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On the CHSH Form of Bell's Inequalities

Justo Pastor Lambare[1,2]

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Abstract A common mistake present in the derivation of the usually known as the CHSH form of Bell's inequalities is pointed out. References and comments to the correct approach are given. This error does not alter the final result and only affects the logical consistency of the derivation, but since it seems to be a widespread misconception regarding the roll and interpretation of the of use of hidden variables in Bell's theorem it is considered to be of general interest.

Keywords EPR paradox · Bell's theorem · Hidden variables · Determinism · Realism

1 Introduction

In many accounts of Bell's theorem, the Bell's inequalities are derived using only algebraic methods and given in a generalized form first derived by Clauser, Horne, Shimony and Holt [1], however a common mistake seems to have passed unnoticed through many of these proofs. That mistake is not present in the original article written by the named authors neither it is in Bell's original paper [2] and it only arises when a special form, said to be more appropriate for experiments, is given [3–5].

Since a debate has been going on in relation to what Bell's theorem really means and what its assumptions are, that seems to stem from the fact that there are two different theorems by Bell [6], the first proved in 1964 [2] and the second in 1976 [7] , and that

✉ Justo Pastor Lambare
jupalam@gmail.com

1 Facultad de Ingenieria, FIUNI, Abg. Lorenzo Zacarias, N° 255 c/ Ruta N° 1 Km 2.5, Encarnacion, Paraguay

2 Departamento de Matematica, FaCEN-UNA, Acceso Avda. Mcal. Lopez a Campus UNA, San Lorenzo, Paraguay

this article has no intention entering that debate, it is stated from the outset that these notes deal exclusively with the 1964 version of Bell's theorem.

2 Background

In 1935 a famous paper by Einstein,Podolsky and Rosen [8], known as the EPR paradox or theorem, intended to prove the incompleteness of quantum theory, however the almost equally famous response by Bohr [9] the same year set the validity of the EPR theorem to the field of personal taste or epistemological point of view.

All this changed in 1964 when a seminal work written by Bell [2] on the Einstein–Podolsky–Rosen paradox elicited a mathematical proof of the incompatibility of quantum mechanics with local hidden variables(HV) theories, that would supposedly complete quantum theory, rendering the EPR argument imposible.

Bell's theorem however does not decide whether quantum mechanics or a local HV theory is correct since it only states its incompatibility.

Given this scenario the stage was set for the experimentalists to decide the validity between quantum mechanics and local HV theories.

3 Bell's Inequality and Theorem

Using the Bhom–Aharonov [10] version of EPR consider two one half spin particles in the entangled state

$$|\psi >= \frac{1}{\sqrt{2}}\,[\,|+,- > \, - |-,+ >\,]$$

This pair is emitted by a source S, then propagate in opposite directions, when they are far apart reaching distant locations, Alice located at A on the left and Bob located at B on the right each performs a spin measurement with Stern–Gerlach apparatuses oriented along directions defined by unitary vectors **a**,**b** and angles a,b respectively.

The states $|+>$ and $|->$ are eigenkets of the spin operator along Oz,that is, for $a = b = \frac{\pi}{2}$.

Whatever the orientations of the apparatuses at A and B each measurement gives the numbers $+1$ or -1 so, if it is supposed that these numbers depend only on the orientation freely chosen(free will assumption) at each location(locality assumption), these results can be represented by symbols $A(a)$ and $B(b)$.

Bell made explicit the hypothesis of realism introducing HV with the symbol λ to represent a set of unknown and uncontrollable parameters that make possible the EPR *elements of physical reality* through the elimination of quantum indeterminism.

The two basic assumptions or hypotheses(free will can be considered implicit) of Bell's theorem, which are locality and the existence of elements of physical reality, are made possible through the use of the symbols $A(a, \lambda)$ and $B(b, \lambda)$ that can be thought of as functions of two variables giving the result of a certain measurement: $A, B : [0, 2\pi] \times \mathbf{R} \to \{-1, 1\}$.

Let $\rho(\lambda)$ be the probability distribution of λ, then the expectation value of the product $A(a, \lambda)B(b\lambda)$ is $E(a, b) = \int \rho(\lambda)A(a, \lambda)B(b, \lambda)d\lambda$

The Bell's inequality is

$$|E(a, b) - E(a, c)| \leq 1 + E(b, c) \tag{1}$$

Bell showed that this inequality is violated by the predictions of quantum mechanics thus proving the incompatibility of this theory with any one theory of local HV.

4 The Chsh Generalization

In order to obtain the CHSH inequality it is necessary to measure, at least, four numbers, two at Alice's location with orientations a and a' and two at Bob's with angles b and b', the usual way to represent these numbers is by the symbols $A(a, \lambda)$,$A(a', \lambda)$,$B(b, \lambda)$ and $B(b', \lambda)$

$$C = A(a, \lambda)B(b, \lambda) - A(a, \lambda)B(b', \lambda) + A(a', \lambda)B(b, \lambda) + A(a', \lambda)B(b', \lambda) \tag{2}$$

$$C = A(a, \lambda)[B(b, \lambda) - B(b', \lambda)] + A(a', \lambda)[B(b, \lambda) + B(b', \lambda)] \tag{3}$$

From the last equation it is immediate to check that C can take only two possible values $+2$ or -2 because each measurement is equal to ± 1 so

$$-2 \leq C \leq +2 \tag{4}$$

taking the expectation value of Eq. (2) we obtain the CHSH inequality

$$-2 \leq E(C) = E(a, b) + E(a, b') + E(a', b) + E(a', b') \leq +2 \tag{5}$$

It can be shown that for certain values of the angles a, a', b and b' the expectation value $E_{qm}(C)$ predicted by quantum mechanics for C is

$$E_{qm}(C) = 2\sqrt{2} \ > 2 \tag{6}$$

hence the conflict with quantum theory.

5 The Mistake

If the emission of an entangled pair is called an event, then a more careful analysis of Eq. (2) reveals an inadequacy of the notation since one must realize that each time an event takes place the HV asume a certain value of which we don't have any control or knowledge, still it is supposed to exist and be well defined.

Considering that it is imposible to obtain the four numbers in (2) with a single event and that in an actual experiment each pair of values in the terms of that equation is generated by one emission of an entangled pair, then four events are needed, yielding eight numbers, so it should be written as

$$C = A(a, \lambda_1)B(b, \lambda_1) - A(a, \lambda_2)B(b', \lambda_2) + A(a', \lambda_3)B(b, \lambda_3) + A(a', \lambda_4)B(b', \lambda_4) \tag{7}$$

The realization of this fact precludes the passage from Eqs. (2) to (3) and one can no longer conclude that

$$-2 \leq C \leq +2 \tag{8}$$

neither

$$-2 \leq E(C) \leq +2 \tag{9}$$

in fact, from (7) one can only infer that $C \in \{-4, -2, 0, +2, +4\}$

6 The Correct Derivation

The error explained in this notes was not committed by the original authors [1] because they arrived at the desired inequality without assuming the bounding of an individual evaluation of Eq. (2) before taking mean values. There is also another derivation by Bell himself [11] that may be more accesible than the original one by Clauser, Horn, Shimony and Holt.

A correct use of Eq. (2) may be accomplished through the inversion of the order of the derivation,i.e., starting from the mean values.

$$|E(C)| = |E(a, b) + E(a, b') + E(a', b) + E(a', b')|$$

$$|E(C)| = |\int \rho(\lambda)A(a, \lambda)B(b, \lambda)d\lambda + \int \rho(\lambda)A(a, \lambda)B(b', \lambda)d\lambda + \dots \tag{10}$$

$$|E(C)| = |\int \rho(\lambda)[A(a, \lambda)B(b, \lambda) + A(a, \lambda)B(b', \lambda) + \dots]d\lambda| \tag{11}$$

$$\begin{aligned} |E(C)| \leq & \int \rho(\lambda)|[A(a, \lambda)B(b, \lambda) + A(a, \lambda)B(b', \lambda) \\ & + A(a', \lambda)B(b', \lambda) + A(a', \lambda)B(b', \lambda)]|d\lambda \end{aligned}$$

$$|E(C)| \leq \int \rho(\lambda)\, 2\, d\lambda = 2 \int \rho(\lambda)d\lambda = 2$$

proceeding in this way there is no logical objection to the expression inside the integral with same value of λ and the bounding of it can now be correctly realized as in (3) and (4).

7 Discussion

1. It is important to stress once more that the origen of the mistake consists in the failure to recognize that λ is supposed to take different and well defined values each time an entangled pair of particles is emitted at the source, since this parameters are what materialize the elements of physical reality that presumably assign a deterministic character to the measurement once an angle is chosen at the place of observation.

2. It is commonly argued that the free will principle, implicit in the reasoning, allows us to use the same value of λ in (7) because the settings of the measuring devices are not known in advance. However such an argument, besides other logical inconsistencies, contains a misconception with respect the meaning of the HV concept for the following reasons:
   - Fixing the value of λ does not imply fixing the result of a future measurement.
   - The roll of λ is to fix $A(a, \lambda)$ as a function of the the angle a rendering the phenomenon under study deterministic and realistic which does not mean predetermined . The experimenter can still freely choose the angular variable a just before the particle arrives at his/her location and after λ has been fixed when the particle was emitted at the source.
   - According to the previous item the elements of physical reality are represented by the functions $A(a, \lambda)$ once λ is fixed at the source for each generated pair, so the value taken by λ at the moment of pair emission does not interfere with the free election of the angle by the experimenter at a later moment.
   - λ is supposed to give the particle all the information it needs to adopt a concrete value upon encountering a certain angle. This is how the HV are supposed to restore determinism and locality to the theory.
3. In an experiment the terms in (2) are obtained through four events so eight numbers are measured by the partners at A and B. Let's call these numbers $\hat{A}(a, b)$, $\hat{A}(a, b')$, $\hat{A}(a', b)$, $\hat{A}(a', b')$ and similarly for B

$$C = \hat{A}(a, b)\hat{B}(a, b) - \hat{A}(a, b')\hat{B}(a, b') + \hat{A}(a', b)\hat{B}(a', b) + \hat{A}(a', b')\hat{B}(a', b') \tag{12}$$

   this last equation represents what is actually measured in a real experiment and Eq. (2) is supposed to represent it with only four different numbers in it, whence if we accept (2) as correct, the following constraints on the measured values are obtained

$$\hat{A}(a, b) = \hat{A}(a, b')$$
$$\hat{A}(a', b) = \hat{A}(a', b')$$

   so to reject the viability of HV theories suffices to obtain two results such that

$$\hat{A}(a, b) \neq \hat{A}(a, b') \tag{13}$$

   but we know this is posible since quantum mechanics only gives probabilities of results.
   According to this it is not necessary to take mean values in (2) to produce a contradiction with quantum mechanics neither is it necessary to design especial experiments since it is well known that (13) actually holds in real situations.
4. The use of only one value of λ in (2) amounts to not using any value at all, i.e., we can eliminate altogether the symbol from the equation and though is not the intention of these notes to asses the meaning of excluding the $\lambda's$ in the demonstration of the CHSH inequality [12], it is however, to point out the logical error of including such hypothesis when in fact it is not used.

5. Why is the use of Eq. (2) correct inside the integral while it is not when used in isolated form outside it? Seeing the integral as an abbreviated form of addition the passage from (10) to (11) can be interpreted as consequence of the application of the associative and commutative properties of the sum over all possible values of λ thus the same values can be chosen to appear in the displayed order.
   On the other hand when written in isolated form it is supposed to represent the values obtained in four consecutive runs of the experiment.

8 Conclusions

The roll of the $\lambda' s$ variables in the 1964 version of Bell's Theorem is not to directly fix the result of a measurement, it is simply to make such a measurement deterministic by setting $A(a, \lambda)$, $B(b, \lambda)$ as functions of the angular variables a and b.

The lack of a correct interpretation of the roll of HV as agents of the elements of physical reality, as well as their use in an equation that can not represent the outcome of a real and meaningful experiment, has led to the use of a widely spread and wrong derivation of the CHSH's inequality.

Finally we recognize that many of the arguments have been used somehow redundantly and that whole point could be deemed trivial but the experience of the author has revealed that the issue is rather subtle and it seems to be harder to grasp than could in principle appear owing to the challenge posed to very ingrained preconceptions.

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