Foundations of Physics (2018) 48:1349–1363
https://doi.org/10.1007/s10701-018-0187-7

Black Hole Perturbations: A Review of Recent Analytical Results

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Received: 3 October 2017 / Accepted: 11 June 2018 / Published online: 16 June 2018

Abstract
We review the gravitational self-force program to analytically compute first-order metric perturbations in a Schwarzschild black hole spacetime in the case of a perturbing (small) mass moving on a slightly eccentric equatorial orbit. The perturbed metric components should then be combined into gauge-invariant quantities to be associated with physical observables. In this way, for example, one determines the various "potentials" entering the Effective-One-Body model, i.e., a powerful formalism for the description of the gravitational interaction of two masses, which is currently successfully used for the analysis of gravitational wave signals.

Keywords Schwarzschild black hole · Gravitational self-force · Eccentric orbits

1 Introduction

When a particle moves on a given background at the zero-order approximation level it follows a (timelike) geodesic of the spacetime itself. At the first-order it perturbs/modifies the background and the corrections (encoded in ten metric components functions of the spacetime coordinates) are proportional to the (small) mass of the particle. The full metric, background plus perturbation, satisfies the (linearized) non-vacuum Einstein field equations, i.e., a coupled set of partial differential equations for the various metric components. Moreover, the perturbation (each metric component) is singular at the location of the particle and unfortunately this is exactly the place where one needs to evaluate it (or some associated gauge-invariant quantity). A regularization procedure, familiar from high-energy physics, is thus necessary in order to extract physical information from the singular field. We will review each of these

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steps in the case of a perturbation due to a particle moving along a slightly eccentric equatorial orbit in a Schwarzschild spacetime.

The first explicit analytical computation of a gauge-invariant quantity was performed in 2013 in Ref. [1], where Detweiler's redshift invariant (i.e., the linear-in-mass-ratio change in the time component of the particle's 4-velocity) was calculated for a particle in circular orbit around a Schwarzschild black hole. This result was then transcribed into the knowledge of the first order correction to the main radial potential of the Effective-One-Body (EOB) model [2–4]. According to this formalism the two-body problem is reduced to the dynamics of a single point particle moving in an effective metric, whose potentials are determined taking results from post-Newtonian (PN) theory, black hole perturbation theory and numerical relativity. Later on several other quantities have been computed following the scheme described above or with some interesting variation, for both circular and slightly eccentric orbits in Schwarzschild and Kerr black hole backgrounds, including corrections to the circular limit of the periastron advance, change in the innermost stable circular orbit, spin precession, tidal invariants [5–16]. The (new) branch of general relativistic research in this sector is called Gravitational Self-Force (GSF) and represents one the most stimulating challenges in general relativity today.

2 GSF Calculations in a Schwarzschild Spacetime

Through $O(m_1)$, a small mass m_1 follows a geodesic in a regularly perturbed spacetime

$$g^{\rm R}_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \frac{m_1}{m_2} h^{\rm R}_{\alpha\beta}, \tag{1}$$

where $g^{(0)}_{\alpha\beta}$ is the background Schwarzschild spacetime

$$g^{(0)}_{\alpha\beta} dx^\alpha dx^\beta = -f dt^2 + \frac{1}{f} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2), \qquad f = 1 - \frac{2m_2}{r}, \tag{2}$$

and $(m_1/m_2) h^{\rm R}_{\alpha\beta}$ is the (regularized) metric perturbation. Henceforth, we shall omit the superscript R and denote $m_1/m_2 = q$. Furthermore, we will denote the large mass by M (instead of m_2) and the small mass by μ (instead of m_1), to follow a standard notation in perturbation theory.

The main steps to analytically compute first-order metric perturbations in a Schwarzschild spacetime are listed below.

1. Decompose the perturbed metric and the energy momentum tensor associated with the perturbing mass in tensor harmonics in some gauge, here the Regge-Wheeler (RW) gauge.
2. Separate even-parity and odd parity quantities/equations. Odd-parity waves satisfy the simple RW equation with odd-parity source terms; even-parity waves satisfy instead the more complicated Zerilli equation with even-parity source terms. Next, using results by Chandrasekhar [17], the Zerilli equation can be mapped onto a

RW equation with different source terms, so that one has to solve in both cases a single RW equation, with appropriate (odd-parity or even-parity) source terms.

3. Solve the homogeneous (radial) Regge–Wheeler equation in various ways: PN approximation (generic l, weak-field and slow motion), Wentzel–Kramers–Brillouin (WKB) approximation (large l), Mano, Suzuki and Takasugi (MST) technique [18,19] (specified values of l).
4. Use the Green method to compute the solutions of the inhomogeneous RW equation, in both cases of even and odd parity source terms, which are singular at the location of the particle (being proportional to the Dirac delta function and its derivatives).
5. Construct the gauge-invariant quantity one is interested in, depending on the azimuthal number m and the orbital angular momentum number l.
6. Sum over $m \in [-l, l]$ by using spherical harmonic identities.
7. Sum over $l \in [0, \infty)$ after performing the regularization of the otherwise divergent summation.

The description of each of these points would necessitate a lot of space to be accounted in detail. We will limit below to a minimal amount of information, pointing out the main literature where these details can be found.

3 The Regge–Wheeler Equation and Its PN Form

The decomposition of the perturbed metric into spherical harmonics allows to separate the angular part of the perturbation. Taking then the Fourier transform of the metric components reduces the Einstein field equations to a set of coupled ordinary differential equations for 6 unknown radial functions of different parity: 3 odd equations (for 2 unknowns) and 7 even equations (for 4 unknown). However, the perturbation functions of both parity can be expressed in terms of a single unknown for each sector, satisfying the same Regge–Wheeler equation

$$\mathcal{L}^{(r)}_{(\mathrm{RW})}[R^{(\mathrm{even/odd})}_{lm\omega}] = S^{(\mathrm{even/odd})}_{lm\omega}(r). \tag{3}$$

Here $\mathcal{L}^{(r)}_{(\mathrm{RW})}$ denotes the RW operator

$$\begin{aligned}\mathcal{L}^{(r)}_{(\mathrm{RW})} &= f(r)^2 \frac{d^2}{dr^2} + \frac{2M}{r^2} f(r) \frac{d}{dr} + [\omega^2 - V_{(\mathrm{RW})}(r)] \\ &= \frac{d^2}{dr_*^2} + [\omega^2 - V_{(\mathrm{RW})}(r)],\end{aligned} \tag{4}$$

with $d/dr_* = f(r) d/dr$, and the RW potential

$$V_{(\mathrm{RW})}(r) = f(r) \left(\frac{l(l+1)}{r^2} - \frac{6M}{r^3} \right). \tag{5}$$

The source terms are singular at the location $r = r_0(t)$ of the particle, namely

$$S_{lm\omega}^{(\text{even/odd})}(r) = \sum_k s_k^{(\text{even/odd})}(t)\delta^{(k)}(r - r_0(t)), \tag{6}$$

with $\delta^{(k)}(r - r_0(t))$ denoting the kth-order derivative of the Dirac delta function.

3.1 Perturbing Particle's World Line

Let the particle move along an eccentric geodesic orbit on the equatorial plane of the Schwarzschild spacetime. The associated 4-velocity $\mathcal{U}_0$ ($\mathcal{U}_0 \cdot \mathcal{U}_0 = -1$) is given by

$$\mathcal{U}_0 = \frac{E_0}{f}\partial_t + \dot{r}_0\partial_r + \frac{L_0}{r_0^2}\partial_\phi, \tag{7}$$

where

$$\dot{r}_0^2 = \left(\frac{dr_0}{d\tau_0}\right)^2 = E_0^2 - f\left(1 + \frac{L_0^2}{r^2}\right). \tag{8}$$

The orbit can be parametrized either by the proper time τ_0 or by the relativistic anomaly $\chi \in [0, 2\pi]$, so that

$$r_0 = \frac{Mp}{1 + e\cos\chi}, \tag{9}$$

which are related by

$$\frac{d\tau_0}{d\chi} = \frac{Mp^{3/2}}{(1 + e\cos\chi)^2}\left[\frac{p - 3 - e^2}{p - 6 - 2e\cos\chi}\right]^{1/2}. \tag{10}$$

The (dimensionless) orbital parameters semi-latus rectum p and eccentricity e are defined by writing the minimum (pericenter, r_{peri}) and maximum (apocenter, r_{apo}) values of the radial coordinate along the orbit as

$$r_{\text{peri}} = \frac{Mp}{1 + e}, \qquad r_{\text{apo}} = \frac{Mp}{1 - e}. \tag{11}$$

They are in correspondence with the conserved energy $E_0 = -u_t$ and angular momentum $L_0 = u_\phi$ per unit mass of the particle, via

$$E_0^2 = \frac{(p - 2)^2 - 4e^2}{p(p - 3 - e^2)}, \qquad L_0^2 = \frac{M^2p^2}{p - 3 - e^2}. \tag{12}$$

The reciprocal of p, $u_p = p^{-1}$, is also a useful "small" quantity corresponding to $r_0/M \ll 1$. Equation (10) can be used to solve the equations for t and ϕ as functions of χ, which are expressible in terms of elliptic integrals.

As it is well known, eccentric orbits are characterized by two fundamental frequencies, $\Omega_{r0} = 2\pi/T_{r0}$ and $\Omega_{\phi 0} = \Phi_0/T_{r0}$, where $\Phi_0 = \oint d\phi$ is the angular advance during one radial period $T_{r0} = \oint dt$ (with $\oint d\xi = \int_0^{2\pi} (d\xi/d\chi) d\chi$). To second order in e we find

$$M\Omega_{r0} = u_p^{3/2}(1-6u_p)^{1/2}\left[1 - \frac{3}{4}\frac{2-32u_p+165u_p^2-266u_p^3}{(1-2u_p)(1-6u_p)^2}\,e^2 + O(e^4)\right],$$
$$M\Omega_{\phi 0} = u_p^{3/2}\left[1 - \frac{3}{2}\frac{1-10u_p+22u_p^2}{(1-2u_p)(1-6u_p)}\,e^2 + O(e^4)\right]. \tag{13}$$

The motion is then specified by

$$\frac{r_0(t)}{Mp} = \frac{1}{1-e} + (\cos\Omega_{r0}t - 1)\,e + b_2(\cos(2\Omega_{r0}t) - 1)\,e^2 + O(e^3),$$
$$\phi_0(t) = \Omega_{\phi 0}t + c_1\sin(\Omega_{r0}t)\,e + c_2\sin(2\Omega_{r0}t)\,e^2 + O(e^3), \tag{14}$$

where

$$b_2 = -\frac{1}{2}\frac{1-11u_p+26u_p^2}{(1-2u_p)(1-6u_p)},$$
$$c_1 = -2\frac{1-3u_p}{(1-2u_p)(1-6u_p)^{1/2}}, \quad c_2 = \frac{1}{4}\frac{5-64u_p+250u_p^2-300u_p^3}{(1-2u_p)^2(1-6u_p)^{3/2}}. \tag{15}$$

In the perturbed spacetime, we consider a timelike geodesic parametrized by its proper time τ (different from τ_0) and *the same orbital parameters* p, e as the reference geodesic of the background. This implies that we are comparing at the same coordinate radius r (χ is the same in both spacetimes), though not the same t and ϕ coordinates. Any such difference is not gauge-invariant, in general.

3.2 Source Terms

For a particle moving on a slightly eccentric orbit in the equatorial plane of the Schwarzschild spacetime the source terms can be easily written.

In the case of odd-parity metric perturbations the source terms are given by

$$S_{lm\omega}^{(\text{odd})} = \left[s_0^{(\text{odd})}\delta(r - r_0(t)) + s_1^{(\text{odd})}\delta'(r - r_0(t))\right] Y^{*\prime}{}_{lm}\left(\frac{\pi}{2}, \phi_0(t)\right), \tag{16}$$

with

$$s_0^{(\text{odd})} = -\frac{4\pi\mu L_0}{E_0\lambda(\lambda+1)r_0(t)^4} f_0(t)\left[2E_0\lambda r_0(t)\frac{dr_0(t)}{dt} - imL_0 f_0(t)\right],$$
$$s_1^{(\text{odd})} = -\frac{4i\pi\mu m L_0^2}{E_0\lambda(\lambda+1)r_0(t)^3} f_0(t)^3. \tag{17}$$

Here we have denoted $f_0(t) = f(r_0(t))$ and

$$\lambda = \frac{1}{2}(l-1)(l+2) \tag{18}$$

is such that $2(\lambda+1) = l(l+1)$, which is unchanged under the transformation $l \to -l-1$.

In the case of even-parity metric perturbations we have instead

$$S^{(\mathrm{even})}_{lm\omega} = \Big[s_0^{(\mathrm{even})}\delta(r-r_0(t)) + s_1^{(\mathrm{even})}\delta'(r-r_0(t)) + s_2^{(\mathrm{even})}\delta''(r-r_0(t))\Big] Y^{*\prime}_{lm}\left(\frac{\pi}{2}, \phi_0(t)\right), \tag{19}$$

with

$$\begin{aligned}
s_0^{(\mathrm{even})} &= -\frac{8i\pi\mu}{\omega E_0\lambda(\lambda+1)r_0(t)^5(\lambda r_0(t)+3M)^3} f_0(t) \\
&\times\Big\{E_0\lambda r_0(t)^3 \frac{dr_0(t)}{dt}[2\lambda(\lambda+1)E_0 r_0(t) + m\omega L_0(\lambda r_0(t)+3M)] \\
&\times[\lambda(\lambda+1)r_0(t)(\lambda r_0(t)+3M) + 3M^2(2\lambda+3)] \\
&+i(\lambda r_0(t)+3M)r_0(t) f_0(t)\{-r_0(t)^4\omega\lambda^2(\lambda+1)^2(\lambda r_0(t)+3M) \\
&-\omega\lambda(\lambda+1)[L_0^2\lambda^2(2(\lambda+1)-m^2) + 3M^2(2\lambda+3)]r_0(t)^3 \\
&+M\lambda^2(\lambda+1)[-m(2\lambda-3)E_0 + 3L_0\omega(-3\lambda-3+2m^2)]L_0 r_0(t)^2 \\
&+3M^2\lambda L_0[-2m(2\lambda^2-3)E_0 + L_0\omega(-7\lambda^2+5\lambda m^2-16\lambda+6m^2 \\
&-9)]r_0(t) + 9M^3L_0[\omega(-\lambda+m^2-1)L_0 - 2\lambda m E_0](2\lambda+3)\}\Big\}, \\
s_1^{(\mathrm{even})} &= \frac{8i\pi\mu}{\omega E_0\lambda(\lambda+1)r_0(t)^4(\lambda r_0(t)+3M)^2} f_0(t)^2 \\
&\times\Big\{3E_0\lambda M r_0(t)^3\frac{dr_0(t)}{dt}\,[2E_0\lambda(\lambda+1)r_0(t) + m\omega L_0(\lambda r_0(t)+3M)] \\
&-i(\lambda r_0(t)+3M)r_0(t) f_0(t)\{mL_0E_0\lambda[\lambda(\lambda+1)r_0(t)^2 + 18M^2] \\
&-3M\omega L_0^2[m^2(\lambda r_0(t)+3M) - (\lambda+1)(2\lambda r_0(t)+3M)] \\
&+3M\omega\lambda(\lambda+1)r_0(t)^3\}\Big\}, \\
s_2^{(\mathrm{even})} &= -\frac{24\pi\mu m M L_0}{\omega(\lambda+1)r_0(t)(\lambda r_0(t)+3M)} f_0(t)^4.
\end{aligned} \tag{20}$$

The expansion of the source-terms in powers of e gives rise to multiperiodic coefficients involving the combined frequencies

$$\omega_{m,n} = m\Omega_{\phi 0} + n\Omega_{r0}, \tag{21}$$

with $n = 0, \pm1, \pm2$ when working, e.g., up to order e^2.

We proceed to explicitly determine analytical solutions to the homogeneous RW equation.

3.3 PN Solutions to the Homogeneous RW Equation

Restoring the factors of c, the homogeneous RW equation $\mathcal{L}^{(r)}_{(\mathrm{RW})} X_{l\omega}(r) = 0$ reads

$$0 = X''_{l\omega}(r) + \frac{2M\eta^2}{r^2}\left(1 - \frac{2M\eta^2}{r}\right)^{-1} X'_{l\omega}(r) + \left(1 - \frac{2M\eta^2}{r}\right)^{-2}\left\{\omega^2\eta^2 - \frac{1}{r^2}\left(1 - \frac{2M\eta^2}{r}\right)\left[l(l+1) - \frac{6M\eta^2}{r}\right]\right\} X_{l\omega}(r). \tag{22}$$

Expanding the above equation in series of $\eta = 1/c$ and solving order by order one can obtain two independent PN-type solutions: the ingoing ("in") solution and the upgoing ("up") solution. The "in" PN solution behaves as r^{l+1} and is regular at the origin $r = 0$, whereas the "up" PN solution decays at infinity with a power law (r^{-l}). They share the property that the PN "up" solution can be obtained from the "in" solution simply by replacing $l \to -l-1$. This follows from the fact that the homogeneous RW equation itself depends only on the product $l(l+1)$, which is invariant under this transformation.

The structure of the "in" solution is

$$X^{\mathrm{in(PN)}}_{l\omega}(r) = r^{l+1}\left(1 + A_2^{\mathrm{in}\,(PN,l)}\eta^2 + A_4^{\mathrm{in}\,(PN,l)}\eta^4 + \cdots\right), \tag{23}$$

where the first few coefficients $A_k^{\mathrm{in}\,(PN,l)}$ are listed below using the notation $X_1 = M/r$, $X_2 = (\omega r)^2$:

$$\begin{aligned}
A_2^{\mathrm{in}\,(PN,l)} &= -\frac{(l-2)(l+2)}{l}X_1 - \frac{1}{2(2l+3)}X_2,\\
A_4^{\mathrm{in}\,(PN,l)} &= \frac{(l-2)(l-3)(l+2)(l+1)}{(-1+2l)l}X_1^2 + \frac{(l^3-5l^2-14l-12)}{2(l+1)(2l+3)l}X_2X_1\\
&\quad + \frac{1}{8(2l+5)(2l+3)}X_2^2,\\
A_6^{\mathrm{in}\,(PN,l)} &= -\frac{(l-2)(l-3)(l-4)(l+2)(l+1)}{3(-1+2l)(l-1)}X_1^3\\
&\quad - \frac{2(15l^4+30l^3+28l^2+13l+24)}{(-1+2l)(2l+1)(l+1)l(2l+3)}X_1^2X_2\ln(r/R)\\
&\quad - \frac{(3l^4-27l^3-154l^2-220l-120)}{24(l+1)l(2l+5)(l+2)(2l+3)}X_1X_2^2\\
&\quad - \frac{1}{48(2l+5)(2l+7)(2l+3)}X_2^3.
\end{aligned} \tag{24}$$

The length (and complexity) of these coefficients increases with higher powers of η. Logarithmic terms ("PN-logs") first appear in the coefficient $A_6^{\mathrm{in}\,(PN,l)}$ and involve a priori arbitrary length scales (denoted as R there). They are associated with r-independent combinations of the type $\eta^6 X_1^2 X_2 = (\eta^2 M/r)^2(\eta\omega r)^2$, and its higher powers (for higher order coefficients). However, the dependence of $X_{l\omega}^{\mathrm{in(PN)}}$ on the choice of these scales is spurious, because changing these scales only modify the solution by a constant multiplicative factor.

The PN "in" solution cannot be used for $l = 0, 1$ multipoles, since many of the above coefficients are singular for these values of l. This is physically related to the fact that the $l = 0$(monopole) and $l = 1$ (dipole) solutions correspond to gauge terms associated with changes in the background mass and angular momentum, respectively. Hence, they must be separately added.

Furthermore, when $l = 2$ the PN "up" solution cannot be used as is at (and beyond) order η^8. Indeed, the coefficient $A_8^{\mathrm{in}\,(PN,l)}$ contains a factor $\sim 1/(l+3)$ which, when converted into the corresponding up coefficient (by replacing $l \to -l-1$), gives a term $\sim 1/(l-2)$ which is singular for $l = 2$. This situation is quite general and the formally singular terms $\sim 1/(l-l_0)$ entering the PN "up" solution at $O(\eta^{6+l_0})$ signal the appearance of new types of logarithmic terms.

3.4 MST Solutions to the Homogeneous RW Equation

In order to resolve this type of ambiguities arising in the the PN "up" solutions a useful technique has been developed by Mano, Suzuki and Takasugi [18,19]. They introduced a hypergeometric (HG) expansion solution such that $X_{l\omega}^{\mathrm{in}}$ is incoming from $r = +\infty$ (and purely ingoing on the horizon), whereas $X_{l\omega}^{\mathrm{up}}$ is upgoing from the horizon (and purely outgoing at infinity). Their analytic expressions are given by

$$
\begin{aligned}
X_{l\omega}^{\mathrm{in(HG)}}(r) &= C_{\mathrm{(in)}}^{\nu}(x) \sum_{n=-\infty}^{\infty} a_n^{\nu} \overline{F}(n+\nu-1-i\epsilon, -n-\nu-2-i\epsilon, 1-2i\epsilon; x],\\
X_{l\omega}^{\mathrm{up(HG)}}(r) &= C_{\mathrm{(up)}}^{\nu}(z) \sum_{n=-\infty}^{\infty} a_n^{\nu} (-2iz)^n \overline{\Psi}(n+\nu+1-i\epsilon, 2n+2\nu+2; -2iz),
\end{aligned} \tag{25}
$$

where the hypergeometric functions are of the usual Gauss-type for the "in" solution and of the confluent type for the "up" solution. Here, $x = 1 - c^2 r/2GM$, $z = \omega r/c$, $\epsilon = 2GM\omega/c^3$,

$$
\begin{aligned}
\nu = l + \frac{1}{2l+1}\Bigg[&-2 - \frac{4}{l(l+1)} + \frac{[(l+1)^2-4]^2}{(2l+1)(2l+2)(2l+3)}\\
&-\frac{(l^2-4)^2}{(2l-1)2l(2l+1)}\Bigg]\epsilon^2 + O(\epsilon^4),
\end{aligned} \tag{26}
$$

and

$$
\begin{aligned}
C^{\nu}_{(\mathrm{in})}(x) &= c_{(\mathrm{in})} e^{i\epsilon[(x-1)-\ln(-x)]}(1-x)^{-1},\\
C^{\nu}_{(\mathrm{up})}(z) &= c_{(\mathrm{up})} e^{iz} z^{\nu+1}\left(1-\frac{\epsilon}{z}\right)^{-i\epsilon} 2^{\nu} e^{-\pi\epsilon} e^{-i\pi(\nu+1)},\\
\overline{F}(a,b,c;x) &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} \quad F(a,b,c;x),\\
\overline{\Psi}(a,b;\zeta) &= \frac{\Gamma(a-2)\Gamma(a)}{\Gamma(a^*+2)\Gamma(a^*)}\Psi(a,b;\zeta),
\end{aligned} \tag{27}
$$

with a^* denoting the complex conjugate of a and Ψ the second Kummer function. The quantities $c_{(\mathrm{in})} = \eta^{\alpha^{(l)}_{(\mathrm{in})}}$ and $c_{(\mathrm{up})} = \eta^{\alpha^{(l)}_{(\mathrm{up})}}$ are some l-dependent powers of $\eta = 1/c$ defined so that $C^{\nu}_{(\mathrm{in})}(x)$ and $C^{\nu}_{(\mathrm{up})}(z)$ both start with zeroth order in η. Finally, the two-sided sequence of coefficients a^{ν}_n entering both series (25) are obtained by solving the three-term recursion relation $\alpha^{\nu}_n a^{\nu}_{n+1} + \beta^{\nu}_n a^{\nu}_n + \gamma^{\nu}_n a^{\nu}_{n-1} = 0$. To solve this three-term relation between $n = -N$ and $n = +N$ (included) one initiates the recursion with $a_{N+1} = 0$ and $a_{-N-1} = 0$ and chooses $a_0 = 1$ to have a fully determined set of algebraic equations.

The expansions of $X^{\mathrm{in(HG)}}_{l\omega}$ and $X^{\mathrm{up(HG)}}_{l\omega}$ in powers of η thus have the general form

$$
\begin{aligned}
X^{\mathrm{in(HG)}}_{l\omega}(r) &= \tilde{c}_{\mathrm{in}} r^{l+1} \sum_{k=0}^{k_{\max}} A_k^{\mathrm{in}\,(HG,l)} \eta^k,\\
X^{\mathrm{up(HG)}}_{l\omega}(r) &= \tilde{c}_{\mathrm{up}} r^{-l} \sum_{k=0}^{k_{\max}} A_k^{\mathrm{up}\,(HG,l)} \eta^k,
\end{aligned} \tag{28}
$$

with $A_0^{\mathrm{in}\,(HG,l)} = 1 = A_0^{\mathrm{up}\,(HG,l)}$ and $\tilde{c}_{\mathrm{in}}$ and $\tilde{c}_{\mathrm{up}}$ are some normalization coefficients. Modulo some logarithmic dependence in r, the coefficients A_k are polynomials in the two quantities $X_1 = GM/r$ (linked to the weak-field expansion), and $\sqrt{X_2} = \omega r$ (linked to the near-zone expansion).

Let us discuss the structure of the η expansion of the hypergeometric "up" solutions for the particular case $l = 2$. Equation (28) then gives

$$
X^{\mathrm{up(HG)}}_{(l=2)\omega}(r) = -\frac{i}{16\omega^2 r^2} \sum_k A_k^{\mathrm{up}\,(HG,l=2)} \eta^k, \tag{29}
$$

with

$$
\begin{aligned}
A_0^{\mathrm{up}\,(HG,l=2)} &= 1,\\
A_1^{\mathrm{up}\,(HG,l=2)} &= 0,\\
A_2^{\mathrm{up}\,(HG,l=2)} &= \frac{1}{6}X_2 + \frac{5}{3}X_1,
\end{aligned}
$$

$$
\begin{aligned}
A_3^{\mathrm{up}\,(HG,\,l=2)} &= \left(6i\gamma - 2\pi - \frac{43}{6}i\right) X_1\sqrt{X_2},\\
A_4^{\mathrm{up}\,(HG,\,l=2)} &= \frac{7}{6}X_1X_2 + \frac{1}{24}X_2^2 + \frac{20}{7}X_1^2,\\
A_5^{\mathrm{up}\,(HG,\,l=2)} &= \left[\left(10i\gamma - \frac{10}{3}\pi - \frac{215}{18}i\right) X_1^2\right.\\
&\quad \left. + \left(-\frac{43}{36}i + i\gamma - \frac{1}{3}\pi\right) X_1X_2 + \frac{1}{45}iX_2^2\right]\sqrt{X_2},\\
A_6^{\mathrm{up}\,(HG,\,l=2)} &= 5X_1^3 + \left(-18\gamma^2 + \frac{7}{3}\pi^2 - 12i\pi\gamma - \frac{272551}{8820} + \frac{214}{105}\ln(2\omega r\eta)\right.\\
&\quad \left. + \frac{4729}{105}\gamma + \frac{43}{3}i\pi\right) X_1^2X_2 + \frac{7}{24}X_1X_2^2 - \frac{1}{144}X_2^3,
\end{aligned}
\tag{30}
$$

where γ is the Euler's constant. Logarithmic terms of the type $\ln(2\omega r\eta)$ first appear at the 3PN level (i.e., A_6^{up}), whereas squared logarithms like $(\ln(2\omega r\eta))^2$ at the 6PN level (i.e., A_{12}^{up}). These logarithms have different physical meanings, being related either to gauge effects or to far-zone effects or even to tail modifications of conservative effects as well as radiation-reaction effects.

4 Solutions of the Inhomogeneous Regge–Wheeler Equation: The Green's Function Method

The Green's function method is used to solve the inhomogeneous Eq. (3). The retarded Green's function is expressed in terms of the two independent *homogeneous* solutions $X_{l\omega}^{\mathrm{in}}$ and $X_{l\omega}^{\mathrm{up}}$ of the RW operator (discussed above) as

$$
\begin{aligned}
G(r, r') &= \frac{1}{W}\Big[X_{l\omega}^{\mathrm{in}}(r)X_{l\omega}^{\mathrm{up}}(r')H(r'-r) + X_{l\omega}^{\mathrm{in}}(r')X_{l\omega}^{\mathrm{up}}(r)H(r-r')\Big]\\
&\equiv G_{(\mathrm{in})}(r, r')H(r'-r) + G_{(\mathrm{up})}(r, r')H(r-r'),
\end{aligned}
\tag{31}
$$

where $W_{l\omega}$ denotes the (constant) Wronskian

$$
W_{l\omega} = \left(1 - \frac{2M}{r}\right)\left[X_{l\omega}^{\mathrm{in}}(r)\frac{d}{dr}X_{l\omega}^{\mathrm{up}}(r) - \frac{d}{dr}X_{l\omega}^{\mathrm{in}}(r)X_{l\omega}^{\mathrm{up}}(r)\right] = \text{const.}
\tag{32}
$$

and $H(x)$ is the Heaviside step function. Both even-parity and odd-parity solutions are then given by integrals over the corresponding (distributional) sources as

$$
R_{lm\omega}^{(\mathrm{even/odd})}(r) = \int dr'\, \frac{G(r, r')}{f(r')} S_{lm\omega}^{(\mathrm{even/odd})}(r').
\tag{33}
$$

5 Metric Reconstruction

Once the radial function is known for both parities, the perturbed metric components are then computed by multiplying by the angular part and summing over $m \in [-l, l]$ and $l \in [0, \infty]$.

5.1 Summation Over *m* and *l*

One has first to perform the summation over the magnetic number m. This is done by using standard spherical harmonic identities namely

$$S_{k,l} = \sum_{m=-l}^{l} m^k \left| Y_{lm}\left(\frac{\pi}{2}, 0\right)\right|^2, \qquad S'_{k,l} = \sum_{m=-l}^{l} m^k \left| Y'_{lm}\left(\frac{\pi}{2}, 0\right)\right|^2, \tag{34}$$

as shown in detail for example in Appendix F of Ref. [20].

The summation over l from 0 to infinity is also straightforward once one takes the regular part of it. This process is called regularization and consists in removing the divergent part of the summation (i.e., the large-l limit) due to the fact that one is computing quantities at the source location, singular by definition.

5.2 The $l = 0, 1$ Multipoles

The contribution of the non-radiative modes $l = 0, 1$ (when seen from outside, i.e., for $r > r_0(t)$) comes from the changes in the mass and angular momentum of the black hole due to the presence of the orbiting particle of mass μ. The exterior Schwarzschild metric perturbed in mass and angular momentum acquires the following nonzero components

$$h_{tt} = -\frac{2\delta M}{r}, \qquad h_{t\phi} = \frac{2\delta J}{r}\sin^2\theta, \qquad h_{rr} = -\frac{2\delta M r}{(r-2M)^2}, \tag{35}$$

with $\delta M = \mu E_0$ and $\delta J = \mu L_0$, where E_0 and L_0 are given by Eq. (12).

6 Computation of Gauge-Invariant Metric Quantities

The perturbed black hole metric constructed by following the procedure outlined above can then be used to compute, in analytical form, gauge-invariant quantities. A useful such quantity for perturbations induced by a particle orbiting a Schwarzschild black hole is the inverse redshift invariant function, as suggested by Detweiler in 2008 [21] for circular orbits and then generalized to eccentric orbits by Barack and Sago [22]. It is defined as

$$U\left(m_2\Omega_r, m_2\Omega_\phi, q\right) = \frac{\oint dt}{\oint d\tau} = \frac{T_r}{\mathcal{T}_r}, \tag{36}$$

where all quantities refer to the perturbed spacetime metric (1). As stated above, the symbol $\oint$ denotes an integral over a radial period (from periastron to periastron), so that $T_r = \oint dt$ denotes the coordinate-time period and $\mathcal{T}_r = \oint d\tau$ the proper-time period. The first-order SF contribution δU to the function (36), i.e.,

$$U\left(m_2\Omega_r, m_2\Omega_\phi, q\right) = U_0\left(m_2\Omega_r, m_2\Omega_\phi\right) + q\delta U\left(m_2\Omega_r, m_2\Omega_\phi\right) + O(q^2),$$

is given in terms of the $O(q)$ metric perturbation $h_{\mu\nu}$ by the following *coordinate* time average (equivalent to the original definition of Ref. [22] in terms of a *proper* time τ average)

$$\delta U = \frac{1}{2} U_0^2 \langle h_{uk}\rangle_t, \tag{37}$$

where $h_{uk} = h_{\mu\nu}u^\mu k^\nu$, with $u^\mu \equiv u^t k^\mu$, $u^t = dt/d\tau$ and $k^\mu \equiv \partial_t + (dr/dt)\partial_r + (d\phi/dt)\partial_\phi$, and U_0 denotes the proper-time average of u^t along the unperturbed orbit, i.e., the ratio $U_0 = T_{r0}/\mathcal{T}_{r0}$.

Starting in 2013, the analytical values of the PN expansion coefficients of δU have been computed in the case of circular motion in a Schwarzschild spacetime from 4PN [1] to very high PN orders [23–26]. For eccentric orbits δU is currently known up to the 9.5PN order for the e^2 and e^4 contributions, and up to the 4PN order for the higher eccentricity contributions through e^{20} [27–30]. We quote below some of the lowest-order PN coefficients to second order in the eccentricity, i.e.,

$$\delta U(u_p, e) = \delta U^{e^0}(u_p) + e^2\delta U^{e^2}(u_p) + O(e^4), \tag{38}$$

with

$$\begin{aligned}
&\delta U^{e^0}(u_p)\\
&= -u_p - 2u_p^2 - 5u_p^3 + \left(-\frac{121}{3} + \frac{41}{32}\pi^2\right)u_p^4\\
&\quad + \left(-\frac{1157}{15} + \frac{677}{512}\pi^2 - \frac{128}{5}\gamma - \frac{256}{5}\ln(2) - \frac{64}{5}\ln(u_p)\right)u_p^5\\
&\quad + \left(\frac{1606877}{3150} - \frac{60343}{768}\pi^2 + \frac{1912}{105}\gamma + \frac{7544}{105}\ln(2) + \frac{956}{105}\ln(u_p) - \frac{243}{7}\ln(3)\right)u_p^6\\
&\quad - \frac{13696}{525}\pi u_p^{13/2}\\
&\quad + \left(\frac{17083661}{4050} - \frac{1246056911}{1769472}\pi^2 + \frac{102512}{567}\gamma + \frac{372784}{2835}\ln(2) + \frac{51256}{567}\ln(u_p)\right.\\
&\quad \left. + \frac{1215}{7}\ln(3) + \frac{2800873}{262144}\pi^4\right)u_p^7\\
&\quad + \frac{81077}{3675}\pi u_p^{15/2} + O(u_p^8),
\end{aligned} \tag{39}$$

and

$$
\begin{aligned}
&\delta U^{e^2}(u_p)\\
&= u_p + 4u_p^2 + 7u_p^3 + \left(-\frac{5}{3} - \frac{41}{32}\pi^2\right) u_p^4\\
&+\left(-\frac{11141}{45} + \frac{29665}{3072}\pi^2 - \frac{592}{15}\gamma + \frac{3248}{15}\ln(2) - \frac{296}{15}\ln(u_p) - \frac{1458}{5}\ln(3)\right) u_p^5\\
&+\left(-\frac{2238629}{1575} - \frac{73145}{1536}\pi^2 + \frac{17392}{105}\gamma - \frac{167696}{105}\ln(2) + \frac{8696}{105}\ln(u_p)\right.\\
&\left.+\frac{42282}{35}\ln(3)\right) u_p^6\\
&-\frac{232618}{1575}\pi u_p^{13/2}\\
&+\left(\frac{2750367763}{198450} - \frac{13433142863}{3538944}\pi^2 + \frac{5102288}{2835}\gamma + \frac{41285072}{2835}\ln(2)\right.\\
&\left.+\frac{2551144}{2835}\ln(u_p) - \frac{673353}{280}\ln(3) + \frac{9735101}{262144}\pi^4 - \frac{9765625}{4536}\ln(5)\right) u_p^7\\
&+\frac{2687231}{4410}\pi u_p^{15/2} + O(u_p^8).
\end{aligned}
\tag{40}
$$

The Detweiler–Barack–Sago redshift function has been computed very recently also in a rotating (Kerr) spacetime for both circular and eccentric orbits in Refs. [31–34].

The interest for this kind of self-force computations is related to the possibility to convert such information in the PN expansion of several EOB potentials, e.g., a, q, ρ [27–34], etc., which proved to be very useful in creating fast, accurate analytic templates largely used in the analysis of the gravitational wave signals recently discovered by LIGO.

7 Concluding Remarks

The first detection of gravitational-wave signals emitted by a coalescing binary system by LIGO [35] has provided a strong incentive for further improving our analytical knowledge of the relativistic gravitational interaction of a two-body system. During the last years several analytical methods have been actively pursued, mainly post-Newtonian theory and black hole perturbation theory. The former expands the equations of motion in the binary separation, thus providing extremely accurate models at large separations where the gravitational field is weak enough. The latter expands instead in the mass-ratio of the two bodies, so that it works well in the case of systems with a very small mass ratio, e.g., the extreme mass ratio inspirals formed by a stellar mass compact object spiralling towards a black hole. In the strong field regime and for objects with comparable masses both such approaches fail and numerical relativity techniques are necessary. The Effective-One-Body model then analytically interpolates between various regimes by taking results from PN theory, black hole perturbation theory and numerical relativity, and currently is playing an important role in modeling the gravitational wave emission from the merger of binary black holes in all phases of its evolution.

We have presented here some recent analytical results of first-order gravitational self-force computations around black holes. In particular, we have reviewed the procedure to calculate the redshift invariant for a particle moving along a slightly eccentric equatorial orbit around a non-rotating Schwarzschild black hole. Computing gauge-invariant quantities allows to extract all physically meaningful information and compare results from different approaches. Furthermore, it can be used to calibrate the EOB model, which is constantly improving from self-force high-PN-order calculations.

Acknowledgements We thank Prof. T. Damour for useful discussions. D.B. thanks ICRANet for partial support and the organizers of the meeting for a high scientific level conference framed in a very unique place.

References

1. Bini, D., Damour, T.: Analytical determination of the two-body gravitational interaction potential at the fourth post-Newtonian approximation. Phys. Rev. D **87**, 121501 (2013). arXiv:1305.4884 [gr-qc]
2. Buonanno, A., Damour, T.: Effective one-body approach to general relativistic two-body dynamics. Phys. Rev. D **59**, 084006 (1999). arXiv:gr-qc/9811091
3. Buonanno, A., Damour, T.: Transition from inspiral to plunge in binary black hole coalescences. Phys. Rev. D **62**, 064015 (2000). arXiv:gr-qc/0001013
4. Damour, T.: Coalescence of two spinning black holes: an effective one-body approach. Phys. Rev. D **64**, 124013 (2001). arXiv:gr-qc/0103018
5. Barack, L., Damour, T., Sago, N.: Precession effect of the gravitational self-force in a Schwarzschild spacetime and the effective one-body formalism. Phys. Rev. D **82**, 084036 (2010). arXiv:1008.0935 [gr-qc]
6. Barack, L., Sago, N.: Gravitational self-force correction to the innermost stable circular orbit of a Schwarzschild black hole. Phys. Rev. Lett. **102**, 191101 (2009). arXiv:0902.0573 [gr-qc]
7. Isoyama, S., Barack, L., Dolan, S.R., Le Tiec, A., Nakano, H., Shah, A.G., Tanaka, T., Warburton, N.: Gravitational self-force correction to the innermost stable circular equatorial orbit of a Kerr black hole. Phys. Rev. Lett. **113**, 161101 (2014). arXiv:1404.6133 [gr-qc]
8. Dolan, S.R., Warburton, N., Harte, A.I., Le Tiec, A., Wardell, B., Barack, L.: Gravitational self-torque and spin precession in compact binaries. Phys. Rev. D **89**, 064011 (2014). arXiv:1312.0775 [gr-qc]
9. Bini, D., Damour, T.: Two-body gravitational spin-orbit interaction at linear order in the mass ratio. Phys. Rev. D **90**, 024039 (2014). arXiv:1404.2747 [gr-qc]
10. Bini, D., Damour, T.: Analytic determination of high-order post-Newtonian self-force contributions to gravitational spin precession. Phys. Rev. D **91**, 064064 (2015). arXiv:1503.01272 [gr-qc]
11. Dolan, S.R., Nolan, P., Ottewill, A.C., Warburton, N., Wardell, B.: Tidal invariants for compact binaries on quasicircular orbits. Phys. Rev. D **91**, 023009 (2015). arXiv:1406.4890 [gr-qc]
12. Bini, D., Damour, T.: Gravitational self-force corrections to two-body tidal interactions and the effective one-body formalism. Phys. Rev. D **90**, 124037 (2014). arXiv:1409.6933 [gr-qc]
13. Nolan, P., Kavanagh, C., Dolan, S.R., Ottewill, A.C., Warburton, N., Wardell, B.: Octupolar invariants for compact binaries on quasicircular orbits. Phys. Rev. D **92**, 123008 (2015). arXiv:1505.04447 [gr-qc]
14. Akcay, S., Dempsey, D., Dolan, S.R.: Spin-orbit precession for eccentric black hole binaries at first order in the mass ratio. Class. Quant. Grav. **34**, 084001 (2017). arXiv:1608.04811 [gr-qc]
15. Kavanagh, C., Bini, D., Damour, T., Hopper, S., Ottewill, A.C., Wardell, B.: Spin-orbit precession along eccentric orbits for extreme mass ratio black hole binaries and its effective-one-body transcription. Phys. Rev. D **96**, 064012 (2017). arXiv:1706.00459 [gr-qc]

16. Bini, D., Damour, T., Geralico, A.: Spin-orbit precession along eccentric orbits: improving the knowledge of self-force corrections and of their effective-one-body counterparts. Phys. Rev. D **97**, 104046 (2018). arXiv:1801.03704 [gr-qc]
17. Chandrasekhar, S.: On the equations governing the perturbations of the Schwarzschild black hole. Proc. R. Soc. Lond. A **343**, 289 (1975)
18. Mano, S., Suzuki, H., Takasugi, E.: Analytic solutions of the Regge-Wheeler equation and the post-Minkowskian expansion. Prog. Theor. Phys. **96**, 549 (1996). arXiv:gr-qc/9605057
19. Mano, S., Suzuki, H., Takasugi, E.: Analytic solutions of the Teukolsky equation and their low frequency expansions. Prog. Theor. Phys. **95**, 1079 (1996). arXiv:gr-qc/9603020
20. Nakano, H., Sago, N., Sasaki, M.: Gauge problem in the gravitational self-force. 2. First post-Newtonian force under Regge-Wheeler gauge. Phys. Rev. D **68**, 124003 (2003). arXiv:gr-qc/0308027
21. Detweiler, S.L.: A consequence of the gravitational self-force for circular orbits of the Schwarzschild geometry. Phys. Rev. D **77**, 124026 (2008). arXiv:0804.3529 [gr-qc]
22. Barack, L., Sago, N.: Beyond the geodesic approximation: conservative effects of the gravitational self-force in eccentric orbits around a Schwarzschild black hole. Phys. Rev. D **83**, 084023 (2011). arXiv:1101.3331 [gr-qc]
23. Bini, D., Damour, T.: High-order post-Newtonian contributions to the two-body gravitational interaction potential from analytical gravitational self-force calculations. Phys. Rev. D **89**, 064063 (2014). arXiv:1312.2503 [gr-qc]
24. Bini, D., Damour, T.: Analytic determination of the eight-and-a-half post-Newtonian self-force contributions to the two-body gravitational interaction potential. Phys. Rev. D **89**, 104047 (2014). arXiv:1403.2366 [gr-qc]
25. Bini, D., Damour, T.: Detweiler's gauge-invariant redshift variable: analytic determination of the nine and nine-and-a-half post-Newtonian self-force contributions. Phys. Rev. D **91**, 064050 (2015). arXiv:1502.02450 [gr-qc]
26. Kavanagh, C., Ottewill, A.C., Wardell, B.: Analytical high-order post-Newtonian expansions for extreme mass ratio binaries. Phys. Rev. D **92**, 084025 (2015). arXiv:1503.02334 [gr-qc]
27. Akcay, S., Le Tiec, A., Barack, L., Sago, N., Warburton, N.: Comparison between self-force and post-Newtonian dynamics: beyond circular orbits. Phys. Rev. D **91**, 124014 (2015). arXiv:1503.01374 [gr-qc]
28. Bini, D., Damour, T., Geralico, A.: Confirming and improving post-Newtonian and effective-one-body results from self-force computations along eccentric orbits around a Schwarzschild black hole. Phys. Rev. D **93**, 064023 (2016). arXiv:1511.04533 [gr-qc]
29. Hopper, S., Kavanagh, C., Ottewill, A.C.: Analytic self-force calculations in the post-Newtonian regime: eccentric orbits on a Schwarzschild background. Phys. Rev. D **93**, 044010 (2016). arXiv:1512.01556 [gr-qc]
30. Bini, D., Damour, T., Geralico, A.: New gravitational self-force analytical results for eccentric orbits around a Schwarzschild black hole. Phys. Rev. D **93**, 104017 (2016). arXiv:1601.02988 [gr-qc]
31. Le Tiec, A.: First law of mechanics for compact binaries on eccentric orbits. Phys. Rev. D **92**, 084021 (2015). arXiv:1506.05648 [gr-qc]
32. Bini, D., Damour, T., Geralico, A.: Spin-dependent two-body interactions from gravitational self-force computations. Phys. Rev. D **92**, 124058 (2015) (Erratum: Phys. Rev. D **93**, 109902 (2016)) arXiv:1510.06230 [gr-qc]
33. Kavanagh, C., Ottewill, A.C., Wardell, B.: Analytical high-order post-Newtonian expansions for spinning extreme mass ratio binaries. Phys. Rev. D **93**, 124038 (2016). arXiv:1601.03394 [gr-qc]
34. Bini, D., Damour, T., Geralico, A.: High post-Newtonian order gravitational self-force analytical results for eccentric equatorial orbits around a Kerr black hole. Phys. Rev. D **93**, 124058 (2016). arXiv:1602.08282 [gr-qc]
35. Abbott, B.P., et al.: LIGO scientific and virgo collaborations: GW151226: observation of gravitational waves from a 22-solar-mass binary black hole coalescence. Phys. Rev. Lett. **116**, 241103 (2016). arXiv:1606.04855 [gr-qc]