Weak Values and Quantum Properties

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Abstract

We investigate in this work the meaning of weak values through the prism of property ascription in quantum systems. Indeed, the weak measurements framework contains only ingredients of the standard quantum formalism, and as such weak measurements are from a technical point of view uncontroversial. However attempting to describe properties of quantum systems through weak values—the output of weak measurements—goes beyond the usual interpretation of quantum mechanics, that relies on eigenvalues. We first recall the usual form of property ascription, based on the eigenstate-eigenvalue link and the existence of "elements of reality". We then describe against this backdrop the different meanings that have been given to weak values. We finally argue that weak values can be related to a specific form of property ascription, weaker than the eigenvalues case but still relevant to a partial description of a quantum system.

Keywords Measurement in quantum mechanics · Properties of quantum systems · Weak measurements · Post-selected quantum systems

1 Introduction

Weak values and weak measurements were introduced by Aharonov, Albert and Vaidman 30 years ago [1] as a tool to understand the properties of quantum systems at intermediate times between preparation and a final state obtained by measuring a chosen observable. Initially applied to elucidate apparently paradoxical behavior in quantum systems, such as the three-box-paradox [2], weak measurements have become increasingly popular in the last 10 years, in part due to several experiments that were able to observe weak values [3–5], and also due to promising technological

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applications, in particular regarding the amplification of weak signals [6,7]. Weak measurements have also been claimed to play a role in reformulating quantum theory [8,9].

Although the theoretical framework of weak measurements (WM) and weak values (WV) only involves ingredients of standard quantum mechanics, WM and WV were criticized [10,11] very soon after their inception, and the criticism has persisted to this day. We will leave aside the substantial fraction of the criticism that is rooted in misunderstandings or erroneous readings of the formalism. Instead the bulk of the arguments, while recognizing the formalism is correct, denies that WM have anything to say concerning the properties of quantum systems at intermediate times. Actually, in the first published criticism of the WM framework, Leggett had already asserted that in his view, weak measurements could not be qualified to be measurements at all [10].

The reason WM and WV have remained controversial is that although the formalism is unambiguous, in the sense that all the ingredients come from standard quantum mechanics, the latter remains silent on many interpretative issues, and in particular has no provision to account for the properties of a system without making a projective measurement – but doing so renders the very question adressed by WM (to understand the properties at an intermediate time) meaningless, since a projective measurement radically modifies the system evolution. Hence the idea at the basis of WM: induce a weak coupling between a system observable and a quantum pointer, a coupling so weak that the evolving state vector is minimally modified and that the probability of obtaining a given outcome when making the final measurement is not modified relative to the no coupling situation. The weak value is precisely the number characterizing the motion of the quantum pointer due to the weak interaction.

The question is then whether this weak value (or rather, its real part, since, as we will recall below, WV are complex) can be taken as a generalized form of eigenvalue as advocated in the original paper [1], or are instead meaningless or arbitrary numbers or at any rate useless to describe the properties of a quantum system at an intermediate time. The rationale for the latter position is that WV can lie outside the eigenvalue spectrum, so that for example the weak value of a projector can be negative. Should this be taken as yet another strange quantum feature, or does it mean there is an irremediable flaw when attempting to attribute a value to a quantum property through weak values? What is at stake here is not only the status of weak values, but more fundamentally the relevance of the results obtained within the weak measurements framework in order to understand the physical nature of quantum systems. Indeed, WM open a new observational window into the quantum world, allowing to acquire information on a system without substantially modifying its evolution. It is crucial to assess the nature of this information, viz. whether it is related to the properties that are weakly measured.

In the present manuscript, we investigate these questions by reexaming how a property value is ascribed to a quantum system. We start by discussing the eigenstate-eigenvalue link, which is the basis of property ascription in quantum systems. We introduce the notion of pre-selection and post-selection and examine how the eigenstate-eigenvalue link ascribes properties in such circumstances, i.e. state preparation (pre-selection) followed by an intermediate projective measurement and finally post-selection (filtering of a particular outcome of a final projective measurement of a different observable). We then introduce the Weak Measurements framework (Sect.

3) and give a few properties of weak values that are important in the present context. Section 4 critically examines the different meanings that have been given to weak values. Indeed, by construction, property ascription for weak values cannot rely on the eigenstate-eigenvalue link, and WV have therefore been related to other features (such as conditional averages over statistical ensembles or response functions to a small perturbation). We will nevertheless argue (Sect. 4.6) that there is room to relate weak values to quantum properties but in a very specific, elusive manner, in a much weaker way than what is provided by the eigenstate-eigenvalue link. We then expose our view on the meaning of weak values (Sect. 4.7) and finally present our Conclusions in Sect. 5.

2 Properties in Quantum Systems

2.1 The Eigenstate-Eigenvalue Link

The standard approach to quantum mechanics is to ascribes a property to a quantum system when the system is in an eigenstate of a given observable. If a given property is represented by an observable A with eigenstates $|a_k\rangle$ and eigenvalues a_k, ie

$$A |a_k\rangle = a_k |a_k\rangle \tag{1}$$

(assumed here discrete and non-degenerate), then if the system is in a state $|\psi\rangle$, that can generally be represented as

$$|\psi\rangle = \sum_k c_k |a_k\rangle , \tag{2}$$

the value of the property represented by A is not defined, unless $|\psi\rangle$ is an eigenstate of A (in which case all the c_k vanish except one).

The fact that a definite value cannot be ascribed to an observable in an arbitary state was already quite clearly stated in Dirac's early textbook (see Sects. 9 and 10 of Ref. [12]): "*The expression that an observable 'has a particular value' for a particular state is permissible in quantum mechanics in the special case when a measurement of the observable is certain to lead to the particular value, so that the state is an eigenstate of the observable*". Otherwise Dirac writes that "*a disturbance involved in the act of measurement causes a jump in the state*" of the system ([12], p. 36). This approach, a cornerstone of the orthodox interpretation, is often known as the eigenstate-eigenvalue link (see Ref. [13] for a historical account of the term).

In his textbook *Quantum Mechanics*, a masterly exposition of the orthodox approach, Bohm explains in addition that a given property value only appears when the system is actually measured, after it has interacted with a measuring apparatus [14]. The physical underlying model is due to von Neumann [15]. In von Neumann's impulsive measurement model, the quantum states of a measuring pointer are explicitly introduced. Suppose that initially (at $t = t_i$) the system is prepared into the state $|\psi(t_i)\rangle$. Let $|\varphi(t_i)\rangle$ designate the initial state of the quantum pointer. The total initial quantum state is the product state

$$|\Psi(t_i)\rangle = |\psi(t_i)\rangle\,|\varphi(t_i)\rangle. \tag{3}$$

We assume the pointer state is initially compactly localized around some position x_0; we will use the notation $\langle x |\,\varphi_{x_0}\rangle = \varphi_{x_0}(x)$. Assume further that the system and the pointer interact during a brief time interval τ. Let A be the measured system observable. The interaction between the system and the quantum pointer is given by the coupling Hamiltonian

$$H_{int} = g(t)AP. \tag{4}$$

$g(t)$ is a smooth function that vanishes for times $t \leqslant t_i$ or $t \geqslant t_i + \tau$ and such that $g \equiv \int_{t_i}^{t_i+\tau} g(t)dt$ appears as the effective coupling constant. If g is large we can neglect the self Hamiltonians of the system and of the pointer and consider that the evolution during the short time interval τ is solely driven by H_{int}. This leads to

$$|\Psi(t_i + \tau)\rangle = e^{-igAP/\hbar}\,|\psi(t_i)\rangle\,\big|\varphi_{x_0}\big\rangle \tag{5}$$

$$= \sum_k |a_k\rangle\,\langle a_k|\,\psi(t_i)\rangle\,e^{-iga_k P/\hbar}\,\big|\varphi_{x_0}\big\rangle \tag{6}$$

$$= \sum_k \langle a_k|\,\psi(t_i)\rangle\,|a_k\rangle\,\big|\varphi_{x_0-ga_k}\big\rangle, \tag{7}$$

where we have used in the last line the properties of the translation operator.

Equation (7) associates each pointer state $\big|\varphi_{x_0-ga_k}\big\rangle$ (shifted relative to the initial pointer state by a distance proportional to the eigenvalue a_k) with the corresponding eigenstate $|a_k\rangle$. At this post-interaction stage, we still have an entangled state: the interaction Hamiltonian (4) drives the system to the observable eigenstates, but not yet to a definite eigenvalue. In some sense, the system has acquired the property (the one represented by the measured observable) relative to the pointer, but not yet its value. A definite value only appears when the linear superposition (7) is replaced by a single term corresponding to the observed value. There is no consensus on the origin or nature of this collapse (that can be taken as apparent or fundamentally real, depending on the specific interpretation [16]), though it has to do with some irreversible amplification that takes place at the macroscopic scale when the pointer is measured. The overall process described by von Neumann's model is known by the rather syncretic term of "projective measurement".

The eigenstate-eigenvalue link calls therefore for an interaction between the system and the pointer with a large coupling constant (large meaning that the shift is larger than the spatial width of the initial state $\varphi_{x_0}(x)$) and a collapse to a final pointer state unambiguously correlated with an eigenstate of the measured observable.

2.2 Element of Reality

The eigenstate eigenvalue link is intimately related to the notion of "elements of reality", as introduced by Einstein, Podolsky and Rosen (EPR) [17]: "*If, without in any way disturbing a system, we can predict with certainty (i. e. , with probability equal*

to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity".

Indeed, it can be noticed that Eq. (7) is an entangled state, not unlike an EPR pair. Measuring the pointer shift to be ga_{k_1} immediately correlates with the system eigenstate being $|a_{k_1}\rangle$. This can be checked by repeating the same measurement (with a second, identical pointer). We know with certainty that the pointer will be shifted by the quantity ga_{k_1}. Hence, because of the correlation encapsulated in the entangled state, after the measurement the system is with certainty in the state $|a_{k_1}\rangle$. This implies that the corresponding eigenvalue is an "element of reality", and property ascription follows from there since we know with certainty the system property and its value.[1]

The relation between the eigenstate-eigenvalue link and an element of reality was already noted by Redhead [18], who coins this relation the "Eigenvector rule" (see Chapter 3 of Ref. [18]). Redhead also notes that the "no disturbance" condition in EPR's definition of "elements of reality" is unnecessary (and even potentially confusing) as far as the Eigenvector Rule is concerned. Hence we can state that when the eigenvalue-eigenstate link holds the corresponding property can be ascribed to a quantum system. This property is then an element of reality.

2.3 Expectation Values

As is well known, there is no consensus as to whether the state vector provides a description (complete or incomplete) of an individual system, or describes instead an ensemble of similarly prepared systems, although the standard view has increasingly tilted toward the statistical approach to state vectors (see eg Chapter 9 of Ref. [19]). Expectation values however are never assumed to refer to properties of a single system. An expectation value is instead obtained when the system is prepared in state $|\psi(t_i)\rangle$ and the measurement of the property represented by A, as described by the von Neumann model given above, is repeated several times, with random outcomes a_k obtained with probability $p_k = |\langle a_k | \psi(t_i)\rangle|^2$, leading to the standard expression

$$\langle A \rangle_{|\psi(t_i)\rangle} = \langle \psi(t_i) | A | \psi(t_i)\rangle = \sum p_k a_k. \tag{8}$$

In each run, the system ends up in the eigenstate $|a_k\rangle$ – the system has the property given by the corresponding eigenvalue – but the expectation value is obviously not an "element of reality".

2.4 Counterfactuals

It is intuitively tempting to go beyond the eigenstate-eigenvalue link and attempt to ascribe properties to a quantum system as the system evolves from its initial state to the final state obtained as the result of a projective measurement. This can only be done by counterfactual reasoning. Indeed, ascribing a value to a property would involve performing a projective measurement at some intermediate time, but doing so

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[1] Note that this reasoning assumes the pointer states $|\varphi_{x_0-ga_k}\rangle$ of Eq. (7) are orthogonal. Otherwise a pointer state does not correlate with a single eigenstate and no element of reality can be defined.

would modify the original experimental arrangement and affect the system evolution dramatically (the system may not even reach the original final state).

Counterfactual definiteness conflicts with quantum mechanics on the general ground [20] that it leads to ascribe to quantum systems joint properties that can never be simultaneously measured. This point was made early on by Bohr, in particular in his reply [21] to EPR [17]. Bohr writes there that "*we have in each experimental arrangement suited for the study of proper quantum phenomena not merely to do with an ignorance of the value of certain physical quantities, but with the impossibility of defining these quantities in an unambiguous way* ", which can be seen as vindicating the eigenstate-eigenvalue link in order to ascribe properties. Bohr further pointed out that countefactual reasoning usually leads to paradoxes.

2.5 Properties in Pre and Post-selected Systems: the ABL rule

Pre-selected and post-selected systems are systems for which not only is the initial state prepared in a known state (this is the pre-selected state) but also the final state is fixed (this state is known as the post-selected state). In practice post-selection is performed by filtering the outcome of the final projective measurement. This is particularly useful when starting from a preselected state $|\psi(t_i)\rangle$, an intermediate standard projective measurement of some observable A is made before a final measurement of a different observable B takes place. The ABL rule [22] states how to compute probabilities for the outcomes a_k of A when the system has been preselected in state $|\psi(t_i)\rangle$ and will finally be found in the post-selected eigenstate $|b_f\rangle$ of B. The probability of obtaining a_n in the intermediate measurement is given by

$$P(a_n|\psi(t_i), b_f) = \frac{\left|\langle b_f | a_n \rangle \langle a_n | \psi(t_i)\rangle\right|}{\sum_k \left|\langle b_f | a_k \rangle \langle a_k | \psi(t_i)\rangle\right|^2}. \tag{9}$$

The ABL rule is a standard quantum mechanical result that follows from the Bayes rule and the Born rule. It illustrates that ascribing properties to a quantum system is a delicate task. Consider indeed a particle that is allowed to take three paths, eg a spin-1 charged particle in a Stern-Gerlach like setup.[2] Let the pre and post-selected states be given by

$$|\psi(t_i)\rangle = (|\psi_1\rangle + |\psi_2\rangle + |\psi_3\rangle)/\sqrt{3} \tag{10}$$

$$|b_f\rangle = (|\psi_1\rangle - |\psi_2\rangle + |\psi_3\rangle)/\sqrt{3}, \tag{11}$$

where $|\psi_j\rangle$ denotes the state vector on path j. We want to compute the probability of finding the particle on path 1 (conditioned on obtaining the final state $|b_f\rangle$). The result depends however on how the measurement is implemented (and hence how the observable is defined). If a projective measurement is made on *each path* then the eigenstates $|a_k\rangle$ that can be obtained are $|\psi_1\rangle$, $|\psi_2\rangle$ and $|\psi_3\rangle$ and Eq. (9) yields $P(a_1) = 1/3$. If instead we measure path-1 vs. non-path-1 (the latter being measured

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[2] This implementation of the three-box paradox [2] has been described in details elsewhere [23,24].

for instance by connecting paths 2 and 3 together and placing a measurement apparatus at that point, see Fig. 1 of [23]), then the eigenstates $|a_k\rangle$ that can be obtained are now $|\psi_1\rangle$ and $|\psi_2\rangle + |\psi_3\rangle$ and Eq. (9) leads to $P(a_1) = 1$; this implies that $P(a_{2+3}) = 0$, a straightforward consequence of the fact that the eigenstate $|\psi_2\rangle + |\psi_3\rangle$ obtained at the intermediate time is orthogonal to the post-selected state $|b_f\rangle$ given by Eq. (11). This means that in this situation, we are certain to find the particle on path 1.

Note that according to our analysis in Sect. 2, the fact that the system will be on path 1 with unit probability is an unambiguous property of the system, an element of reality. This is not an innocuous remark, because as is well-known [2], we can repeat exactly the same argument for a path 3 versus non-path 3 measurement (we now have a measurement apparatus on path 3 and another apparatus at a point where paths 1 and 2 are connected). That is if we compute the probability $P(a_3)$ to find the particle on path 3 on a path 3 versus non-path 3 measurement we find $P(a_3) = 1$ and $P(a_{1+2}) = 0$. We are thus certain to find the system on path 3. This apparent paradox has triggered vivid discussions on counterfactuals in pre and post-selected systems [20,27–30]. We will just note here that both properties following from $P(a_1) = 1$ and $P(a_3) = 1$ are well-defined, but each in its own configuration, involving different measurements and experimental arrangements. For instance when measuring path 1 versus non-path 1, $|\psi_3\rangle$ is not an eigenstate of the corresponding measurement, and no value can be ascribed to the property "the particle is on path 3" (contrarily to a path 3 vs. non-path 3 measurement). A paradox only appears if counterfactuals are employed, and value assignment is made without reference to the eigenstate-eigenvalue link. Conversely, embracing the eigenstate-eigenvalue link dispels the paradox but evades the question concerning the value of the path projectors at intermediate times. This is the difficulty that weak measurements aim to bypass.

3 Weak Measurements

3.1 Weak Measurement Protocol

Weak measurements [1] deal with extracting information about a given property, represented by an observable A, as the system evolves from a prepared initial state towards the final eigenstate obtained after measuring a different observable B. The context is identical to the one exposed above concerning the ABL rule, Sect. 2.5: the system is prepared in the pre-selected state $|\psi(t_i)\rangle$, a weak measurement of A takes place and finally B is measured and outcomes corresponding to the post-selected state $|b_f\rangle$ of B are filtered. The difference is that A is not measured through a standard projective measurement that would bring the system to one of the eigenstates. Instead, a very weak interaction is established between the system and a quantum pointer, so as to leave the system state "essentially undisturbed", meaning that the perturbation is so small that the post-selection probabilities are not affected by the weak interaction.

Let us therefore represent the initial system-pointer state as in Eq. (3) by

$$|\Psi(t_i)\rangle = |\psi(t_i)\rangle \, |\varphi(t_i)\rangle . \tag{12}$$

We will take the system-pointer interaction to be given again by $H_{int} = g(t)AP$ as in Eq. (4). Let us assume that the interaction takes place in a time window $[t_w - \tau/2, t_w + \tau/2]$, i.e. t_w is the average interaction time and τ the duration. If τ is small relative to the system evolution timescale, the interaction can be simply taken to take place precisely at t_w (for a proof of this "midpoint rule", see Ref. [31]). As in von Neumann's impulsive measurement scheme (see below Eq. (4)), $g \equiv \int_{t_w-\tau/2}^{t_w+\tau/2} g(t)dt$ appears as the effective coupling constant, but we now require g to be very small. Finally, we will allow for the system to evolve from t_i to t_w and denote $U(t_w, t_i)$ the corresponding unitary operator, but disregard instead the self-evolution of the pointer. After the interaction ($t > t_w + \tau/2$) the initial uncoupled state (12) becomes:

$$|\Psi(t)\rangle = U(t, t_w)e^{-igAP}U(t_w, t_i)\,|\psi(t_i)\rangle\,|\varphi(t_i)\rangle \tag{13}$$

$$= U(t, t_w)e^{-igAP}\,|\psi(t_w)\rangle\,|\varphi(t_i)\rangle \tag{14}$$

$$= U(t, t_w)\sum_k e^{-iga_kP}\,\langle a_k|\,\psi(t_w)\rangle\,|a_k\rangle\,|\varphi(t_i)\rangle\,. \tag{15}$$

At time t_f the system undergoes a standard projective measurement of the observable B. Filtering the results of this projective measurement by keeping only projections to the postselected state $|b_f\rangle$ yields

$$|\varphi(t_f)\rangle = \sum_k \left[\langle b_f(t_w)|\,a_k\rangle\,\langle a_k|\,\psi(t_w)\rangle\right]e^{-iga_kP}\,|\varphi(t_i)\rangle\,, \tag{16}$$

where we have used $\langle b_f(t_w)| = \langle b_f(t_f)|\,U(t_f, t_w)$. $\varphi(x, t_f)$ is then given by a superposition of shifted initial states We now use the fact that the coupling g is small, so that $e^{-iga_kP} \approx 1 - iga_kP$ holds for each k. Equation (16) takes the form

$$|\varphi(t_f)\rangle = \langle b_f(t_w)|\,\psi(t_w)\rangle\left(1 - igP\frac{\langle b_f(t_w)|\,A\,|\psi(t_w)\rangle}{\langle b_f(t_w)|\,\psi(t_w)\rangle}\right)|\varphi(t_i)\rangle \tag{17}$$

$$= \langle b_f(t_w)|\,\psi(t_w)\rangle \exp\left(-igA_f^wP\right)|\varphi(t_i)\rangle \tag{18}$$

where

$$A_f^w = \frac{\langle b_f(t_w)|\,A\,|\psi(t_w)\rangle}{\langle b_f(t_w)|\,\psi(t_w)\rangle} \tag{19}$$

is the weak value of the observable A given pre and post-selected states $|\psi\rangle$ and $|b_f\rangle$ respectively. We will drop the index f and write A^w instead whenever the postselection state is uniquely fixed and no confusions may arise.

Note that here we have not explicitly included the pointer coupled to B, which is a standard von Neumann pointer described in Sect. 2.1. Hence when referring to a pointer in the remainder of the text, we will usually mean the weakly coupled quantum pointer that registers the weak measurement. This pointer is a quantum system that will need to be measured in order to extract the weak value.

3.2 Weak Values: Properties

The interpretative questions relative to the property of the system at the intermediate time t_w will be examined below in Sect. 4. Here we note a few basic properties of weak values that will be useful in our discussion below.

3.2.1 Real Part

The first point to note is that in general the weak value is a complex quantity. For an initially localized pointer state $\left|\varphi_{x_0}\right\rangle$, Eq. (18) can be written as

$$\left|\varphi(t_f)\right\rangle = \left\langle b_f(t_w)\right| \psi(t_w)\rangle \exp\left(g \operatorname{Im} A_f^w P\right) \left|\varphi_{x_0 - g\operatorname{Re} A^w}\right\rangle. \tag{20}$$

The real part $\operatorname{Re} A^w$ induces a shift $\left|\varphi_{x_0 - g\operatorname{Re} A^w}\right\rangle$. This is similar to the first step, Eq. (7), of the standard projective measurement, except that here g is small: the original and the shifted pointer states are almost overlapping, so that extracting $\operatorname{Re} A^w$ cannot be done by performing a single measurement of the pointer, contrary to the case of strong g which discriminates pointer states correlated with different eigenvalues a_k. Note that if the pre or post-selected states $|\psi(t_w)\rangle$ or $\left|b_f(t_w)\right\rangle$ is an eigenstate of A, the weak value is real – it is actually the corresponding eigenvalue of A. In the general case, $\operatorname{Re} A^w$ is different from the eigenvalues and can lie outside the spectrum of A. From Eq. (19) it is straightforward to obtain

$$\operatorname{Re} A^w = \frac{\langle \psi(t_w)| \frac{1}{2}\left(\rho_{b_f(t_w)} A + A\rho_{b_f(t_w)}\right) |\psi(t_w)\rangle}{\langle \psi(t_w)| \rho_{b_f(t_w)} |\psi(t_w)\rangle} \tag{21}$$

where $\rho_{b_f(t_w)} \equiv \left|b_f(t_w)\right\rangle\left\langle b_f(t_w)\right|$ is the density matrix to the post-selected state evolved backward in time to the time t_w of interaction. Equation (21) has the form of a conditional expectation value when the system is in state $|\psi(t_w)\rangle$: the denominator is the average of the projector $\rho_{b_f(t_w)}$ (i.e., the probability of post-selection) while the numerator is the average of the symmetrized operator $\rho_{b_f(t_w)} A + A\rho_{b_f(t_w)}$ (measurement of A and projection to $\left|b_f(t_w)\right\rangle$).

Note that in the special case $\left|b_f(t_w)\right\rangle = |\psi(t_w)\rangle$ (this happens in particular when there is no self-evolution and the pre and postselected states are the same) $\operatorname{Re} A^w = \langle \psi(t_w)| A |\psi(t_w)\rangle$ becomes a standard expectation value.

3.2.2 Imaginary Part

The imaginary part can be put in the form

$$\operatorname{Im} A^w = \frac{\langle \psi(t_w)| \frac{1}{2i}\left(\rho_{b_f(t_w)} A - A\rho_{b_f(t_w)}\right) |\psi(t_w)\rangle}{\langle \psi(t_w)| \rho_{b_f(t_w)} |\psi(t_w)\rangle}. \tag{22}$$

The numerator represents the average backaction of the measurement of A on the post-selection projector. This can be seen from the Liouville equation, where the

commutator $-i[\rho_{b_f(t_w)}, A]$ appears as generating the evolution of $\rho_{b_f(t)}$ due to the interaction Hamiltonian Eq. (4) coupling A to the quantum pointer. For the case $\left|b_f(t_w)\right\rangle = |\psi(t_w)\rangle$, Im $A^w = 0$.

3.2.3 Expectation Value

The expectation value of A in state $|\psi(t_w)\rangle$, written in the standard form

$$\langle\psi(t_w)|\, A\, |\psi(t_w)\rangle = \sum_k a_k\, p_k(a_k) \tag{23}$$

when A is measured through a projective measurement, with $p_k(a_k) \equiv |\langle a_k|\, \psi(t_w)\rangle|^2$, can also be written as

$$\langle\psi(t_w)|\, A\, |\psi(t_w)\rangle = \sum_k |\langle b_k(t_w)|\, \psi(t_w)\rangle|^2\, \mathrm{Re}\, \frac{\langle b_k(t_w)|\, A\, |\psi(t_w)\rangle}{\langle b_k(t_w)|\, \psi(t_w)\rangle} \tag{24}$$

after some manipulations (see Eqs. (12)–(15) of [32]), by which it can also be seen that the weighted sum over the imaginary parts vanishes, so that Eq. (24) can equivalently be written as

$$\langle\psi(t_w)|\, A\, |\psi(t_w)\rangle = \sum_k A_k^w\, p_k(b_k), \tag{25}$$

with $p_k(b_k) \equiv |\langle b_k(t_w)|\, \psi(t_w)\rangle|^2$. Equations (24)–(25) involve a projective measurement of B and a weak measurement of A. Relative to Eq. (23), the probabilities are now those of obtaining a given post-selected state $|b_k\rangle$ while the eigenvalues are replaced by the real part of the weak values associated with the post-selected state $|b_k\rangle$.

4 What Do Weak Values Stand For?

4.1 Preliminary Remarks

As mentioned in the Introduction, since its inception, weak values have remained controversial, stirring much discussion. The fact that experimentally the predictions of the weak measurements framework are verified is beyond discussion. This is why the debate has centered on the meaning and significance of the weak values. The viewpoint developed in this paper is to frame this issue under the question: "Is a weak value related to a property of the system?". To this end we recalled in Sect. 2.1 the eigenstate-eigenvalue link, the basis of property ascription in standard quantum mechanics. We have then seen in Sect. 3 that the weak value appears as a shift in the pointer state [Eq. (18)], pretty much like an eigenvalue [Eq. (7)]; the analogy is also patent when comparing the expressions for the observable average (23) and (25) in terms of eigenvalues and weak values respectively. We have also seen however that the real and imaginary parts of a weak value can be written in terms of conditional expectations, Eqs. (21) and (22), making weak values look like an

average. On the other hand, from its definition, Eq. (19) the weak value is seen to be the ratio of two transition matrix elements, hence weak values are akin to amplitudes.

We will further analyze here the different meanings that the weak values can take. An important point to keep in mind, obvious for practitioners of weak measurements but potentially confusing for others, is that previous to the the measurement of B the system state is undefined, as Eq. (16) represents an entangled system-pointer state. After post-selection, at $t = t_f$, the final system state $|b_f\rangle$ is an eigenstate of B, and according to the eigenstate-eigenvalue link, the system has at that point acquired the property value b_f. The weak value also becomes instantiated at $t = t_f$, although, as is clear from the definition (19), the weak value depends on the physical interaction that took place at time t_w (when the system interacted with the pointer). The weak value is hence defined retroactively, as if the post-selected state had propagated backwards in time. This does not call for any sort of retrocausation (except if one endorses [33] a time-symmetric formulation of quantum mechanics, such as the Two State Vector Formalism [34,35]), but is a peculiar feature arising from quantum correlations (see Sects. 4.6 and 4.7).

4.2 Weak Values and the Eigenstate-Eigenvalue Link

By construction, weak measurements do not respect the eigenstate-eigenvalue link. Indeed, the rationale is that the coupling between A and the quantum pointer should minimally disturb the system state, that is the coupling must leave the post-selection probability $|\langle b_k(t_w)| \psi(t_w)\rangle|^2 = \left|\langle b_k(t_f)| \psi(t_f)\rangle\right|^2$ unchanged (relative to the situation without interaction). Therefore, if the eigenstate-eigenvalue link is deemed necessary in order to ascribe a value to a quantum system, then very clearly weak values will not be able to ascribe quantum properties. Although to our knowledge, the status of weak measurements has not been up to now explicitly discussed in terms of property ascription relying on the eigenstate-eigenvalue link, it seems to us that much of the criticism raised against weak values is implicitly relying on this point.

For example for Leggett [10] a weak measurement does not qualify as "*a true measurement process*", true meaning here that the pointer states should be orthogonalized, hence leading to the standard measurement described by Eq. (7). Sokolovski [36] requests that measurements should create real pathways (calling for orthogonal pointer states correlated with orthogonal eigenstates) as opposed to virtual pathways (that take place when the system states in the pointer basis are not orthogonal, leaving the property undefined). Svensson concludes his analysis [37] by asserting that weak values cannot represent "*ordinary properties*", on par with eigenvalues. While Svensson does not discuss property ascription in quantum mechanics nor mentions the eigenstate-eigenvalue link, it turns out (for reasons that will become clear in Sects. 4.6 and 4.7) that his requirement of "bona fide" properties can only be fulfilled when the system ends up in an eigenstate of the measured observable.

4.3 Weak Values as Ensemble Expectation Values

The most common way of introducing weak values is to state they represent some sort of expectation value in pre and post-selected ensembles; a detailed exposition of this approach is given in [38,39]. The first argument in favor of this thesis is that experimentally, a weak value can only be determined by measuring an ensemble of identically prepared and post-selected systems. The shift is indeed very small and can therefore not be meaningfully measured for a single system; the weak value appears statistically as the average taken over the ensemble. Second, as we have seen above, Eqs. (21)–(22), the real and imaginary parts of a weak value are formally equal to conditional expectation values of different operators. Third, it can be shown [40] that the weak values define an operator that is the best estimate of an observable A when not only the initial state but the final state is known.[3]

In our view, none of these reasons are compelling. The first point appears as a practical issue in which statistics are employed to reduce the measurement uncertainties, and has no bearing on fundamental aspects.

The second argument relies on a numerical equivalence: the value of the shift, given by $\operatorname{Re} A^w$, is equal to a conditional expectation value, but this does not imply that $\operatorname{Re} A^w$ is itself an expectation value, i.e. a statistical quantity relevant to ensembles. This can be seen very easily in the particular case in which the pre and post-selected states are arbitrary but identical. Then

$$A^w = \langle \psi(t_w) | A | \psi(t_w) \rangle, \tag{26}$$

so the weak value is numerically given by the expectation value. In this case the pointer state (16) is given by

$$\left| \varphi(t_f) \right\rangle = \sum_k p_k(a_k) e^{-iga_k P} | \varphi(t_i) \rangle, \tag{27}$$

with $p_k(a_k) \equiv |\langle a_k | \psi(t_w) \rangle|^2$ as above. When g is small it is easy to see that the weighted superposition (27) over the shifted pointer states ga_k results in the shift $g \langle \psi(t_w) | A | \psi(t_w) \rangle$. This is the shift of a single pointer, obtained in a single run.

The third point is an interesting observation, but depends on the choice of a specific distance in Hilbert space (arguments based on the choice of a different distance have been put forward to show the opposite, namely that weak values do not behave as averages, see Sect. 4.4). Moreover, it is difficult to explain how a physical pointer can be shifted by an optimal estimator, which is by definition an epistemic quantity.

Therefore, leaving aside commitments to a fully epistemic interpretation of the quantum formalism, for instance if one adheres to the statistical interpretation of quantum mechanics [19] (by which it is assumed that the quantum formalism intrinsically

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[3] The estimate minimizes a specific distance d in Hilbert space, namely $d = \operatorname{Tr}\left[|\psi(t_i)\rangle \langle \psi(t_i)| (A - A_{\text{est}})^2 \right]$, and the resulting best estimate is [40] $A_{\text{est}} = \sum_f \operatorname{Re} A_f^w |b_f\rangle \langle b_f| \operatorname{Re} A_f^w$ where A_f^w is the weak value (19).

describes ensembles) there is no ground to assert that weak values only characterize ensembles with post-selection.

4.4 Weak Values as Generalized Eigenvalues in a Single System

Weak values were originally introduced [1] as a generalized form of eigenvalues, or rather "*a new kind of value for a quantum variable*" [1]. In our context, we will take this to mean that (i) a weak value is a quantity relevant to a single system (as opposed to an ensemble property); and (ii) a weak value is relevant to a property of the quantum system, namely it gives the value of a quantum observable correlated with a given post-selection. There are several arguments in favor of this thesis. First, the pointer motion that is generated by the weak value, see Eq. (20), is taken to be analogous to the pointer motion proportional to an eigenvalue in the case of a projective measurement. Second, the expressions (23) and (25) give the same observable average in terms of eigenvalues and weak values respectively; in the latter case, the probability $p_k(b_k)$ appearing in Eq. (25) is the probability of post-selecting to state $|b_k\rangle$ assuming the disturbance induced by the weak interaction can be neglected. An additional argument, that can be seen as a consequence of the first, was recently given by Vaidman and co-workers [41]: they examine the effect on the pointer dynamics when the shift is induced by an eigenvalue, a weak value, or an average (the pointer is then in a mixed state) and find that for short times the pointer with a weak value shift behaves much more like an eigenvalue shifted pointer than the mixed pointer state corresponding to an average value.

It is not difficult, if one agrees that an eigenvalue is a property of a single system, to admit point (i) above. Indeed, upon post-selection the observable B undergoes a standard projective measurement and the corresponding pointer at first entangled with the system ends up indicating the eigenvalue b_f which we have assumed to be a property of a single system. Since the weakly coupled pointer is entangled with the system, which in turn becomes entangled with the post-selection pointer, the weakly coupled pointer undergoes a small shift upon post-selection. This shift must also be the property of a single system, since there is no reason to interpret the entanglement involving the weakly coupled pointer differently than the entanglement involving the post-selection pointer. In other words, this shift is an "element of reality", and hence the "*mechanical effect*" [8] of the system on the weakly coupled pointer is therefore established as being relevant to a single overall system.

Whether this mechanical effect indicates a generalized eigenvalue representative of a system property is not so straightforward. In the specific case in which A^w is indeed an eigenvalue—implying that either the pre-selected or the post-selected state is an eigenstate of the weakly measured observable A—one relies indirectly on the eigenstate-eigenvalue link: the eigenstate is either the pre or post-selected state, and the eigenvalue comes out of the weak measurement by orthogonalization (the pointer states are indeed orthogonalized despite their overlap).

In the general case, when both the pre and post-selected states are arbitrary, the real and imaginary parts of A^w, given by Eqs. (21) and (22) involve the ratio of averages because as we have seen, due to the weakness of the interaction, the pointer captures

the entire spectrum of the weakly measured observable A. Moreover the expression does not involve the sole weakly measured observable A, but the projector to the post-selected state Π_{b_f}. Finally, we will argue below (Sects. 4.6 and 4.7) that the system has no element of reality corresponding to A^w. For these reasons the term "generalized eigenvalue" might not be very appropriate to characterize a weak value.

4.5 Weak Values as Perturbation Amplitudes

The formal definition of the weak value (Eq. 19) is given by a ratio of amplitudes. This point has often been been put forward [37,42,43] in order to assert that weak values cannot have any meaningful relevance to physical properties. We have already stated that any approach that relies, albeit implicitly, on the eigenstate-eigenvalue link in order to ascribe properties to a quantum system will consistently deny that amplitudes, and hence weak values, can represent values of quantum properties.

Sokolovski goes further [43] in arguing that amplitudes are ubiquitous when perturbation theory is applied, and sees weak measurements as a specific output of perturbation theory. This is of course indisputable from a technical point of view, but such arguments do not take into account the peculiar character of this form of perturbation theory, that is almost identical to a standard measurement process and induces pointer shifts. In this sense, this type of criticism appears as incomplete [44].

4.6 Weak Values as Weak Values

A standard projective measurement of a property represented by the observable A, of the type described above (see Sect. 2.1), involves a correlation between the pointer position and an eigenvalue of A. The entangled state (7) between the pointer and the system correlates each eigenvalue with an unambiguously discriminate pointer state. At the end of the measurement process (after the projective collapse), the pointer indicates the value of the system property.

In a weak measurement, the weakly coupled pointer similarly indicates $\mathrm{Re}\,A^w$, but the shift is small and appears after the post-selection collapse, whereby the post-selection pointer indicates the value b_f of the property corresponding to the observable B. Hence unlike an eigenvalue, $\mathrm{Re}\,A^w$ does not reflect the value of the sole property A, but the value of A correlated with the system having the eigenvalue b_f for the property B. Moreover, although $\mathrm{Re}\,A^w$ depends on the time t_w and on the location of the interaction zone with the weakly coupled pointer, the weak value only appears at the post-selection time t_f. But at t_f the system has a value b_f for the property B and no value can be ascribed to the property A. Strictly speaking the (real part of the) weak value does not ascribe a property to the system, in the sense that there is no corresponding element of reality in the system.[4]

Nevertheless the state of the weakly coupled pointer upon post-selection can be predicted with certainty and is an element of reality for the pointer. This results from a mechanical effect of the coupling interaction on the pointer, that we derived in Sect.

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[4] We have already noticed above that even in the absence of postselection, no element of reality can be defined if the pointer states are not orthogonal.

3.1 and that can also be shown to follow from the dynamics of the pointer variable in the Heisenberg picture [8]. This mechanical effect characterizes the value of A when the system is filtered to an eigenstate of a different observable.

This is exactly how the expression giving Re A^w [Eq. (21)] can be read: the relevant observable is $\frac{1}{2}\left(\rho_{b_f(t_w)}A + A\rho_{b_f(t_w)}\right)$, a symmetrized operator describing the measurement of A followed by ρ_{b_f}. A well-known quantity employed in standard quantum mechanics that has this form is the Schrödinger current $j_\psi(x,t)$, with the corresponding operator being given by [45] $J = \frac{1}{2m}\left(|x\rangle\langle x|P + P|x\rangle\langle x|\right)$, where P is the momentum operator.[5] The denominator in Eq. (21) accounts for the renormalization of the density $\rho = |\psi(t_w)\rangle\langle\psi(t_w)|$, as only the fraction of ρ that reaches post-selection is to be taken into account.

Note that in a purely classical context, the expression equivalent to Eq. (21) would represent [46] the motion of a pointer coupled to the system through the classical interaction Hamiltonian (4), when a filter is implemented. This filter selects the classical particles that will have a specific value b_f at some final time t_f, after the weak interaction.[6] Quantum mechanically the filter is the post-selection, and the apparent retrodictive aspect arises upon post-selection from the quantum correlations imprinted in the entangled state (16) between the system and the pointer.

4.7 The Meaning of Weak Values

We have argued that, despite similarities with eigenvalues, property ascription for weak values is not straightforward. Indeed, the state of the weakly coupled pointer after post-selection (at time t_f) can be predicted with certainty—it is an element of reality as per Sect. 2.2—but regarding the system only the post-selected state $|b_f\rangle$ is an element of reality. Hence for the system there is no element of reality corresponding to the weak value, neither at the time t_w of the interaction, nor at post-selection. This is hardly surprising since the system state is minimally disturbed by the interaction at t_w and has acquired the property value b_f after post-selection.

Despite the lack of an element of reality in the system corresponding to a weak value, it remains possible to link the shift Re A^w to a form of system property. As we have seen in Sect. 4.6, this link is embodied in the correlations encapsulated in the entangled system-pointer state (16). The weak value—that is the mechanical effect on the weakly coupled pointer described in Sect. 4.4—reflects retrodictively the value of A due to the coupling (that took place at the earlier time t_w) compatible with

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[5] Unsurprisingly, the current density appears in the numerator of the following weak value of the momentum, Re $\frac{\langle x|P|\psi(t)\rangle}{\langle x|\psi(t)\rangle} = \frac{mj(x,t)}{\rho(x,t)}$.

[6] The corresponding classical expression is

$$\int_{\mathcal{B}_f} A(q,t_w)\frac{\rho(q,t_w)}{\int_{\mathcal{B}_f}\rho(q',t_w)dq'}dq \qquad (28)$$

where $\rho(q)$ is the configuration space classical distribution. The integral is taken over $\mathcal{B}_f$ which is the set of all q's taken at t_w such that at the final time t_f we have $B(q,t_f) = b_f$. In a classical setting, the filtering needs to be done before the weak interaction takes place. Note that the denominator is simply the normalization constant for the density due to the filtering (see [46] for details).

post-selection. In this sense it is a partial property of the system, relative to a specific space-time region (defined by the location of the weakly coupled pointer) and relative to a choice of post-selected observable and eigenvalue. This form of property ascription is considerably weaker than the one for eigenvalues, which holds for the entire system and is grounded on the existence of a corresponding element of reality.

Nevertheless, this weaker form of property ascription can be meaningful and useful. We have mentioned above the Schrödinger current density as a well known quantity in standard quantum mechanics having the same structure as weak values. It can hardly be maintained that the current density at a particular space-time point does not characterize a partial property of the system at that particular space-time point. We have amply discussed elsewhere [24–26,32] the case of null weak values. In the case of a projector $A = |a_k\rangle \langle a_k|$, a null weak value $A^w = 0$ means that the property represented by A cannot be registered by the weakly coupled pointer for the given post-selection. Such a result stems from the quantum correlations between the weakly coupled and post-selection pointers, and also holds for a strongly coupled intermediate pointer. Null weak values have been used to interpret phenomena like the Quantum Cheshire Cat [23,47] or to account for discontinuous trajectories in the proposals investigating the past of a quantum particle [48–51].

Anomalous weak values (that is WV falling outside the eigenvalue range, such as $A^w = -1$ for a projector) are a consequence of the non-commutativity of the projectors into the pre-selected and post-selected states and the observable A. They are intimately linked to interference effects and cannot be obtained with classical probability distributions [40]. As we have argued in this paper, the interpretation of anomalous weak values as *bone fide* properties on par with eigenvalues [37] cannot hold: there is no corresponding element of reality in the system, as a weak value describes a partial property at a given space-time point, characterizing amplitudes and depending on interference effects. Anomalous weak values still have explanatory power when the system is considered as a whole. For instance a negative projector value or a negative particle number on a given path may not be particularly illuminating by itself, but comparing with weak values of the analogous projectors on other paths gives an explanation—in terms of experimentally measurable quantities—of the dynamics of interference, and further explains the outcomes obtained when projective measurements are made at an intermediate time. Last but not least, weak values give an additional experimentally observable confirmation of the validity of the standard quantum formalism at the level of transition amplitudes, as measured by weakly coupled pointers.

5 Conclusion

We have investigated in this work the meaning of weak values through the prism of the description of the properties of a quantum system that evolves from an initially prepared state to a final post-selected one. We first recalled how properties are ascribed to quantum systems, namely through the eigenstate-eigenvalue link. We focused on pre and post-selected systems to examine how the eigenstate-eigenvalue links works when attempting to understand the property of a quantum system at an intermediate time. The emerging picture is somewhat limited, since such intermediate properties

depend on the measurements that are made, while any attempt to unify the physical picture by counterfactual reasoning leads to paradoxes.

The weak measurements framework bypasses these limitations by implementing a minimally perturbing interaction with a quantum pointer. The weak value, quantifying the imprint of the interaction and the subsequent post-selection on the pointer, shares some similarities with eigenvalues, in particular the fact that, if an eigenvalue is assumed to be relevant to a single system (and not an ensemble), then this is also the case for a weak value. We examined property ascription to a system observable based on weak values. This turned out to be a subtle issue, as a weak value A^w characterizes the system observable A filtered by post-selection in a retrodictive manner, mediated by entanglement and without a corresponding system element of reality.

We discussed several interpretations that have been given to weak values, and argued that weak values can indeed be seen as ascribing properties to a system but in a partial way, certainly not on par with the standard property ascription based on the eigenstate-eigenvalue link. The explanatory power afforded by the weak measurements framework not only concerns the outcomes obtained in standard projective measurements when quantum interferences play a prominent role, but confirm the validity of the standard formalism at the level of amplitudes. In turn, this could lead to novel fascinating implications concerning the physical nature of the formalism described by quantum theory.

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