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Theory of Stochastic Schrödinger Equation in Complex Vector Space

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Abstract A generalized Schrödinger equation containing correction terms to classical kinetic energy, has been derived in the complex vector space by considering an extended particle structure in stochastic electrodynamics with spin. The correction terms are obtained by considering the internal complex structure of the particle which is a consequence of stochastic average of particle oscillations in the zeropoint field. Hence, the generalised Schrödinger equation may be called stochastic Schrödinger equation. It is found that the second order correction terms are similar to corresponding relativistic corrections. When higher order correction terms are neglected, the stochastic Schrödinger equation reduces to normal Schrödinger equation. It is found that the Schrödinger equation contains an internal structure in disguise and that can be revealed in the form of internal kinetic energy. The internal kinetic energy is found to be equal to the quantum potential obtained in the Madelung fluid theory or Bohm statistical theory. In the rest frame of the particle, the stochastic Schrödinger equation reduces to a Dirac type equation and its Lorentz boost gives the Dirac equation. Finally, the relativistic Klein–Gordon equation is derived by squaring the stochastic Schrödinger equation. The theory elucidates a logical understanding of classical approach to quantum mechanical foundations.

Keywords Foundations of quantum mechanics · Stochastic electrodynamics · Zeropoint field · Complex vector algebra

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1 Introduction

In classical electrodynamics, the incoming radiation field is normally chosen as zero and it is considered that all the radiation comes from somewhere at a finite time and the universe in the remote past contains only radiation. However, the presence of a ubiquitous zeropoint field changes the situation and the classical electrodynamics combined with the zeropoint radiation field is known by the name stochastic electrodynamics [1]. The main aim in developing stochastic electrodynamics was to find a reasonable classical approach to the foundations of quantum mechanics and to certain extent quantum electrodynamics. A complete account of stochastic studies has been considered in several reviews [2–4]. The prime effort of all stochastic electrodynamics theories is to find a classical approach to quantum mechanics and hence deriving the fundamental Schrödinger equation in the treatment of stochastic electrodynamics solves the desired goal to a certain extent.

The wave equation of a quantum system was derived by Schrödinger [5] from the classical Hamilton–Jacobi equation. The stochastic interpretation of quantum mechanics in the Markov process was initially done by Fenyes [6] and it was further developed by Nelson [7,8]. Nelson's approach was based on the assumption that every particle of mass m moves in a random environment due to Brownian motion. Similar studies were also reported in the independent research work of Della Riccia and Wiener [9], and Favella [10]. However, Nelson's theory was criticised on the grounds that the formulation merely recapitulated the results of quantum theory. However, a valid quantum theory from the stochastic process was derived by de la Peña and Cetto [11] by introducing zeropoint field into stochastic theory. The theory of stochastic process as a generalization of Newtonian mechanics was formulated by de la Peña and the Schrödinger equation had been derived for specific values of certain parameters [12]. In this formalism, the usual quantum mechanical operators and commutator relations emerge in a natural way. The stochastic theory was further extended to the particles with spin by considering the particle as a spinning rigid body [13–15] and in the presence of external electromagnetic field, Pauli–Schrödinger equation, the Feynman–Gellman type and Dirac equations were derived. In the non-Markovian stochastic interpretation, Cavalleri et al. [16–18] extended the density gradient expansion and derived the Schrödinger equation along with higher order correction terms resembling quantum electrodynamics radiative correction terms due to vacuum fluctuations. In this interpretation, the quantum formalism turns out to be a classical motion with superimposed zitterbewegung motion.

In 1932, Wigner [19] showed that the phase space evolution of an ensemble of particles can be described by probability distribution function and it satisfied the classical Liouville equation. Thus the Wigner distribution function provided the formulation of quantum mechanics and revealed the similarity between Schrödinger equation and classical Liouville equation. In the presence of zeropoint fields, Dechoum et al. [20] derived the Liouvillian form of time evolution of particles and showed that the classical probability amplitude is related to the Wigner distribution function. Using the Fourier transform of Wigner distribution function in the classical Liouville equation Faria et al. [21] arrived at the Schrödinger equation. Further, based on classical trajectories and continuous spin orientation the classical aspects of Pauli–Schrödinger equation were discussed [22]. Considering the Wigner–Moyal infinitesimal transformation, and using the conserved probability density function of a statistical equilibrium system containing fluctuations in momentum and position, Olavo [23,24] derived the Schrödinger equation and gave a new approach to quantum formalism. Hall and Reginatto [25] showed that the strength of momentum fluctuations is inversely correlated with uncertainty in position, from which they formulated the exact uncertainly principle and derived Schrödinger equation from classical equations of motion. Recent studies on Schrödinger equation by Schleich et al. [26,27] reveal the fact that the linearity of quantum mechanics is intimately connected to the strong coupling between amplitude and phase of quantum wave.

Merging the non-equilibrium thermodynamics with classical wave mechanics, Grössing [28–30] derived the Schrödinger equation. A unified treatment of arbitrary large fluctuations in small systems has been achieved by the formulation of so called fluctuation theorems. A particle is treated as a fluctuating quantity in the vacuum dominated by the thermal fluctuations. These fluctuations of a quantum system lead to kinetic energy equal to the quantum potential of the system. The gradient of such quantum potential has been described by thermalized fluctuating force field, where the origin of such fluctuations are identical to the zeropoint fluctuations. The theoretical considerations show that the employment of non-equilibrium thermodynamics gives a deeper understanding of quantum mechanics.

In all these aforesaid studies, the fundamental equation of quantum mechanics has been derived mainly considering fluctuations of the system under consideration. The quantum potential in the Schrödinger equation reveals the fact that particles must have some internal structure. The above theoretical considerations suggest that the quantum behaviour of particles or quantum systems has an underlying classical fluctuation mechanism which may be formulated as a physical theory.

In the zitterbewegung model of Dirac electron, Schrödinger showed that the motion of electron contains internal oscillations with amplitude of the order of Compton wavelength [31,32]. The theoretical formulation leads to particle internal structure described by separated center of mass and center of charge. The center of charge rotates around the center of mass point in a helical fashion. The frequency of rotation is equal to $\omega_0 = mc^2/\hbar$ and the radius of rotation is equal to the Compton wavelength of the particle of mass m, where c is the velocity of light and $\hbar$ is the reduced Planck's constant. A classical relativistic theory of spinning particles interacting with electromagnetic fields was developed by Bhabha and Corben [33]. In this theory, a free particle describes a helix and the rotational motion gives the particle magnetic moment. This internal helical motion is analogous to the zitterbewegung of Dirac electron [34,35]. In the classical spinning electron model of Mathisson [36], the electron motion is separated into free motion of center of mass and the internal rotational motion due to zitterbewegung. The internal motion has the characteristic radius of the order of Compton wave length. In the approach of geometric algebra using multivector valued Lagrangian, Barut and Zanghi [37] studied the classical analogue of zitterbewegung. The invariant proper time is connected with the center of mass and not with the center of charge and the electron spin appears as the orbital angular momentum associated with the internal motion and the rest mass energy appears as the internal energy.

In the Madelung fluid theories [38–40], the spinning particle appears as an extended object while the quantum potential is tentatively related to internal motion. The quantum potential, the non-classical energy term in the Hamiltonian, is simply related to the non-classical energy term to the zitterbewegung and spin. In the absence of spin the quantum potential term vanishes and the motion becomes classical or Newtonian. Therefore, the particle spin ensures the presence of quantum potential in the Schrödinger equation. The proper time is connected with the center of mass and not with the center of charge because the orbital motion of charge is expected to be light-like. All this carries further evidence that quantum mechanics of micro-systems may be a direct consequence of spin [41,42]. These considerations elucidate the fact that an elementary particle (like electron or quark) contains a sub-structure described by point charge rotating in circular motion with spin angular momentum and the frequency of rotation is equal to the zitterbewegung frequency. Such internal circular motion of center of charge is mainly responsible for the deviations in the average path of the particle, spin angular momentum of the particle and observed magnetic moment of the charged particle. The zitterbewegung motion or oscillation of the particle is considered to be a manifestation of the presence of fluctuating zeropoint fields.

Recently, the role of spin and the internal structure of the particle in complex vector formalism were studied by the author [43]. The oscillations of the particle in the presence of zeropoint field are considered as complex rotations characteristic of internal spin angular momentum in complex vector space. Considering the complex null vectors in the particle rest frame, the energy of particle oscillator is derived and it has been shown that the existence of spin converts classical oscillator into quantum oscillator. The mass of the particle may be interpreted as local complex rotation in the rest frame. It has been found that the complex vector formalism has an added advantage of separating scalar, vector and bivector parts of a multivector expansion in such a way it gives a better classical approach to the quantum mechanical phenomena [44].

The present article mainly deals with the derivation of a generalized stochastic Schrödinger equation from the consideration of extended particle structure in the complex vector space. The total derivative is expressed as a sum of convective derivative and stochastic derivatives in complex vector space. A short account of extended structure of a charged particle in the complex vector space and its connection with spin are discussed in Sect. 2. The wave function of a particle and its meaning in the complex vector space are explored in Sect. 3. The convective and stochastic derivatives are considered in Sect. 4. The correspondence between quantum potential and internal structure of the particle is revealed in Sect. 5. The derivation of stochastic Schrödinger equation and the consequences of higher order correction terms are presented in Sect. 6. Derivations of Dirac and Klein–Gordon equations from stochastic Schrödinger equation are given in Sects. 7 and 8 respectively. Finally, conclusions are given in Sect. 9. The mathematical language required for understanding the article, the algebra of complex vectors is given in the Appendix . Throughout this article, a charged particle means a particle like electron.

2 Particle Structure and Bivector Spin in Complex Vector Space

A charged particle immersed in the fluctuating zeropoint field oscillates from its mean position and such particle oscillator absorbs energy from the zeropoint field at a single frequency which is the characteristic frequency of oscillation. Let us consider a particle of mass m and charge e in a fluctuating zeropoint field and also consider it as an oscillator with angular frequency ω_0. The particle deviates from its mean position due to oscillations induced by the zeropoint field. A stochastic average of all such deviations leads to a mean deviation of the particle from its path. In other words, at the particle level, the mean position of oscillations may be considered as the center of mass of the particle. The average separation of center of mass and center of charge appears as mean deviation in the path of the particle. Such deviations are described by local complex rotations in complex vector space. The average deviation from the path of the particle corresponds to the radius of complex internal rotation. Considering the average complex rotations in a plane normal to the mean path of the particle and denoting the center of mass position by a vector $\mathbf{x}$ and the average radius of rotation by a vector $\boldsymbol{\xi}$, the extended particle system can be expressed by a complex vector [43,44].

$$X = \mathbf{x} + \mathbf{i}\boldsymbol{\xi}, \tag{1}$$

where $\mathbf{i}$ is a pseudoscalar and it commutes with all vectors in three dimensional Euclidean space. The bivector $\mathbf{i}\boldsymbol{\xi}$ represents an oriented plane. The average magnitude of radius of rotation is equal to half the Compton wave length and $\xi = \hbar/2mc$. Differentiating X with respect to time gives velocity complex vector.

$$U = \mathbf{v} + \mathbf{i}\mathbf{u}, \tag{2}$$

where $\mathbf{v} = d\mathbf{x}/dt$ is the velocity of center of mass and $\mathbf{u} = d\boldsymbol{\xi}/dt$ is the instantaneous velocity of internal rotation of the center of charge and the corresponding internal momentum $\boldsymbol{\pi} = m\mathbf{u}$. When the particle system is observed from an arbitrary frame, the trajectory of center of charge traces a helical path resembling zitterbewegung motion. The angular momentum of internal complex rotation is represented by a bivector S which corresponds to particle spin [45,46].

$$S = \boldsymbol{\xi} \wedge \boldsymbol{\pi} \tag{3}$$

If we choose a unit vector $\boldsymbol{\sigma}_s$ normal to the spin plane, then the bivector particle spin angular momentum can be expressed as $S = \frac{\hbar}{2}\mathbf{i}\boldsymbol{\sigma}_s$, where $\mathbf{i}\boldsymbol{\sigma}_s$ is a unit bivector in the spin plane. Since, the geometric structure of a bivector contains two possible opposite orientations, the orientation of S can be either positive or negative in the bivector spin plane. The above description of spin univocally suggests that the spin is not a fundamental notion of quantum mechanics but at a deeper level it corresponds to the angular momentum of zeropoint oscillations of the particle.

3 The Wave Function

In the complex vector formalism, the mass of the particle has been interpreted as local complex rotation in the rest frame of the particle [43]. The spatial rotation in the rest frame of an extended particle can be represented by a general rotor

$$R_0 = \exp\left(\frac{mc^2 t}{2S}\right). \tag{4}$$

The rotor R_0 is a consequence of complex rotation in a local space and actually represents an average rotation on a complex plane obtained from many number of entities of stochastic rotation. Because of the bivector nature of spin angular momentum, the rotor R_0 satisfies the relation $R_0\bar{R}_0 = 1$, where an over bar represents reversion operation. If the particle center of mass moves with velocity $\mathbf{v}$, say along the direction of $\mathbf{x}$, then the rotor R_0 will be subjected to a Lorenz boost. The Lorentz boost is in general expressed in terms of rapidity factor η and a unit vector $\hat{\mathbf{v}}$ along the direction of velocity of the particle.

$$L = \exp\left(\frac{-\hat{\mathbf{v}}\eta}{2}\right) \tag{5}$$

The Lorentz boost on the rotor R_0 is then expressed as

$$R = LR_0L^{-1} = \exp\left(\frac{Et - \mathbf{p}\cdot\mathbf{x}}{2S}\right), \tag{6}$$

where $E = \gamma mc^2$, $\mathbf{p} = \gamma m\mathbf{v}$ and γ is the Lorentz factor. This rotor represents the state of the particle. In quantum mechanics, the state of a particle is represented by the wave function. According to the conventional explanation, the wave function represents the probability of finding the particle in an ensemble of particles. Because the rotor satisfies the relation $R\bar{R} = 1$, any addition of a phase factor does not change the action of R. In other words, the rotor is said to be gauge invariant. In view of these properties of the rotor, the wave function of the particle may be expressed as

$$\psi(\mathbf{x}, t) = \rho^{1/2}R. \tag{7}$$

The product $\psi\bar{\psi} = \rho$. The above analysis reveals that the meaning of wave function is intricately connected to the average internal complex rotations. Thus the wave function with its probability interpretation gives the most complete possible specification of an individual system in quantum mechanics. The epistemological feature of this interpretation is possible only from the consideration of zeropoint field. To endorse particle and antiparticle correspondence, a phase factor $e^{\mathbf{i}\epsilon/2}$ is introduced in (7) and the final wave function is represented by [47]

$$\psi(\mathbf{x}, t) = e^{\mathbf{i}\epsilon/2}\rho^{1/2}R. \tag{8}$$

The particle and antiparticle states correspond to $\epsilon = 0$ and $\epsilon = \pi$ respectively.

4 Convective and Stochastic Derivatives in Complex Vector Space

Consider the position of an extended particle system that is denoted by the complex vector X. The convective or total derivative is normally expressed as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{\mathbf{x}}\boldsymbol{\nabla}, \tag{9}$$

where $\boldsymbol{\nabla} = \frac{\partial}{\partial \mathbf{x}}$. To induce stochastic nature of the particle, the vector $\mathbf{x}$ may be replaced by the complex vector X in the above equation.

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{X}\frac{\partial}{\partial X} \tag{10}$$

The partial derivative $\frac{\partial}{\partial X}$ can be expanded in the following manner.

$$\frac{\partial}{\partial(\mathbf{x}+\mathbf{i}\boldsymbol{\xi})} = \frac{\Delta}{\Delta\mathbf{x}+\mathbf{i}\boldsymbol{\xi}\Delta} = \frac{\Delta}{\Delta\mathbf{x}}\left(1+\mathbf{i}\boldsymbol{\xi}\frac{\Delta}{\Delta\mathbf{x}}\right)^{-1} = \frac{\partial}{\partial\mathbf{x}}\left(1+\mathbf{i}\boldsymbol{\xi}\frac{\partial}{\partial\mathbf{x}}\right)^{-1} \tag{11}$$

In the above equation, the magnitude of $\boldsymbol{\xi}$ is very small when compared to $\mathbf{x}$ and the vector $\boldsymbol{\xi}$ is not a function of $\mathbf{x}$. Now using a simple binomial expansion, the partial derivative $\frac{\partial}{\partial X}$ can be expressed as

$$\frac{\partial}{\partial(\mathbf{x}+\mathbf{i}\boldsymbol{\xi})} = \boldsymbol{\nabla} - \mathbf{i}\boldsymbol{\xi}\nabla^2 - \xi^2\nabla^2\boldsymbol{\nabla} + \mathbf{i}\boldsymbol{\xi}\xi^2\nabla^4 - (\xi^2\nabla^2)^2\boldsymbol{\nabla} + \cdots . \tag{12}$$

Substituting the above expansion in (10) gives the total derivative in the following form.

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{v}+\mathbf{i}\mathbf{u})[\boldsymbol{\nabla} - \mathbf{i}\boldsymbol{\xi}\nabla^2 - \xi^2\nabla^2\boldsymbol{\nabla} + \mathbf{i}\boldsymbol{\xi}\xi^2\nabla^4 - (\xi^2\nabla^2)^2\boldsymbol{\nabla} + \cdots] \tag{13}$$

Using the relation $S = m(\boldsymbol{\xi} \wedge \mathbf{u})$, the total derivative can be written as a sum of five terms.

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\Sigma + \mathbf{i}\mathbf{u} \wedge \boldsymbol{\nabla}\Sigma - \mathbf{v} \wedge \mathbf{i}\boldsymbol{\xi}\nabla^2\Sigma - \frac{S}{m}\nabla^2\Sigma, \tag{14}$$

where $\Sigma = \sum_{k=0}^{\infty}(-\xi^2\nabla^2)^k$. In the above expression, it is clear that the term $\mathbf{v}\cdot\boldsymbol{\nabla}\Sigma$ is a scalar quantity, the terms $\mathbf{i}\mathbf{u} \wedge \boldsymbol{\nabla}\Sigma$ and $\mathbf{v} \wedge \mathbf{i}\boldsymbol{\xi}\nabla^2\Sigma$ are vectors and the last term is a bivector. Thus one can call the expansion of total derivative as a multivector expansion.

In the point particle limit, the total derivative reduces to the convective derivative. In the stochastic derivation of Schrödinger equation, the convective derivative is called as systematic derivative and the velocity connected to the convective derivative is known as systematic velocity. When $\mathbf{v} = 0$, the total derivative is written as follows.

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{i}\mathbf{u} \wedge \boldsymbol{\nabla}\Sigma - \frac{S}{m}\nabla^2\Sigma \tag{15}$$

Neglecting the higher order correction terms in the above equation, we have

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{iu} \wedge \boldsymbol{\nabla} - \frac{S}{m}\nabla^2. \tag{16}$$

This derivative is quite similar to the stochastic derivative when the diffusion tensor D is identified with $|S|/m$. The corresponding velocity $\mathbf{u}$ of this derivative is called stochastic velocity in the stochastic approach developed by de la Peña et al. [14]. However, the approach adopted here is quite different from the analysis of stochastic approach using Brownian motion. In this approach, the diffusion tensor is absent and it may be noted that the velocity $\mathbf{u}$ is not the diffusive or the osmotic velocity but it is the internal rotational velocity whose magnitude is equal to the velocity of light.

5 Internal Structure Connection to Quantum Potential and Operators

In the point particle limit, the total derivative reduces to the normal convective derivative and multiplying (9) from right by the probability density ρ gives the continuity equation.

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}(\mathbf{v}\rho) = 0 \tag{17}$$

Multiplying (16) from right with probability density ρ gives

$$\frac{\partial \rho}{\partial t} + \mathbf{i}\boldsymbol{\nabla}\left(\mathbf{u}\rho - \frac{|S|}{m}\nabla\rho\right) = 0. \tag{18}$$

The first term in the above equation is equal to zero when $\mathbf{v} = 0$ substituted in the continuity equation (17), and then the term in brackets is like current.

$$C = \mathbf{u}\rho - \frac{|S|}{m}\nabla\rho \tag{19}$$

The average value of this current must be zero in the rest frame of the particle. Squaring the above expression and equating the scalar quantities to zero gives

$$u^2 = -\left(\frac{|S|}{m}\right)^2\left(\frac{\nabla\rho}{\rho}\right)^2. \tag{20}$$

Now, the square of internal momentum can be written as follows.

$$\pi^2 = -|S|^2(\nabla \ln \rho)^2 \tag{21}$$

Since the magnitude of spin $|S| = \xi\pi$, from the above equation, the square of the radius of rotation can be expressed in terms of probability density.

$$\xi^2 = -(\nabla \ln \rho)^{-2} \tag{22}$$

In most of the fluctuation theories [25,26], the average fluctuations in momentum and position are similar to (21) and (22) respectively and thus, these equations are the most fundamental equations connected with the uncertainty in quantum mechanics. Now, the internal kinetic energy of the extended particle can be obtained from (21).

$$\frac{\pi^2}{2m} = -\frac{|S|^2}{2m}(\nabla \ln \rho)^2 \tag{23}$$

We have seen in the complex vector approach that the internal velocity $\mathbf{u}$ is not a function of the mean position vector $\mathbf{x}$ and therefore, the scalar product $\boldsymbol{\nabla}.\mathbf{u} = \nabla u = 0$. The square root of (20) gives the magnitude of internal velocity and differentiating with respect to x gives

$$\nabla u = -i\left(\frac{|S|}{m}\right)\nabla\left(\frac{\nabla\rho}{\rho}\right) = -i\left(\frac{|S|}{m}\right)\left[\frac{\nabla^2\rho}{\rho} - \left(\frac{\nabla\rho}{\rho}\right)^2\right] = 0 \tag{24}$$

and

$$\frac{\nabla^2\rho}{\rho} = \left(\frac{\nabla\rho}{\rho}\right)^2. \tag{25}$$

In Bohm's hidden variable approach, the quantum potential is given by [48,49]

$$Q = \frac{\hbar^2}{2m}\left[\frac{1}{2}\frac{\nabla^2\rho}{\rho} - \frac{1}{4}\left(\frac{\nabla\rho}{\rho}\right)^2\right]. \tag{26}$$

With the magnitude of spin $|S| = \hbar/2$ and using (25), the internal kinetic energy in (23) is equal to the quantum potential obtained in Bohm's hidden variable approach to quantum mechanics or in the theories of Madelung fluid [39–41]. It is quite interesting to observe that, Bohm considered a fluctuating ψ-field that can act on the particle to produce fluctuations. The nature of ψ-field had been considered analogous (not identical) to the electromagnetic field. In the present context, such fluctuating field is analogous to the fluctuating electromagnetic zeropoint field. From (21) and (22), the internal momentum and radius of rotation can be expressed as follows.

$$\boldsymbol{\pi} = \mathbf{i}\boldsymbol{\sigma}_\xi|S|\boldsymbol{\nabla}\ln\rho(\mathbf{x}), \tag{27}$$

$$\boldsymbol{\xi} = \mathbf{i}\boldsymbol{\sigma}_u(\boldsymbol{\nabla}\ln\rho(\mathbf{x}))^{-1}, \tag{28}$$

where $\boldsymbol{\sigma}_\xi$ and $\boldsymbol{\sigma}_u$ are unit vectors along the directions of $\boldsymbol{\xi}$ and $\boldsymbol{\pi}$ respectively. Writing the probability density $\rho(\mathbf{x}) = \psi\bar{\psi}$, the internal momentum can be expressed in the following form.

$$\boldsymbol{\pi}\psi\bar{\psi} = \mathbf{i}\boldsymbol{\sigma}_\xi|S|\boldsymbol{\nabla}\psi\bar{\psi} \tag{29}$$

This equation shows that the internal momentum corresponds to the momentum operator.

$$\boldsymbol{\pi} \to -i\hbar\boldsymbol{\nabla} = \hat{p} \tag{30}$$

Similarly, from the commutation relation one can expect a correspondence between internal radius of rotation and position operator.

$$\boldsymbol{\xi} \to \hat{x} \tag{31}$$

The correspondence between internal momentum and radius of rotation to the momentum and position operators has been studied recently [50]. The above correspondence shows that the necessity of introducing operators into quantum mechanics is to account for the influence of zeropoint field on the charged particle. Thus in quantum mechanics, even though the particles are treated as point particles, the operator approach takes care of the internal structure of the particle in disguise. In other words, considering the internal particle complex structure leads to the quantum results of the system. This will be clarified in the next section with the derivation of generalized Schrödinger equation.

6 Stochastic Schrödinger Equation

In the point particle limit, the total derivative is equal to the normal convective derivative and multiplying (9) from right by $\mathbf{p} = m\mathbf{v}$, we have

$$\frac{d\mathbf{p}}{dt} = \mathbf{v}\nabla m\mathbf{v} = \nabla\left(\frac{1}{2}\mathbf{p}\cdot\mathbf{v}\right) = \nabla E_k, \tag{32}$$

where E_k is the kinetic energy. The force in the above equation is like a convective force and it can be generically expressed as a gradient of an external potential V'. Then from the above equation we have, $E_k = V'$. In the non-relativistic limit, let the wave function of the particle is defined by $\psi = \rho^{1/2}R$, where the momentum $\mathbf{p} = m\mathbf{v}$ and energy $E = E_k$. Now, multiplying (14) from right by ψ gives

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \mathbf{v}\cdot\nabla\Sigma\psi + \mathbf{iu}\wedge\nabla\Sigma\psi - \mathbf{v}\wedge\mathbf{i}\boldsymbol{\xi}\nabla^2\Sigma\psi - \frac{S}{m}\nabla^2\Sigma\psi. \tag{33}$$

In the rest frame of the particle, the wave function $\psi = \rho^{1/2}R_0$ and the first term in (33) becomes

$$\frac{d\psi}{dt} = \frac{mc^2}{2S}\psi. \tag{34}$$

However, when the particle is in motion the term $\frac{d\psi}{dt}$ represents the total energy which is a sum of kintetic energy and rest mass energy.

$$\frac{d\psi}{dt} = \frac{E_k + mc^2}{2S}\psi \tag{35}$$

Now, the term $\mathbf{v}\cdot\nabla\Sigma\psi$ can be written as follows.

$$\begin{aligned}\mathbf{v}\cdot\nabla\Sigma\psi &= \mathbf{v}\cdot\nabla(1-\xi^2\nabla^2+\xi^4\nabla^4+\cdots)\psi \\ &= -\frac{\mathbf{p}\cdot\mathbf{v}}{2S}\left(1-\frac{\xi^2p^2}{(2S)^2}+\frac{\xi^4p^4}{(2S)^4}+\cdots\right)\psi,\end{aligned} \tag{36}$$

where the probability density ρ is taken as a constant and its functional dependence on position will be considered later in this section. Using the values $\xi = \hbar/2mc$, $(2S)^2 = -\hbar^2$ and $\mathbf{p}\cdot\mathbf{v} = 2V'$, we have

$$\mathbf{v}\cdot\nabla\Sigma\psi 2S = \left[-V' - V'\left(1+\frac{\beta^2}{2}+\frac{\beta^4}{8}+\cdots\right)\right]\psi = -V''\psi, \tag{37}$$

where $\beta = v/c$, the additional correction factors are merged into V' and the resulting potential is denoted by V''. The above calculation shows that the term $\mathbf{v}\cdot\nabla\Sigma\psi 2S$ corresponds to a potential. Now, from (35) and (37) we have

$$\frac{d\psi}{dt} - \mathbf{v}\cdot\nabla\Sigma\psi = \frac{mc^2+V}{2S}\psi. \tag{38}$$

In the above equation, we have chosen $V = V' + V''$. The third and fourth terms on right of (33) are vectors. Now, substituting (38) in (33) and equating bivector terms gives

$$\frac{mc^2+V}{2S}\psi = \frac{\partial\psi}{\partial t} - \frac{S}{m}\nabla^2\Sigma\psi. \tag{39}$$

Multiplying throughout by $2S$ and rearranging the above equation finally gives the generalized Schrödinger equation.

$$\frac{\partial}{\partial t}\psi 2S = \frac{(2S)^2}{2m}\nabla^2\Sigma\psi + mc^2\psi + V\psi \tag{40}$$

If the higher order terms and mass term are suppressed in (40), it reduces to the normal time-dependent Schrödinger equation in quantum mechanics.

$$\frac{\partial}{\partial t}\psi 2S = \frac{(2S)^2}{2m}\nabla^2\psi + V\psi \tag{41}$$

For a wide variety of applications, the solution of this equation with different potential functions contains only solving second order differential equation with certain boundary conditions. Notice that, the particle spin is present in the Schrödinger equation. In normal quantum mechanics, energy and momentum operators contain unit imaginary and it has no meaning. However, in (41) the bivector $2S$ replaces $i\hbar$ and the presence of spin gives a geometric meaning to unit imaginary that it represents a unit bivector. The connection between unit imaginary and spin was initially studied by Hestenes [51].

Now, consider $\rho = \rho(\mathbf{x})$ and expand the first term on right of (41) and equating the scalar parts gives

$$\frac{(2S)^2}{2m}\nabla^2\psi = \frac{p^2}{2m}\psi + \frac{\hbar^2}{8m}\left(\frac{\nabla\rho}{\rho}\right)^2\psi. \tag{42}$$

The first term on the right side of this equation is the kinetic energy and the second term is known as quantum potential and it is equal to the internal kinetic energy of the particle. Thus the Schrödinger equation contains internal structure of the particle in disguise and it can be revealed in the form of quantum potential which was obtained in the Madelung fluid theory or Bohm statistical theory. The quantum potential is usually merged with the external potential term and the Schrödinger equation is solved in the point particle limit that represents the center of mass motion.

6.1 Higher Order Correction Terms

The higher order terms in the expansion Σ represent correction terms to the classical kinetic energy and also to the quantum potential. The first correction term can be estimated as follows.

$$-\frac{(2S)^2}{2m}\xi^2\nabla^4\psi = -\frac{(2S)^2}{2m}\xi^2\left[\rho^{1/2}\nabla^4 R + \frac{1}{16}\left(\frac{\nabla\rho}{\rho}\right)^4\psi + \frac{5}{4}\left(\frac{\nabla\rho}{\rho}\right)^4\rho^{1/2}\nabla^2 R\right], \tag{43}$$

where the terms proportional to $\boldsymbol{\nabla}\rho\boldsymbol{\nabla}R = \boldsymbol{\nabla}\rho\mathbf{p}/2S$ and $\boldsymbol{\nabla}\rho\boldsymbol{\nabla}^3 R \propto \boldsymbol{\nabla}\rho\mathbf{p}/2S$ are bivectors and the above equation is obtained by equating the scalar terms. Using the values $\xi = \hbar/2mc$ and $(2S)^2 = -\hbar^2$ in (43) gives

$$-\frac{(2S)^2}{2m}\xi^2\nabla^4\psi = \frac{p^4}{8m^3c^2}\psi + \frac{\hbar^4}{128m^3c^2}\left(\frac{\nabla\rho}{\rho}\right)^4\psi - \frac{5\hbar^2}{32m^3c^2}\left(\frac{\nabla\rho}{\rho}\right)^4 p^2\psi. \tag{44}$$

In this equation, the second and third terms on right are the correction terms to the quantum potential and the first term on right turns out to be the relativistic energy correction normally obtained in the Foldy–Wouthuysen transformation of Dirac equation. Since, such correction terms are obtained by considering the internal complex structure of the particle which is a consequence of stochastic average of particle oscillations in the zeropoint field, the generalized Schrödinger equation may be called as stochastic Schrödinger equation.

Further correction terms of the order ξ^4 and above can be estimated in a similar manner as calculated in (44) and to look into it in more detail, (40) can be rewritten as

$$\frac{\partial}{\partial t}\psi 2S = \frac{(2S)^2}{2m}\nabla^2\psi\left[\rho^{-1}\bar{\psi}\Sigma\psi\right] + mc^2\psi + V\psi. \tag{45}$$

Now, considering only the kinetic energy correction terms and solving the term in square brackets gives a series

$$\left[\rho^{-1}\bar{\psi}\Sigma\psi\right] = 1 + a^2p^2 + a^4p^4 + a^6p^6 + \cdots, \tag{46}$$

where $a = \xi/\hbar$. Using the value of $\xi = \hbar/2mc$, this series can be expressed in terms of $\beta = v/c$.

$$\left[\rho^{-1}\bar{\psi}\Sigma\psi\right] = 1 + \left(\frac{\beta^2}{2}\right) + \left(\frac{\beta^2}{2}\right)^2 + \left(\frac{\beta^2}{2}\right)^3 + \cdots = \left[1 - \frac{\beta^2}{2}\right]^{-1} \tag{47}$$

Thus, one can conclude that all higher order correction terms to a certain extent correspond to relativistic effects of the charged particle in motion. Now, the stochastic Schrödinger equation is written as

$$\frac{\partial}{\partial t}\psi 2S = \frac{(2S)^2}{2m}\nabla^2\psi\left[1 - \frac{\beta^2}{2}\right]^{-1} + mc^2\psi + V\psi. \tag{48}$$

In the present calculation, as the particle position is represented by the complex vector X, the potential is not simply a function of center of position $\mathbf{x}$ but the complex vector X. Therefore, the potential $V(X)$ after expansion contains higher order correction terms.

$$V(X) = V(\mathbf{x}) + \mathbf{i}\boldsymbol{\xi}\boldsymbol{\nabla}V(\mathbf{x}) + \frac{1}{2}\xi^2\nabla^2 V(\mathbf{x}) + \cdots \tag{49}$$

The third term on right of the above expression is similar to the Darwin term in relativistic quantum theory.

6.2 Modified Position and Momentum Commutation Relation

In quantum mechanics, we consider the point particles without any size or substructure. When we consider the extended particles, modification to the commutation relation is expected. The procedure considered above allows us to reconsider the quantum commutation relation between position and momentum operators. Now, a modified momentum operator $\hat{p}$ can be written from (48).

$$\hat{p} = -2S\nabla\left[1 - \left(\frac{\beta}{2}\right)^2\right]^{-1/2}. \tag{50}$$

Using this equation, the modified commutation relation can be expressed as

$$[\hat{x}, \hat{p}] = 2S\left[1 - \left(\frac{\beta}{2}\right)^2\right]^{-1/2}. \tag{51}$$

Thus, the commutation relation also contains additional correction terms and to the first order of ξ^2, it is expressed as

$$[\hat{x}, \hat{p}] = 2S\left(1 + \kappa^2 p^2\right), \tag{52}$$

where $\kappa^2 = a^2/2$. A similar modified commutation relation was derived long back by Snyder [52] and showed that the spacetime is not continuous but discrete. Such relations also lead to minimal length uncertainty relations [53]. The modification in the commutation relation actually related to sub-quantum phenomena.

6.3 The Pauli Equation

It is well known that the presence of particle spin was first considered in the Pauli equation and its presence can be simply obtained by replacing momentum with the canonical momentum $\mathbf{p} - \frac{e}{c}\mathbf{A}$ in the minimal interaction approximation of electromagnetic potential $\mathbf{A}$. Now, replacing $(2S)^2\nabla^2$ in (43) by $(2S\boldsymbol{\nabla} - \frac{e}{c}\mathbf{A})^2$ gives a generalized Pauli equation.

$$\frac{\partial}{\partial t}\psi 2S = \frac{(2S)^2}{2m}\nabla^2\Sigma\psi + \frac{ge}{2mc}\mathbf{iB}.S\Sigma\psi + \frac{e^2A^2}{2mc^2}\Sigma\psi + V\psi \tag{53}$$

Neglecting higher order terms and mass term in (53) gives the Pauli equation with additional energy term containing magnetic field.

$$\frac{\partial}{\partial t}\psi 2S = \frac{(2S)^2}{2m}\nabla^2\psi + \frac{ge}{2mc}\mathbf{iB}.S\psi + V\psi \tag{54}$$

where the gyromagnetic ratio g = 2. With the discovery of Lamb shift, it has been expected that the value of gyromagnetic ratio g for electron is slightly more than 2 and such discrepancy from both Pauli and Dirac theories of electron was addressed in the theory of interaction of radiation with matter, the quantum electrodynamics.

7 The Dirac Equation

In the rest frame of the particle, the first term on right of (40) vanishes and in the absence of external potential, the stochastic Schrödinger equation reduces to

$$\frac{\partial}{\partial t}R_0 2S = mc^2 R_0. \tag{55}$$

This equation is Dirac like equation in the rest frame of the particle. In the above equation the spin orientation is taken as positive and the wave function represents a particle state. If the spin orientation is taken as negative, then the wave function represents an antiparticle state. When the particle moves with velocity $\mathbf{v}$, applying Lorentz boost to (55) yields the Dirac equation of the extended particle. Detailed derivation of obtaining the Dirac equation from (55) is given in [43]. Multiplying (55) from left by a unit vector γ_0 which is along the direction of future light cone and taking the Lorentz boost of the equation gives

$$L\gamma_0\partial_0 L^{-1}LR_0 2S - mcL\gamma_0 R_0 = 0. \tag{56}$$

The set $\{\gamma_\mu;\ \mu = 0, 1, 2, 3\}$ are unit vectors in spacetime algebra [54]. The unit vector γ_0 is invariant under spatial rotation, $\bar{R}_0\gamma_0 R_0 = \gamma_0$ and $L\gamma_0\partial_0 L^{-1} = \gamma^\mu\partial_\mu = \partial$. Denoting $\Lambda = LR_0$, Eq. (56) is now expressed as

$$\partial\Lambda 2S - mc\Lambda\gamma_0 = 0. \tag{57}$$

Since the spinor Λ satisfies the relation $\Lambda\bar{\Lambda} = 1$, a general spinor can be expressed in the form $\psi = \rho^{1/2}e^{\mathbf{i}\epsilon/2}\Lambda$. Now, (57) can be written in the following form.

$$\partial\psi 2S - mc\psi\gamma_0 = 0 \tag{58}$$

In the presence of external electromagnetic potential the above equation becomes

$$c\partial\psi 2S - eA\psi - mc^2\psi\gamma_0 = 0. \tag{59}$$

where e is the particle charge. This equation is the well-known Hestenes–Dirac equation and its geometrical properties were discussed elaborately by Hestenes [47].

8 The Klein–Gordon Equation

The relativistic extension of Schrödinger equation is normally considered as the Klein–Gordon equation. To find Klein–Gordon equation from stochastic Schrödinger equation, let us consider $V = 0$ in (40) and multiply it from left by

$$2S\frac{\partial}{\partial t} = \left[\frac{(2S)^2}{2m}\nabla^2\Sigma + mc^2\right] \tag{60}$$

and 2ψ from right. This is equivalent to squaring the stochastic Schrödinger equation in the absence of external potential.

$$(2S)^2\partial_t^2\varphi = m^2c^4\left[\frac{(2S)^2}{2m^2c^2}\nabla^2\Sigma + 1\right]^2\varphi \approx [(2S)^2c^2\nabla^2\Sigma + m^2c^4]\varphi \tag{61}$$

where $\varphi = 2\psi^2$. Using the plane wave form of ψ, φ can be written in the form

$$\varphi = (\rho')^{1/2}\exp[(Et - \mathbf{p}\cdot\mathbf{x})/S] \tag{62}$$

where $(\rho')^{1/2} = 2\rho$ and $\bar{\varphi}\varphi = \rho'$. It may be noted that the transformation $\psi \to \varphi$ does not change the form of wave function but the wave function φ does not represent the state of spin half particles any more. Now, the higher order terms in (61) can be calculated using the wave function in (62) and all these terms converge to give the Lorentz factor.

$$\left(\frac{1}{\rho'^2}\right)\bar{\varphi}\Sigma\varphi = (1 - \beta^2)^{-1} = \gamma^2 \tag{63}$$

Substituting this result in (61) gives

$$\partial_0^2 \varphi = (\gamma^2 \nabla^2 + \mu^2). \tag{64}$$

where $\mu^2 = (mc/2S)^2$ and $x_0 = ct$. Now, (64) can be expressed in the form

$$(\partial^2 - \mu^2)\varphi = 0. \tag{65}$$

In the above equation the Lorentz factor is absorbed into the wave function and it is expressed as

$$\varphi = \varphi_0 \exp(p.x/S), \tag{66}$$

where $p = (E/c, \gamma p)$, $x = (x_0, x)$ and $S = \mathbf{i}\boldsymbol{\sigma}_s\hbar$. Since, the adjoint $\bar{\varphi}$ is also a solution of (65), the wave function may be written as a superposition of φ and $\bar{\varphi}$, $\phi = \varphi + \bar{\varphi}$. Now, the function ϕ is equal to its self adjoint, $\phi = \bar{\phi}$.

$$(\partial^2 - \mu^2)\phi = 0 \tag{67}$$

Thus, the Klein–Gordon equation represents a relativistic equation for particles of integer spin and additionally it is also a field equation. Thus, squaring the stochastic Schrödinger equation gives classical approach to the relativistic Klein–Gordon equation.

9 Conclusions

A charged particle placed in a random fluctuating zeropoint electromagnetic field, may be treated as an oscillator oscillating at very high frequency. Such high frequency oscillations of the particle are in general known as zitterbewegung motion in quantum mechanics. The average of all such oscillations leads to the deviations in the path of the particle. However, such average oscillations may be represented by complex rotations in complex vector space. In the complex structure of the oscillating particle, the center of mass and center of charge are considered as separate and the center of charge is considered to rotate around the center of mass with a radius of rotation of the order of Compton wavelength of the particle. The position of such complex system has been denoted by a complex vector X. Expanding the convective derivative of this complex vector, we find a multivector expansion of total derivative which contains the velocity derivatives of both center of mass and center of charge. It has been shown that the internal velocity of the particle is similar to the stochastic velocity normally obtained in stochastic theories. The internal structure of the particle is found to be intricately connected to the quantum potential and operators of quantum mechanics. It shows that the necessity of introducing operators into quantum mechanics is to account for the influence of zeropoint field on the charged particle.

Considering the multivector expansion of the total derivative, a generalized Schrödinger equation is derived and it is called as stochastic Schrödinger equation. In this equation, the kinetic energy operator contains a series of higher order correction terms which correspond to correction terms to the classical kinetic energy of the particle. When higher order correction terms are neglected, the stochastic Schrödinger equation reduces to normal Schrödinger equation. It is found that the Schrödinger equation contains internal structure in disguise and it can be revealed in the form of internal kinetic energy. The internal kinetic energy is found to be equal to the quantum potential which was obtained in the Madelung fluid theory or Bohm statistical theory. The second order correction terms are found to be similar to the relativistic corrections obtained in quantum mechanics. It may be concluded that the consideration of internal structure of the particle leads to relativistic correction terms to certain extent in quantum mechanics. Consideration of correction terms in the momentum operator of stochastic Schrödinger equation leads to a modified commutation relation. In the rest frame of the particle, the stochastic Schrödinger equation reduces to a Dirac type equation and its Lorentz boost gives the Dirac equation. Finally, the relativistic Klein–Gordon equation is derived by squaring the stochastic Schrödinger equation. The analysis presented here gives an insight into the fundamental aspects of classical formulation of quantum mechanics.

Appendix: Algebra of Complex Vectors

The geometric algebra or Clifford algebra is found to be a superior algebra than the vector algebra and it is being used by a growing number of mathematicians and physicists today [54,55]. A detailed account of geometric algebra and its applications to physics is given in the book by Doran and Lasenby [56]. An introduction to geometric algebra and the algebra of complex vectors are considered in this appendix.

The geometric product of two vectors **a** and **b** is defined as

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a} \wedge \mathbf{b}, \tag{68}$$

where the scalar product or symmetric product is defined as

$$\mathbf{a}\cdot\mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba}), \tag{69}$$

and the wedge product or outer product is defined as

$$\mathbf{a} \wedge \mathbf{b} = \frac{1}{2}(\mathbf{ab} - \mathbf{ba}). \tag{70}$$

Changing the order of vectors is called reversion operation and it is denoted by an over bar.

$$\overline{\mathbf{ab}} = \mathbf{ba} = \mathbf{a}\cdot\mathbf{b} - \mathbf{a} \wedge \mathbf{b}, \tag{71}$$

A set of unit right handed basis vectors $\{\boldsymbol{\sigma}_k; k = 1, 2, 3\}$ is considered to span the three dimensional space. A pseudoscalar in three dimensional space is defined as $\mathbf{i} = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3$ and it represents a unit oriented volume. Multiplying vectors $\boldsymbol{\sigma}_k$ by pseudoscalar form unit bivectors, $B_k = \mathbf{i}\boldsymbol{\sigma}_k = \boldsymbol{\sigma}_i\boldsymbol{\sigma}_j$. Each unit bivector represents an oriented plane. The set of elements $\{1, \boldsymbol{\sigma}_k, B_k, \mathbf{i}; k = 1, 2, 3\}$ form geometric algebra of Euclidean space. A general element in geometric algebra is called a multivector and it is a sum of a scalar, vector, bivector and trivector.

$$M = \alpha + \mathbf{a} + \mathbf{ib} + \mathbf{i}\delta, \tag{72}$$

where α and δ are scalars, $\mathbf{a}$ and $\mathbf{b}$ are vectors and $\mathbf{i}\delta$ is a trivector. A complex vector is defined as a sum of a vector and a bivector.

$$Z = \mathbf{a} + \mathbf{ib} \tag{73}$$

The advantage of this definition of complex vector gives an additional geometric understanding of orientation and rotation in space. A reversion operation on a complex vector changes the sign of bivector.

$$\bar{Z} = \mathbf{a} - \mathbf{ib} \tag{74}$$

The complex vector $\bar{Z}$ is known as complex conjugate of Z. Two complex vectors are equal when their vector and bivector parts are equal. The inner product of two complex vectors $Z = \mathbf{a} + \mathbf{ib}$ and $Y = \mathbf{c} + \mathbf{id}$ can be expressed as

$$Z.Y = (\mathbf{a}\cdot\mathbf{c} - \mathbf{b}\cdot\mathbf{d}) + \mathbf{i}(\mathbf{b}\cdot\mathbf{c} + \mathbf{a}\cdot\mathbf{d}) = \alpha + \mathbf{i}\beta. \tag{75}$$

Thus the scalar product of two complex vectors is a complex scalar. The outer product of complex vectors Z and Y is

$$Z \wedge Y = (\mathbf{a} \wedge \mathbf{c} - \mathbf{b} \wedge \mathbf{d}) + \mathbf{i}(\mathbf{b} \wedge \mathbf{c} + \mathbf{a} \wedge \mathbf{d}). \tag{76}$$

The term $(\mathbf{a}\wedge\mathbf{c}-\mathbf{b}\wedge\mathbf{d})$ is a bivector and the term $\mathbf{i}(\mathbf{b}\wedge\mathbf{c}+\mathbf{a}\wedge\mathbf{d})$ is vector. Thus, the outer product of two complex vectors is a complex vector. From the above two products one can see that the geometric product of two complex vectors is a combination of a scalar, vector, bivector and trivector parts. Thus the geometric product of two complex vectors is a multivector. The inner and outer products of complex vectors are in general known as symmetric and asymmetric products respectively. Two complex vectors $Z = \mathbf{a}+\mathbf{ib}$ and $Y = \mathbf{c}+\mathbf{id}$ are said to be perpendicular when the condition $\mathbf{a}\cdot\mathbf{c} = 0$ is satisfied and they are parallel when the condition $\mathbf{a} \wedge \mathbf{c} = 0$ is satisfied. The square of a complex vector is a complex scalar.

$$Z^2 = (\mathbf{a} + \mathbf{ib})(\mathbf{a} + \mathbf{ib}) = a^2 + b^2 + 2\mathbf{i}(\mathbf{a}\cdot\mathbf{b}) \tag{77}$$

Consider that the vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal to each other. Then the scalar product $\mathbf{a}\cdot\mathbf{b} = 0$. In this case, the complex vector represents an oriented directional ellipse. The bivector $\mathbf{ib}$ represents an oriented plane and the vector $\mathbf{a}$ lies in the plane of $\mathbf{ib}$. The rotation in the plane $\mathbf{ib}$ is counterclockwise for the complex vector Z and clockwise for $\bar{Z}$. The product of a complex vector with its conjugate contains scalar and vector parts. The products $Z\bar{Z}$ and $\bar{Z}Z$ are written in the following form.

$$\bar{Z}Z = a^2 + b^2 + 2\mathbf{i}(\mathbf{a} \wedge \mathbf{b}) \tag{78}$$

$$Z\bar{Z} = a^2 + b^2 - 2\mathbf{i}(\mathbf{a} \wedge \mathbf{b}) \tag{79}$$

Since, $\mathbf{i}$ is a pseudoscalar which commutes with all vectors in three dimensional space, the quantity $2\mathbf{i}(\mathbf{a} \wedge \mathbf{b})$ is a vector and it is normal to the orientation of the bivector $\mathbf{a} \wedge \mathbf{b}$. The scalar part of (78) is equal to the scalar product of $\bar{Z}$ and Z.

$$\bar{Z}.Z = \frac{1}{2}(\bar{Z}Z + Z\bar{Z}) = a^2 + b^2 \tag{80}$$

The vector part of (78) is equal to the outer product of $\bar{Z}$ and Z.

$$\bar{Z} \wedge Z = \frac{1}{2}(\bar{Z}Z - Z\bar{Z}) = 2\mathbf{i}(\mathbf{a} \wedge \mathbf{b}) \tag{81}$$

In the case when the magnitudes of vectors $\mathbf{a}$ and $\mathbf{b}$ are equal, the complex vector represents an oriented directional circle. Then the square of complex vector $Z^2 = \bar{Z}^2 = 0$ and therefore in this case the complex vector may be called complex null vector.

When the third direction is chosen normal to the bivector plane $\mathbf{a} \wedge \mathbf{b}$, the complex vector Z and its conjugate $\bar{Z}$ represent a physical space. If we choose a set of orthonormal right handed unit vectors $\{\boldsymbol{\sigma}_k; k = 1, 2, 3\}$ along the direction of vectors $\mathbf{a}$, $\mathbf{b}$ and $2\mathbf{i}(\mathbf{a} \wedge \mathbf{b})$, the unit vectors $\boldsymbol{\sigma}_k$ can be expressed in terms of complex vectors Z and $\bar{Z}$.

$$\boldsymbol{\sigma}_1 = \frac{Z + \bar{Z}}{2a}; \quad \boldsymbol{\sigma}_2 = \frac{Z - \bar{Z}}{2a}; \quad \boldsymbol{\sigma}_3 = \frac{Z\bar{Z} - \bar{Z}Z}{4ab}; \quad 1 = \frac{Z\bar{Z} + \bar{Z}Z}{4ab} \tag{82}$$

The basis elements $\{1, \boldsymbol{\sigma}_k\}$ form a closed complex four dimensional linear space. A complete version of complex vector algebra is elaborated in [44].

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