Quantum Hydrodynamics: Kirchhoff Equations

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Abstract

In this paper, we show that the Kirchhoff equations are derived from the Schrödinger equation by assuming the wave function to be a polynomial like solution. These Kirchhoff equations describe the evolution of n point vortices in hydrodynamics. In two dimensions, Kirchhoff equations are used to demonstrate the solution to single particle Laughlin wave function as complex Hermite polynomials. We also show that the equation for optical vortices, a two dimentional system, is derived from Kirchhoff equation by using paraxial wave approximation. These Kirchhoff equations satisfy a Poisson bracket relationship in phase space which is identical to the Heisenberg uncertainty relationship. Therefore, we conclude that being classical equations, the Kirchhoff equations, describe both a particle and a wave nature of single particle quantum mechanics in two dimensions.

Keywords Schrödinger equation · Kirchhoff equations · n Point vortices · Paraxial wave equation

1 Introduction

Nine different formulations of non-relativistic quantum mechanics exist [1]. They are, wavefunction formalism, matrix mechanics, path integral formalism, phase space formalism, density matrix formalism, second quantization, variational formalism, pilot wave theory, and Quantum Hamilton–Jacobi (QHJ) formulations. Of these, the wave function formalism and the matrix mechanics are popular and few other formalisms attempt to map the quantum mechanics to classical mechanics. The well known one's are the phase space formalism [2] and pilot wave theory. In the phase space formulation, where Wigner quasi-probability distribution [2] is defined such that it links the wave function that appears in Schrödinger equation to a probability distribution in phase

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space, but this formalism has a draw back of negative probabilities. This mapping of quantum mechanics to classical mechanics is of central importance to the philosophy of physics, and also the interpretation of quantum mechanics.

One of the first attempts to find a classical relationship is given by Erwin Madelung known as Madelung quantum hydrodynamics [3] and the Madelung equations

$$\partial_t \rho_m + \nabla \cdot (\rho_m \mathbf{u}) = 0, \tag{1}$$

and

$$\frac{d\mathbf{u}}{dt} = \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{m} \nabla (Q + V) \tag{2}$$

where $\mathbf{u}$ is the flow velocity, $\rho_m = m\rho = m|\psi|^2$ is the mass density, $Q = -\frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}}$ $= -\frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{\rho_m}}{\sqrt{\rho_m}}$ is the Bohm quantum potential, and V is the potential from the Schrödinger equation. The Kirchhoff equations are given by

$$\frac{d\bar{z}_i}{dt} = \sum_{1 \leq i \leq n, i \neq j}^{n} \frac{i\Gamma_i}{z_i - z_j} + i W(z_i) \tag{3}$$

which describe the evolution of n point vortices in incomprehensible fluid [4], where $z_i = x_i + i y_i$ are position of vortices, Γ_i the circulation strength, $W(z_i)$ is background flow and $\bar{z}_i = x_i - i y_i$. are derived from Euler Eq. (1) and (2), for the complex velocity [5]. Readers should note that in deriving Kirchhoff equations quantum potential Q is not considered.

This Madelung quantum hydrodynamics formalism was later modified by Bohm, known as the pilot wave formulation [6,7]. de Broglie first proposed the pilot wave theory of double solution [8], to explain wave particle duality. In this theory, the Schrödinger equation has two solutions, one the regular wave function $\psi(x, y, z) = ae^{i\phi(x,y,z)}$ where a is a constant, and the other physical wave solution $u(x, y, z) = f(x, y, z)e^{i\phi(x,y,z)}$. The two solutions are related by the phase $\phi(x, y, z)$. The singularities in $u(x, y, z)$ are due to presence of singularities in $f(x, y, z)$ give rise to particle-like nature. These are moving singularities. The drawback of the theory is that it explains only single particle case. This theory was further developed by Bohm [6,7] for a system of many particles. In this theory, the wave particle duality vanishes, quantum system behaves like a particle and a wave, simultaneously, in the same experimental setup. Then the particles are directed by the pilot wave which will guide them to areas of interference.

In this paper, the Kirchhoff equations are derived from the Schrödinger equation by assuming the wave function to be a polynomial like solution. These Kirchhoff equations describe the evolution of n point vortices in hydrodynamics. These Kirchhoff equations are classical equations of motion. Therefore, classical mechanics structure of

the Kirchhoff equations admits both particle nature and wave nature of single particle quantum mechanics in two dimensions.

In literature, the fractional quantum Hall effect ground state evolution is described by Laughlin wave function [9], which is an ansatz, modeled in terms of Kirchhoff equations [10–13]. These Kirchhoff equation (3) are in phase space and admit a Poisson bracket relationship [4] in terms of the complex coordinates given by

$$\{z_i, \bar{z}_j\} = -2i\delta_{ij}\Gamma_i.$$

(4)

For more details, refer to [4]. The identification of $\bar{z} = \partial_{z_i} = P_z$ with canonical momenta [10] allows one to replace Poisson bracket with commutator

$$[z_i, \partial_{z_i}] = i\hbar,$$

(5)

where $\Gamma_i = \hbar$ is the Heisenberg uncertainty relation.

As an illustration we show that the logarithmic derivative of the Laughlin wave function looks like the right-hand-side of the Kirchhoff equation and their one particle solutions are complex Hermite polynomials. Then, we address the wave particle duality through Kirchhoff equation in terms of interference or as a relative phase. We show that the equation for optical vortices is derived from Kirchhoff equation using paraxial wave equation in presence of real constant background.

2 Schrödinger Equation and Kirchhoff Equations

In this section, the Kirchhoff equations are derived from the Schrödinger equation by assuming the wave function to be a polynomial like solution [5]. These equations describe the evolution of n point vortices in Hydrodynamics.
Consider the time dependent Schrödinger equation with potential $V(x) = 0$

$$i\frac{\partial}{\partial t}\psi(x, t) = \Gamma\frac{\partial^2}{\partial x^2}\psi(x, t),$$

(6)

where $\Gamma = \frac{-\hbar}{2m}$. By introducing a polynomial

$$\psi(x, t) = (x - x_1(t))(x - x_2(t))\cdots(x - x_n(t))$$
$$= \prod_{k=1}^{n}(x - x_k(t)),$$

(7)

for $n = 2$ substituting Eq. (7) in Eq. (6)

$$-i\dot{x}_1(x - x_2(t)) - i\dot{x}_2(x - x_1(t)) = 2\Gamma$$

(8)

The equations at $x = x_1$ and $x = x_2$

$$\dot{x}_1 = \frac{2\Gamma i}{(x_1 - x_2)}, \quad \dot{x}_2 = \frac{2\Gamma i}{(x_2 - x_1)}. \tag{9}$$

By following a similar procedure for $n = 3$, the equations at points $x = x_1$, $x = x_2$ and $x = x_3$ are given by

$$\dot{x}_1 = 2\Gamma i \left[\frac{1}{(x_1 - x_2)} + \frac{1}{(x_1 - x_3)} \right]$$

$$\dot{x}_2 = 2\Gamma i \left[\frac{1}{(x_2 - x_1)} + \frac{1}{(x_2 - x_3)} \right]$$

$$\dot{x}_3 = 2\Gamma i \left[\frac{1}{(x_3 - x_1)} + \frac{1}{(x_3 - x_2)} \right]$$

The same procedure for n zeros gives

$$\dot{x}_i = 2\Gamma i \sum_{i \neq j}^{n} \frac{1}{(x_i - x_j)}. \tag{10}$$

Equation (10) are known as Kirchhoff equations which describe the evolution of n point vortices in Hydrodynamics. Therefore, Schrödinger equation can be written as a system of n linear equations (10). Kirchhoff equations (10) with the background flow $\mathcal{W}(x)$ are given by

$$\dot{x}_i = 2\Gamma i \sum_{i \neq j}^{n} \frac{1}{(x_i - x_j)} + i\mathcal{W}(x_i). \tag{11}$$

Solution to the stationary Kirchhoff equations (11), that is $\dot{x}_i = 0$, is found by Stieltjes electrostatic model [14,15]. In this model, there are n unit moving charges between two fixed charges p and q at -1 and 1 respectively, on a real line and it is shown by Stieltjes that the system attains equilibrium at the zeros of Jacobi polynomials. Further, it is proved by the author that, Stieltjes electrostatic model is analogous to the quantum momentum function of quantum Hamilton Jacobi [16]. Here the moving unit charges are replaced by moving poles. They are similar to imaginary charges with $i\hbar$ placed between two fixed poles like fixed charges. In the process, the background flow $\mathcal{W}(x)$ is identified with the superpotential.

In supersymmetry, the superpotential $\mathcal{W}(x)$ is defined in terms of intertwining operators $\hat{A}$ and $\hat{A}^\dagger$ as

$$\hat{A} = \frac{d}{dx} + \mathcal{W}(x), \quad \hat{A}^\dagger = -\frac{d}{dx} + \mathcal{W}(x). \tag{12}$$

from this a pair of factorized Hamiltonians $H^{\pm}$ is defined as

$$H^{+} = \hat{A}^{\dagger}\hat{A} = -\frac{d^2}{dx^2} + V^{+}(x) - E, \tag{13}$$

$$H^{-} = \hat{A}\hat{A}^{\dagger} = -\frac{d^2}{dx^2} + V^{-}(x) - E, \tag{14}$$

where E is the factorization energy. The partner potentials $V^{\pm}(x)$ are related to $\mathcal{W}(x)$ by

$$V^{\pm}(x) = \mathcal{W}^2(x) \mp \mathcal{W}'(x) + E, \tag{15}$$

where prime denotes differentiation with respect to x.

As an illustration, we solve the Harmonic oscillator in natural units using the Kirchhoff equations

$$\sum_{1 \leq j \leq n, j \neq k}^{n} \frac{1}{x_k - x_j} - x_j = 0, \tag{16}$$

where $\mathcal{W}(x) = x_j$ is the superpotential of Harmonic oscillator.
By introducing a polynomial

$$f(x) = (x - x_1)(x - x_2) \cdots (x - x_n), \tag{17}$$

and taking the limit $x \to x_j$ and using l'Hospital rule we obtain

$$\begin{aligned}
\sum_{1 \leq j \leq n, j \neq k} \frac{1}{x_j - x_k} &= \lim_{x \to x_j} \left[\frac{f'(x)}{f(x)} - \frac{1}{x - x_j} \right] \\
&= \lim_{x \to x_j} \frac{(x - x_j)f'(x) - f(x)}{(x - x_j)f(x)} \\
&= \frac{f''(x_j)}{2f'(x_j)}.
\end{aligned} \tag{18}$$

By substituting Eq. (18) in Eq. (16), we obtain

$$f''(x_j) + 2x_j f'(x_j) = 0. \tag{19}$$

Hence Eq. (19) is a polynomial of order n, and is proportional to f(x) which gives Hermite differential equation

$$f''(x) + 2xf'(x) + nf(x) = 0. \tag{20}$$

Here it should be noted that when we solve the problem for general potential say $Q(x_j)$ through Stieltjes electrostatic model we end up with the following polynomial solutions

$$f''(x_j) + Q(x_j)f'(x_j) = 0. \tag{21}$$

Thus, the differential Eq. (21) will have a classical orthogonal polynomial solution only when $Q(x_j) = \mathcal{W}(x_j)$.

As an illustration, consider the Coulomb potential whose superpotential is given by

$$\mathcal{W}_{coul}(r_j) = \frac{1}{2} - \frac{(l+1)}{r_j}. \tag{22}$$

Then the Kirchhoff equations

$$\sum_{\substack{1 \le j \le n, j \ne k}}^{n} \frac{1}{r_k - r_j} - = \frac{1}{2} - \frac{(l+1)}{r_j} = 0, . \tag{23}$$

Using the identity (18) and substituting in Kirchhoff equation (23) gives

$$\frac{f''(r_j)}{2f'(r_j)} - \left(\frac{1}{2} - \frac{(l+1)}{r_j}\right) = 0. \tag{24}$$

Equation (24) is polynomial of order n given by

$$rf''(r) + (2(l+1) - r)f'(r) = 0, \tag{25}$$

proportional to $f(r)$ gives the Laguerre differential equation

$$rf''(r) + (2(l+1) - r)f'(r) + nf(r) = 0. \tag{26}$$

Similarly, for super potential

$$\mathcal{W}(x_j) = \frac{p}{x_j - 1} + \frac{q}{x_j + 1}., \tag{27}$$

the Kirchhoff equations are given by

$$\sum_{\substack{1 \le j \le n, j \ne k}}^{n} \frac{1}{x_k - x_j} - \frac{p}{x_j - 1} - \frac{q}{x_j + 1} = 0. \tag{28}$$

Using the identity (18) and substituting in Kirchhoff equation (28) gives

$$-\frac{f''(x_j)}{2f'(x_j)} - \frac{p}{x_j - 1} - \frac{q}{x_j + 1} = 0. \tag{29}$$

Equation (29) is polynomial of order n given by

$$(1 - x^2)f''(x) + 2[q - p - (p + q)x]f'(x) = 0. \tag{30}$$

is proportional to $f(x)$ gives the Jacobi differential equation

$$(1 - x^2)f''(x) + 2[q - p - (p + q)x]f'(x) + n(n + p + q + 1)f'(x) = 0. \tag{31}$$

Therefore, we can solve all the bound state problems in terms of Kirchhoff equations by using the Stieltjes electrostatic model.

3 Laughlin Wave Function

One of the well known examples of two dimensional systems is fractional quantum Hall effect. Robert Laughlin proposed the following wave function as an ansatz for the ground state wave function of fractional quantum Hall effect [9],

$$\psi = \prod_{N \geq j > i \geq 1} (z_j - z_i)^n \exp\left(-\frac{1}{4l_B^2} \sum_i |z_j|^2\right) \tag{32}$$

where, $z_i = x_i + iy_i$, N is number of electrons, $l_B = \sqrt{\frac{\hbar}{eB}}$ is magnetic length, $\hbar$ is Planck constant, e is electric charge, B is magnetic field, ω_B is cyclotron frequency $\omega_B = \frac{eB}{m}$, z_i and z_j are the position of electrons for the ground state of a two-dimensional electron gas with the lowest Landau level, where n is written in terms of $\nu = 1/n$ and n is an odd positive integer filling numbers. As the Laughlin wave function is a trial wave function, it is not an exact ground state of any potential, in particular, not an exact ground state of Coulomb repulsion problem. But it has been tested numerically, for the Coulomb and several repulsive potentials, that the Laughlin wave function has more than 99% overlap with the true ground state [17]. The gap vanishes at the edges and the fractional charges are calculated using Berry's connection. In literature, several model Hamiltonians have been proposed which can admit Laughlin wave function as a solution and first model of its kind was proposed by Haldane [18]. The Berry connection for N quasi-holes are given by [17]

$$A(\eta_j) = -\frac{i\nu}{2} \sum_{1 \leq k \leq N, j \neq k} \frac{1}{\eta_k - \eta_j} + i\nu \frac{\bar{\eta}_j}{4l_B^2}, \tag{33}$$

here η_i are the positions of the quasi holes and the ν are the filling numbers defined in Eq. (32). A similar equation for $A(\bar{z}_j)$ exist which describes the adiabatic transport of quasihole at z_i when all other quasihole positions are fixed. Taking logarithm of Laughlin wave function (32) and then differentiating with respect to z, equating the derivative to zero gives Kirchhoff equations.

$$i\frac{d}{dz_j}ln\psi(z_j) = \sum_{1 \leq i \leq N, i \neq j} \frac{in}{z_i - z_j} - i\frac{1}{4l_B^2}\bar{z}_j = 0. \tag{34}$$

Equation (34) is Berry's Connection (33) for the positions of quasi hole,s and the ν's are filling numbers defined in Eq. (32). Similarly, by taking the transpose of Eq. (34) we obtain an equation for $A(\bar{z}_j)$

$$\sum_{1 \le i \le N, i \ne j} \frac{in}{\bar{z}_i - \bar{z}_j} - i \frac{1}{4l_B^2} z_j = 0. \tag{35}$$

which describes the adiabatic transport of quasihole at z_i when all other quasihole positions are fixed. Therefore, from Eq. (34), it is clear that the Laughlin wave function (32) represents a Hamiltonian, and the Kirchhoff equations are obtained by taking the minimum of logarithm of the Laughlin wave function (32). This represents time independent Schrödinger equation with background flow $W(\bar{z}_j) = \frac{1}{4l_B^2} \bar{z}_j$.

Equations (34) and (35) are Schrödinger equations, and it immediately follows that the Hamiltonian is not hermitian as the superpotential is complex in nature. In other words, Kirchhoff equations are function of holomorphic coordinates $z = x + iy$ with the corresponding momenta $\partial_{z_i} = \frac{1}{2}(\partial_{x_i} - i\partial_{y_i})$ and antiholomorphic coordinates $\bar{z} = x - iy$ with the corresponding momenta $\partial_{\bar{z}_i} = \frac{1}{2}(\partial_{x_i} + i\partial_{y_i})$. One notices that the Laughlin wave function (32) is a product of the van der Mondes determinant $\prod_{N \ge i > j \ge 1} (z_j - z_i)$ times the Gaussian weight function $\exp\left(-\sum_j \frac{1}{4l_B^2} |z|_j^2\right)$. The van der Mondes determinant is a function of holomorphic coordinates, and the the Gaussian weight function is a function of holomorphic and antiholomorphic coordinates. In literature these kinds of function are studied in Segal-Bargmann space.

It is well known in literature that the Laughlin wave function is obtained using Landau Hamiltonian with interaction, for details readers may refer to [17]. The Hamiltonian is given by

$$H = \frac{1}{2m} \pi \cdot \pi = \hbar \omega_B \left(a^\dagger a + \frac{1}{2} \right) \tag{36}$$

where, the momentum is defined in terms of minimal coupling $\pi = P + eA$ and $A = -\frac{yB}{2}\hat{x} - \frac{xB}{2}\hat{y}$. Then the lowering operator is defined as

$$a = \frac{1}{\sqrt{2e\hbar B}} (\pi_x - i\pi_y) \tag{37}$$

$$= \frac{1}{\sqrt{2e\hbar B}} \left(-i\hbar \left(\partial_x + \frac{yB}{2} \right) + i\hbar \left(i\partial_x - i\frac{xB}{2} \right) \right). \tag{38}$$

Using the complex coordinates $z = x + iy$ and the corresponding momenta $\partial_{z_i} = \frac{1}{2}(\partial_{x_i} - i\partial_{y_i})$ is a holomorphic function and corresponding antiholomorphic coordinates $\bar{z} = x - iy$ and the momenta $\partial_{\bar{z}_i} = \frac{1}{2}(\partial_{x_i} + i\partial_{y_i})$. which allows us to define the following raising and lowering operators

$$a^\dagger = -i\sqrt{2} \left(l_B \partial_z - \frac{1}{4l_B} \bar{z} \right), \qquad a = -i\sqrt{2} \left(l_B \partial_{\bar{z}} + \frac{1}{4l_B} \Omega z \right). \tag{39}$$

Then the lowest landau levels are defined as $a\psi_{LLL} = 0$ where

$$\psi_{LLL} = f(z)\exp\left(-\frac{1}{4l_B^2}|z|^2\right), \tag{40}$$

where $f(z)$ is holomorphic function. For particles $N > 2$ Laughlin proposed

$$\psi_{LLL}(z_1 \cdots z_n) = f(z_1 \cdots z_n)exp\left(-\sum_{i=1}^n \frac{1}{4l_B^2}|z|_i^2\right), \tag{41}$$

and Laughlin proposed Ground state filling fraction $\nu = 1/m$

$$\psi(z_1 \cdots z_n) = \prod_{i<j}(z_i - z_j)^m exp\left(-\sum_{i=1}^n \frac{1}{4l_B^2}|z|_i^2\right). \tag{42}$$

Here it should be noted that these raising and lowering operators defined in Eq. (39) are identical to the intertwining operators $\hat{A}$ and $\hat{A}^\dagger$ defined in Eq. (12).

Then the Hamiltonian reads as

$$H\psi(|z|) = a^\dagger a\psi(|z|) = (-\partial_z + \Omega\bar{z})(\partial_{\bar{z}} + \Omega z)\psi(|z|), \tag{43}$$

where $\Omega = \frac{1}{4l_B^2}$, which gives

$$(\partial_{|z|}^2 + \Omega\bar{z}\partial_{\bar{z}} - \Omega z\partial_z + \Omega^2|z|^2)\psi(|z|) = 0. \tag{44}$$

Equation (44) is the complex Hermite polynomial differential Eq. [19] and its solutions are given by

$$\psi(|z|) = \frac{1}{\sqrt{2^n\,n!}} \cdot (\Omega)^{1/2} \cdot e^{-\Omega^2|z|^2} \cdot H_n(\Omega|z|), \, n = 0, 1, 2, \ldots \tag{45}$$

Therefore, for single particle Laughin wave function is solved using Kirchhoff equations and their solutions are found to be complex Hermite polynomials.

4 Wave Particle Duality

One of the major differences between classical mechanics and quantum mechanics is Heisenberg uncertainty principle in terms of position and momentum given by Eq. (5). It is widely accepted that Heisenberg uncertainty relationship is a restatement of wave particle duality, which is the corner-stone of quantum mechanics. It states that quantum particles are both particles and waves. There are several theories based on wave particle duality, of which the most widely accepted one is the principle of complementarity. The essence of this principle is that quantum systems exhibit both

wave and particle nature. However, the outcome of an event is dependent purely on an experiment. Either the wave or the particle nature is observed but both will not be observed simultaneously in a given experiment. Here, we address the wave particle duality through Kirchhoff equations.

Consider Kirchhoff equations (3) with identical say Γ are given by

$$\frac{d\bar{z}_i}{dt} = \sum_{1 \le i \le n, i \ne j}^{n} \frac{i\Gamma}{z_i - z_j} + iW(z_i). \tag{46}$$

where $z_i = x_i + y_i$. By using the following identity [20] and substituting Eq. (18) in Kirchhoff equations (46) one gets

$$f''(z_j) + \mathcal{W}(z_i)f'(z_j) = u(z_i) + iv(z_i) = 0, \tag{47}$$

where $u(z_i)$ and $v(z_i)$ are real valued function. If a relative phase in the Kirchhoff equations (47) is developed and $v(z_i)$ is not a constant, it gives rise to wave particle duality. As an illustration, we consider the case $v(z_i)$ is constant, which gives rise to optical vortices.

The optical vortices are defined as phase dislocation on the beam axis, the quantised orbital angular momentum, of a Laguerre–Gaussian laser mode [21]. Optical vortex beams are described in terms of Laguerre–Gaussian modes which are good approximation to the vortex modes created from Hermite–Gaussian laser modes [22,23]. It is well known that the Laguerre–Gaussian vortex beam state arises as a solution ot the paraxial approximation of the Helmholtz equation for light or Schrödinger equation for electrons in free space. The vortex states are solutions to the Schrödinger equation, Klein–Gordon equation and Dirac equations [24–27]. The connection between optical vortices and Hydrodynamiccs is studied in Ref. [28].

In the Eq. (47), if $v(z_i) = P$ is a constant. Then, the Eq. (47) is written as

$$\frac{\partial^2 u}{\partial z_i^2} + 2ik \frac{\partial u}{\partial z_i} = 0, \tag{48}$$

where $k = P/\Gamma$ and the position of N vortices given by $z_\alpha = x_i + iy_i$. Under paraxial wave approximation the Eq. (47) admit a wave equation which describes the optical vortices. Expressing the Eq. (48) in terms of coordinates x_i and y_i gives

$$\frac{\partial^2 u}{\partial x_i^2} + \frac{\partial^2 u}{\partial y_i^2} + 2ik \frac{\partial u}{\partial z_i} = 0. \tag{49}$$

In optics the wave Eq. (49) is called paraxial wave equation and Gaussian beams of any beam waist w_0 satisfy this wave equation [29]. In paraxial wave approximation, the term $\frac{\partial^2 u}{\partial z_i^2}$ is neglected. Substituting, the term $\frac{\partial^2 u}{\partial z_i^2}$ back into the Eq. (49) gives

$$\frac{\partial^2 u}{\partial x_i^2} + \frac{\partial^2 u}{\partial y_i^2} + \frac{\partial^2 u}{\partial z_i^2} + 2ik \frac{\partial u}{\partial z_i} = 0. \tag{50}$$

Equation (50) is derived from the Helmholtz equation:

$$\nabla^2 \psi(x_i, y_i, z_i) + k^2 \psi(x_i, y_i, z_i) = 0 \qquad (51)$$

with

$$\psi(x_i, y_i, z) = u(x_i, y_i, z_i) e^{ikz_i}. \qquad (52)$$

Therefore, two dimensional n point vortices evolution described by Kirchhoff equations (46) with the identical circulation strength with the constant imaginary background flow will admit a wave equation under paraxial wave approximation.

In optics, relative phase gives rise to interference. It is well known, that the particle nature of electrons passing through Young's double slit experiment one observe interference pattern on screen which is the wave nature of electrons. It should be noted, that Kirchhoff equations (10) are one dimensional and Kirchhoff equations for the vortices (3) are two dimensional. Therefore, we claim, if Kirchhoff equations in differential form (47) admits a relative phase then it gives rise to wave particle duality.

5 Discussion

The fractional Hall effect is a topological insulator. In case of insulators, as the gap is large, it is characterized in terms of band gap. If the insulators are connected, say from one insulator to another, without changing the band gap, such that the system always remains in the ground state, they are called topologically equivalent insulators. In other words, when some parameter of the Hamiltonian is slowly changed adiabatically, the ground state of the system remains unchanged. Those insulators which cannot be connected by the slowly changing Hamiltonian are called topologically inequivalent insulators. Connecting topologically equivalent insulators gives rise to a phase transition resulting in the vanishing of gap. In this gapless state, topological invariants are quantized giving rise to current. For fractional Hall effect, the Berry connection is given in terms of Eq. (33), and the parameter ν gives topological winding numbers.

It is clear from the above how Kirchhoff equation (11) is derived from one dimensional Schrödinger equation. Hence, Eqs. (11) and (33) both are Kirchhoff equations for one and two dimensions respectively. Therefore, we conclude the Schrödinger equation in terms of Kirchhoff equations is a Berry connection.

In the classical Hamilton Jacobi, equation of motion is governed by the Hamiltonian and the problem is solved by continuously transforming the Hamiltonian from the initial state to the final state through canonical transformations. In quantum mechanics, equation of motion is governed by the Schrödinger equation which is described by a Hamiltonian. It is well known that the Schrödinger equation is related to classical Hamilton Jacobi equation in the limit $\hbar \to 0$. It is not possible to continuously transform Schrödinger equation from the initial state to the final state through a canonical transformation as it has simple poles which are quite evident when the Schrödinger

equation is written in terms of Kirchhoff equations (11). The analogous n point vortices of hydrodynamics with the fractional quantum Hall effect has allowed us to identify the Schrödinger equation as Berry connection. Thus, making the Berry connection exact gives rise to quantisation. Hence, the quantum numbers are topological invariants arising due to singularities in Schrödinger equation . Therefore, we conclude that quantisation arises as it continuously connects the topologically inequivalent Hamiltonians in the Hilbert space.

Mapping of Schrödinger equation to Kirchhoff equations allows us to draw the following conclusions: the n point vortices are integrable unto three vortices [4]. If the vortices circulation strength is identical, then the n point vortices, can be solved with Stieltjes electrostatic model and their solutions are classical orthogonal polynomials [30]. The n point vortices with identical circulation strength corresponds to Schrödinger equation and the solutions are classical orthogonal polynomials, and they are the basis in Hilbert space. Hence, the first postulate: the state of a quantum mechanical system is completely specified by a function $\Psi(\mathbf{r}, t)$ is a vector in complex Hilbert space. The complex Hilbert space arise because the Kirchhoff equations are solved using Stieltjes electrostatic model with imaginary charges [16]. In Stieltjes electrostatic model, the n moving charges between two fixed charges attain the equilibrium at the zeros of classical orthogonal polynomials depending on the position of the fixed charges. That is, if the position of fixed charges are at $\pm\infty$ the system attains equilibrium at zeros of Hermite polynomials. Hence, it is the statistical distribution of these charges is given in terms of probability distribution. The probability distribution in terms of wave function is given by

$$\int_{-\infty}^{\infty} \Psi^*(\mathbf{r}, t)\Psi(\mathbf{r}, t)d\tau = 1. \tag{53}$$

These Kirchhoff equations satisfy a Poisson bracket relationship in phase space which is identical to the Heisenberg uncertainty relationship. Therefore, we conclude that the Kirchhoff equations, being classical equations, describe both a particle and a wave equation nature of single particle quantum mechanics.

6 Conclusion

In this paper, we have shown that the Kirchhoff equations are derived from the Schrödinger equation by assuming the wave function to be a polynomial like solution. These Kirchhoff equations describe the evolution of n point vortices in hydrodynamics. In two dimensions, Kirchhoff equations are used to demonstrate the solution to single particle Laughlin wave function as complex Hermite polynomials. We have also shown that the equation for optical vortices, a two dimentional system, is derived from Kirchhoff equation by using paraxial wave approximation. These Kirchhoff equations satisfy a Poisson bracket relationship in phase space which is identical to the Heisenberg uncertainty relationship. Therefore, we conclude that being classical equations, the Kirchhoff equations, describe both a particle and a wave nature of single particle quantum mechanics in two dimensions.

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