model in GR. It turns out that the Univalence Axiom is not crucial: rather, everything depends on models of GR being taken to be dependent types, and how they differ from the cartesian products used in set theory. According to Erik Curiel, “[w]hen one applies a diffeomorphism, one must apply it to both the manifold and the metric...no other procedure has physical content”. ([1, p. 9]). If he is right then HoTT is a natural way to incorporate that constraint and so to avoid the Hole Argument altogether.

## 4 HoTT

### 4.1 Overview

HoTT is a **dependent type theory** that can be interpreted in homotopy theory, but can also be understood without it.<sup>4</sup> The basic components of the framework are **types** and **tokens** of types (sometimes called ‘terms’ or ‘elements’). For example, there is a type  $\mathbb{N}$ , the natural numbers, whose tokens are individual natural numbers e.g.  $5 : \mathbb{N}$ ,  $24 : \mathbb{N}$ , and so on (in general ‘ $x : A$ ’ means the term  $x$  is a token of the type  $A$ ).

There are also types corresponding to **propositions**. The tokens of these types are called ‘**certificates**’ to the truth of the proposition (roughly, proofs; often also called ‘witnesses’). A proposition may have multiple certificates. Some types can be read in both ways, for example,  $\text{Iso}(A, B)$  can be interpreted as both the proposition expressed by ‘ $A$  and  $B$  are isomorphic’, and the type of isomorphisms between  $A$  and  $B$ . Logical operations such as conjunction, disjunction, and implication correspond to the type constructions product, coproduct, and function types respectively, and quantification is achieved using dependent types. Thus HoTT incorporates (constructive) logic into the framework, rather than being built on top of a separate logic.

HoTT is **intensional**. The precise sense of this is explained in the next subsection, but it also relates to the fact that types are individuated by their definitions not their extensions. So, for example, the Zero type is a special type that is defined to have no tokens, though there may be other types that are also empty.

Something having a property, for example, a number being prime, is expressed in the theory by a *dependent pair* consisting of the thing and a certificate to the proposition

<sup>4</sup> See The Univalent Foundations Program [14] and for a more basic introduction to the language of HoTT, see Ladyman and Presnell [7].

that says it has that property. Hence, subtypes are more complex than their types, unlike subsets that are restrictions of their sets. For example, there is the type of *prime numbers*  $\mathbb{P}$  whose tokens are pairs, such as  $\langle 5, p_5 \rangle : \mathbb{P}$  and  $\langle 17, p_{17} \rangle : \mathbb{P}$ , where  $p_n$  is a certificate to the fact that  $n : \mathbb{N}$  is prime.

In HoTT, and in general, a **Lorentzian manifold** is not just a manifold  $M$  that *can* be equipped with a Lorentzian metric  $g$  (a function from pairs of points of the manifold to the real numbers), but rather a pair consisting of a manifold and a Lorentzian metric defined on it. In HoTT, but not in general, this is a dependent pair  $\langle M, g \rangle$ . The significance of this is explained in the following subsections.<sup>5</sup>

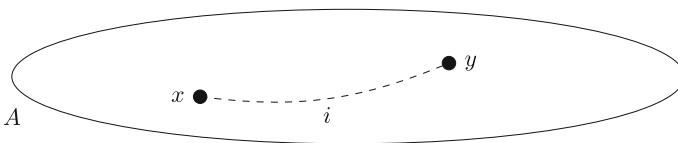
### 4.2 Internal and External Identity

HoTT has two different identity relations:

- **external identity**  $x \equiv y$  says about the language that ‘ $x$ ’ and ‘ $y$ ’ are two synonymous expressions (so this is not a statement in the language of HoTT)
- **internal identity**  $x = y$  is a proposition expressed by the language and represented by a type  $\mathbb{I}d_A(x, y)$  in the theory

Externally identical elements may be substituted for one another in any context, with no restrictions. Internal identity is not so simple, but this is the source of the interest and power of HoTT.<sup>6</sup>

Tokens of the type  $\mathbb{I}d_A(x, y)$  are called ‘**identifications**’ between tokens  $x$  and  $y$  of type  $A$ . In the homotopy interpretation these are represented as *paths* between points  $x$  and  $y$ . As explained below, properties of tokens can be ‘transported along paths’, meaning that tokens of propositions can be constructed to implement the law of the indiscernibility of identicals, that any property of  $x$  is also a property of  $y$  and vice versa. All functions respect internal identity (i.e.  $x = y$  implies  $f(x) = f(y)$ ). Nothing we can express within the language of HoTT can distinguish tokens that are internally identified. This means that the Univalence axiom can be consistently added to the basic framework of HoTT. It says that the equivalence type of two types is itself equivalent to the identity type of those two types. For present purposes, what matters is the consequence of this that two types being equivalent entails that they can be identified (see below).



<sup>5</sup> Thanks to Dan Christensen and Chris Kapulkin for discussion of manifolds as dependent pairs versus products.

<sup>6</sup> For much more on identity in HoTT see [8] and [9].

### 4.3 Multiple Identifications

Importantly, internal identity does not imply external identity because it does not allow for substitution in all contexts. This is the formal sense in which the theory is *intensional*. Thus in HoTT identifying two tokens of a type does not collapse them together because their identity type may itself have structure. Since  $\text{Id}_A(x, y)$  is a type, it may have multiple tokens each of which is an identification of  $x$  and  $y$ . (They may sometimes be thought of as multiple proofs that  $x$  and  $y$  are identical.) Every token of every type is trivially identical to itself, but also non-trivial identifications are allowed between any tokens of a type, including higher identity types. Under Univalence non-trivial automorphisms produce non-trivial identifications. This is an expression of the familiar idea of a non-trivial automorphism as a map from a structure to itself that is not just the identity map. The non-trivial automorphisms of differentiable manifolds are diffeomorphisms, and so in HoTT with Univalence they correspond to non-trivial identifications of manifolds with themselves.

### 4.4 Products and Dependent Pairs

Given any two types  $A$  and  $B$  we can form the **product** type  $A \times B$ , the tokens of which are ordered pairs  $(a, b) : A \times B$ , where  $a : A$  and  $b : B$ . This is one of the basic constructions in HoTT, not derivative as in ZFC set theory (where ordered pairs are sets of a certain form). Identity for products is componentwise so that  $(a, b) = (a', b')$  iff  $a = a'$  and  $b = b'$ .

HoTT is a dependent type theory in the sense that it allows definitions of types that depend upon tokens of other types. A family of types  $P(a)$  is indexed by a token of some other type  $A$ . For example, token  $x : A$  gives type  $P(x)$ , token  $y : A$  gives type  $P(y)$ , and so on. Given a family of types  $P(a)$  indexed over  $A$ , there is the **dependent pair type**  $\sum_{a:A} P(a)$ .

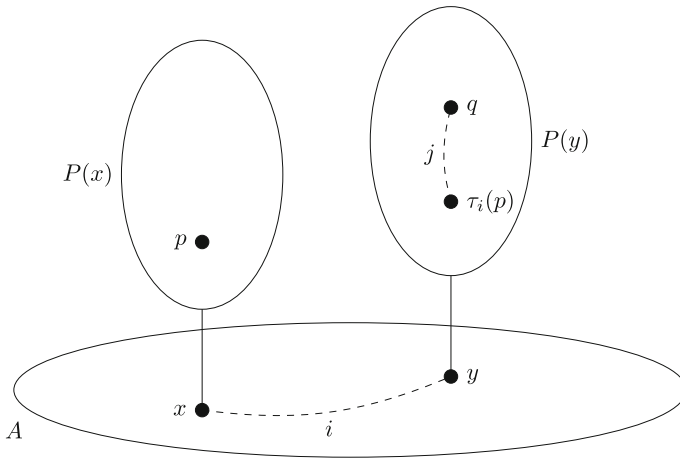
Tokens of  $\sum_{a:A} P(a)$  are **dependent pairs**  $\langle x, p \rangle$  whose first component is a token  $x : A$  and whose second component is a token  $p$  of the corresponding type  $P(x)$ .  $P$  can be thought of as a predicate on tokens of  $A$ . The type  $P(x)$  corresponds to the proposition that  $x$  has property  $P$ . A token  $p : P(x)$  is a certificate to this proposition.

Thus the dependent pair type  $\sum_{a:A} P(a)$  may be regarded (constructively) as the type of all tokens of  $A$  having property  $P$ .

Given an identification  $i$  between tokens in the ‘base type’, we get a **transport** function  $\tau_i : P(x) \rightarrow P(y)$ . Thus given a certificate that  $x$  has property  $P$ , and an identification between  $x$  and  $y$ , we can produce a certificate that  $y$  has property  $P$ .

The identity criterion for dependent pairs is different from that for products. Given two dependent pairs  $\langle x, p \rangle$  and  $\langle y, q \rangle$ , an identification between them consists in an identification  $i : \text{Id}_A(x, y)$  and an identification  $j : \text{Id}_{P(y)}(\tau_i(p), q)$ .

Recall that to identify products  $(a, b) = (a', b')$  we must have the corresponding identities  $a = a'$  and  $b = b'$  of the components. Dependent pairs may be identified even when there is no identity between their second components. In this case,  $p$  cannot be identified with  $q$  because they belong to different types. Rather it is  $\tau_i(p)$ , the transport of  $p$  along the identification  $i$  between  $x$  and  $y$ , that is identified with  $q$ .



### 4.5 Univalence

Univalence is an axiom that can be added to the basic rules of the type theory. Its exact form is a matter of some subtlety but it has the consequence that any two equivalent types can be identified.

Types  $A$  and  $B$  are **isomorphic** iff there are functions  $f : A \rightarrow B$  and  $g : B \rightarrow A$  such that

$$g(f(a)) = a \text{ for every } a : A$$

$$f(g(b)) = b \text{ for every } b : B$$

**Equivalence** is another relation between types that is similar to isomorphism, and for the present discussion the difference between them does not matter (because any equivalence is an isomorphism and vice versa).

Given any identification between types, we can immediately construct an equivalence between them. Thus there is a function  $\text{Id}(A, B) \rightarrow \text{Equiv}(A, B)$ .

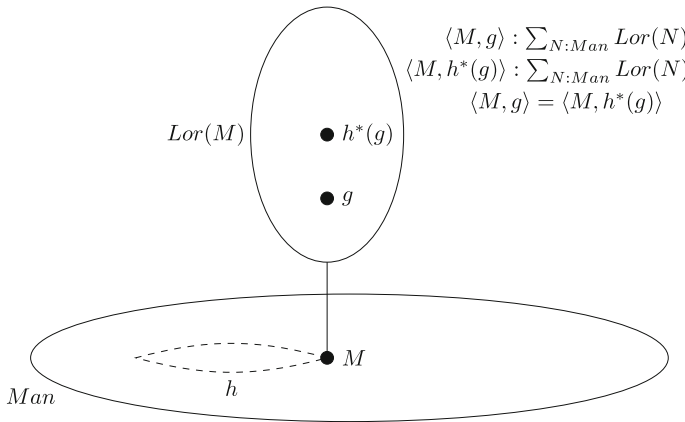
The **Univalence Axiom** entails that there is an inverse function to this  $\text{Equiv}(A, B) \rightarrow \text{Id}(A, B)$ .

### 5 HoTT and the Hole Argument

Consider the type  $\text{Man}$  of manifolds, and the dependent pair type  $\sum_{N:\text{Man}} \text{Lor}(N)$  of manifolds equipped with Lorentzian metrics. Tokens of this type are dependent pairs  $\langle M, g \rangle$ . Diffeomorphism between manifolds is an equivalence relation. Thus, under Univalence, every diffeomorphism  $h$  from  $M$  to itself produces a self-identification of  $M$ .



The transport function takes any self-identification  $h : M = M$  and gives a mapping from a metric  $g$  on  $M$  to a metric  $h^*(g)$  on  $M$ . (That is, the construction of isometries from diffeomorphisms is taken care of automatically by transport).



By the identity criterion for dependent pair types above, a non-trivial diffeomorphism  $h : M = M$  gives a corresponding identity  $\langle M, g \rangle = \langle M, h^*(g) \rangle$ , where  $M$  is identified with itself via  $h$  and  $h^*(g)$  is trivially identified with itself. On the other hand, if  $M$  is mapped to itself by the identity diffeomorphism, which is to say *trivially*, then there is only an identification between the dependent pairs  $\langle M, g \rangle$  and  $\langle M, h^*(g) \rangle$  if  $g = h^*(g)$ , which is not true for general diffeomorphisms  $h$ .

This distinction is made visible in HoTT, which allows us to express the non-trivial self-identity of  $M$ . Shulman’s objection (op. cit.) may be paraphrased as saying that ‘one should not apply  $I$  to  $M$  and  $\phi$  to  $g$ ’; this cannot be done with dependent pairs. However, the fact that a model is taken to be a dependent pair is nothing to do with Univalence. Furthermore, it can be stipulated that the representation function respect equivalence (in this case diffeomorphism) without Univalence. This is just what Leibniz Equivalence says about diffeomorphic models. HoTT enforces this only under assumption that  $\langle M, g \rangle$  is dependent pair, so Univalence alone is not sufficient. The assumption that  $\langle M, g \rangle$  is a dependent pair is sufficient if it is supposed that diffeomorphism is identity for models so Univalence is not necessary. But Univalence enforces this and that the representation function respects equivalence.

It might be possible within HoTT to form the Hole model of GR as a product rather than a dependent pair, by first forming the dependent pair, acting on it with  $h$ , and then projecting out the metric and any other fields and adding them into a product with the manifold. However, this would be a very artificial and unmotivated way to proceed counter to standard practice, and all notions of equivalence and identity for such products would have to be defined component-wise via the relevant definitions for dependent types.

## 6 Conclusion

The individuation of the metric  $g$  in a model of space-time is a subtle matter. Suppose it is specified to be the Lorentz metric. The definite article does not pick out a metric in the sense of a function from points to the real numbers, but rather the form of such a function. Any such actual function must be defined on a particular manifold. Type theory enforces clarity about such matters. It is implicitly assumed in ordinary mathematical practice that  $(M, g)$  is a dependent pair, since equivalence of Lorentzian manifolds is isometry generated by diffeomorphism on  $M$ , but HoTT enforces this.<sup>7</sup>

If the type of Lorentzian manifolds  $\text{LOR}(M)$  is a dependent pair, it is natural to take a model of the field equations of GR to be a dependent tuple. Accordingly the definition of equivalence should be that for a dependent tuple, requiring that to generate a diffeomorphic model the same diffeomorphism is applied to  $M$  as that applied to the metric and other fields defined on  $M$ , and so not allowing the formulation of the Hole Argument.

Nothing about the type theory or Univalence requires generally covariant representation, however the mathematics of manifolds is built for it, and dependent pairs in type theory can enforce it. The diffeomorphisms of a differential manifold automatically generate the pull back and push forward of any fields defined on it, and the equivalence between dependent tuples of manifolds and fields is determined by the underlying equivalences between manifolds, that is by the diffeomorphism group. Under Univalence this implies that the models in the Hole Argument are identical. However, even this is compatible with stipulating that they represent different possible worlds if representation is not required to respect identity. Furthermore, type theory can be used to represent the mathematics of a preferred coordinate system, and so to model physics in a non-generally covariant way if required.

The example of the Hole Argument suggests that HoTT may provide a good framework for tackling foundational problems in physics, for example, concerning the interpretation of gauge fields, concerning what and how mathematical structures should be taken to represent physical structures. Being a type theory, HoTT requires precision about how everything is defined, and being an intensional theory it allows symmetries to be represented as non-trivial identities, respecting the fact they map a structure to itself, while also keeping track of how they do so.

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<sup>7</sup> Note also that here  $M$  is a differentiable manifold and not just a set of points. Diffeomorphisms are permutations of points of the underlying set, that are also lifted to give maps that act on the structure (atlas and charts) defined on the underlying set that makes it a differentiable manifold.

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