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Deformed Entropy and Information Relations for Composite and Noncomposite Systems

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Abstract The notion of conditional entropy is extended to noncomposite systems. The q-deformed entropic inequalities, which usually are associated with correlations of the subsystem degrees of freedom in bipartite systems, are found for the noncomposite systems. New entropic inequalities for quantum tomograms of qudit states including the single qudit states are obtained. The Araki–Lieb inequality is found for systems without subsystems.

Keywords Marginal probability distribution · Composite system · Entropy · Deformation · Conditional entropy · Information relations

1 Introduction

The probability distribution is characterized by Shannon entropy [1]. The q-entropies [2,3] containing an extra parameter q provide extra information on the probability distributions. The Tsallis q-entropy [3] can be expressed in terms of Renyi q-entropy [2]. Both q-entropies for the parameter $q = 1$ coincide with the Shannon entropy. The states of quantum systems, identified with density matrices [4,5] are characterized by von Neumann entropy. The q-entropies also characterize the properties of quantum states. At complete order in a classical system, the Shannon entropy is equal to zero. For composite classical and quantum systems, there exist some inequalities related to entropies of the system and its subsystems [6–8].

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There exist the entropic and information inequalities, e.g., the subadditivity condition, which is the inequality for von Neumann entropies of the bipartite-system state and its two subsystem states [9]. For three-partite systems, there exists the strong subadditivity condition, which is the inequality for the von Neumann entropies of the composite system and its subsystems [10]. The nonnegativity of the Shannon mutual information and quantum mutual information follows from the subadditivity condition valid for composite systems.

Recently [11–13], it was observed that all entropic inequalities known for composite classical and quantum systems with two or several random variables like, e.g., the subadditivity condition, can be found also for the noncomposite system with only one random observable.

It is known (see, e.g., a recent review [14]) that the states of quantum systems can be described in terms of fair probability distributions, called quantum tomograms, which contain complete information on the state density matrices.

The tomographic probability representation of quantum mechanics was suggested in [15]. The tomographic probability representation of classical mechanics was suggested in [16]. Within the framework of this representation, both classical and quantum states are described by the same objects—tomograms: the quantum states are determined by tomographic fair probability distributions, the states of classical systems are determined by classical tomograms. The analogous description of quantum spin states by the probability distributions (spin tomograms) was suggested in [17,18]. In the tomographic probability representation, the standard formulae for classical probability distributions can be easily applied and compared with the corresponding quantum ones [19,20].

The classical random variables are described within the framework of classical probability theory [21]. The model of quantum mechanics based on the classical probability distributions is elaborated in [22–24]. Based on this fact and on the tomographic-probability representation of quantum mechanics, one may use the apparatus of classical probability theory to consider quantum correlations and the entanglement phenomenon in quantum systems. For example, a specific map of the classical probability distribution called the qubit portrait of qudit states was introduced to study the entanglement phenomenon in [25,26]. The quantum correlations were studied within the framework of tomographic probability representation of quantum states [27,28].

There exist different entropic inequalities for composite quantum systems [9,10,29–35]. Using the approach based on the portrait method it was observed in [11,36,37] that the entropic inequalities valid for composite systems can be extended to arbitrary systems including the systems without subsystems. In [31–35], some inequalities associated with positive operators acting in the Hilbert space, which has the structure of tensor product of Hilbert spaces, were studied.

In [38], new entropic inequalities for single qudit states were obtained employing known properties of relative entropy of composite systems. In [39], a new entropic inequality for states of the system of $n \geq 1$ qudits was derived, and a general statement on the existence of the subbaditivity condition for an arbitrary probability distribution and an arbitrary qudit-system tomogram was formulated. In [40], the entropic inequalities and uncertainty relations for q-derformed entropy were studied for noncomposite quantum systems realized by superconducting circuits with the Josephson junction, and possible realizations of various quantum logic gates of noncomposite quantum systems were discussed.

The aim of our work is to obtain new matrix inequalities for density matrices of qudit states of noncomposite quantum systems which do not depend on the tensor-product structure of the Hilbert space. The other goal of this paper is to extend the notion of conditional entropy to the case of noncomposite systems and to obtain the Araki–Lieb inequality for the single-qudit state, as well as to obrain new inequalities for q-deformed entropy in the case of noncomposite systems. Also we obtain a new chain relation for a single qudit state.

This paper is organized as follows.

In the second section, we review the probability distributions and the conditional entropies for one random variable. In the third section, we obtain new entropic relations for qudit-state tomograms. In the fourth section we find an analog of the Araki–Lieb inequality for an arbitrary density matrix ρ and consider an example of the matrix of a single-qudit state. In the fifth section, we discuss the deformed subadditivity condition in the classical and quantum cases and study quantum correlations expressed in terms of the deformed information depending on global unitary transform. In conclusion, we list our main results.

2 The Probability Distributions and Conditional Entropies for One Random Variable

The conditional entropy is the notion related to the properties of the joint probability distribution $P(j,k)$, $(j = 1,2,\ldots,n,\ k = 1,2,\ldots m)$ of two random variables, where the first random variable describes the degrees of freedom of a system A and the second random variable describes the degrees of freedom of a system B. The joint probability distribution $P(j,k)$, determines the conditional probability distributions, in view of Bayes rule,

$$P(j|k) = \frac{P(j,k)}{\sum_{j=1}^{n} P(j,k)}. \tag{1}$$

The marginal probability distributions for the first and second random variables read

$$\mathcal{P}_1(j) = \sum_{k=1}^{m} P(j,k), \quad \mathcal{P}_2(k) = \sum_{j=1}^{n} P(j,k). \tag{2}$$

The Shannon entropies associated to the probability distributions $P(j,k)$, $\mathcal{P}_1(j)$, and $\mathcal{P}_2(k)$, as well as to the conditional probability distribution $P(j|k)$ are

$$\begin{aligned} H(A,B) &= -\sum_{j=1}^{n}\sum_{k=1}^{m} P(j,k)\ln P(j,k), \quad H(A) = -\sum_{j=1}^{n} \mathcal{P}_1(j)\ln \mathcal{P}_1(j), \\ H(B) &= -\sum_{k=1}^{m} \mathcal{P}_2(k)\ln \mathcal{P}_2(k) \end{aligned} \tag{3}$$

and

$$H(A|k) = -\sum_{j=1}^{n} P(j|k) \ln P(j|k). \tag{4}$$

The conditional entropy $H(A|B)$ is given by the average entropy

$$H(A|B) = \sum_{k=1}^{m} \mathcal{P}_2(k) H(A|k) = H(A, B) - H(B) \tag{5}$$

Thus one has the equality

$$H(A, B) = H(A|B) + H(B), \tag{6}$$

which is called the chain relation.

On the other hand, one can obtain an analogous relation for only one random variable described by the probability distribution $P(s)$, $s = 1, 2, \ldots N$, where the integer $N = nm$. To show this possibility, following the approach [11], we use the map of integers $1 \leftrightarrow 11,\ 2 \leftrightarrow 12, \ldots, m \leftrightarrow 1m,\ m+1 \leftrightarrow 21,\ m+2 \leftrightarrow 22, \ldots,\ N-1 \leftrightarrow nm-1,\ N \leftrightarrow nm$; this means that the index s in $P(s)$ is considered as double index jk where $j = 1, 2, \ldots n$ and $k = 1, 2, \ldots m$. Thus probability distribution for one random variable is mapped onto the table $P(j, k)$ of nonnegative numbers, which satisfies the normalisation condition

$$\sum_{s=1}^{N} P(s) = \sum_{j=1}^{n} \sum_{k=1}^{m} P(j, k) = 1. \tag{7}$$

Since all Eqs. (1)–(7) are formally the relations between the $N = nm$ nonnegative numbers given by the table $P(j, k)$, the relations do not depend on the interpretation of these numbers, say, as connected with a joint probability distribution. They are valid also for the numbers $P(s)$ considered as the probabilities describing one random variable but organized as the table of numbers $P(j, k)$.

We give an example of $P(s)$ for four nonnegative numbers $p_1,\ p_2,\ p_3,\ p_4$, such that $\sum_{s=1}^{4} p_s = 4$. One can introduce the notation $P(1, 1) \equiv p_1,\ P(1, 2) \equiv p_2,\ P(2, 1) \equiv p_3,\ P(2, 2) \equiv p_4$. Then one has analogs of all the probabilities given by (1) and (2) as

$$\mathcal{P}_1(1) = p_1 + p_2, \quad \mathcal{P}_1(2) = p_3 + p_4, \tag{8}$$

$$\mathcal{P}_2(1) = p_1 + p_3, \quad \mathcal{P}_2(2) = p_2 + p_4. \tag{9}$$

Let us introduce two artificial subsystems A and B corresponding to indices j and k in the table $P(j, k)$. Then we introduce the analogs of conditional probability distributions. For example, all the numbers

$$P^A(1|1) = \frac{p_1}{p_1 + p_3}, \quad P^A(2|1) = \frac{p_3}{p_1 + p_3} \tag{10}$$

and

$$P^A(1|2) = \frac{p_2}{p_2 + p_4}, \quad P^A(2|2) = \frac{p_4}{p_2 + p_4}, \tag{11}$$

can be considered as conditional probability distributions for subsystem A.

These formulae provide the nonlinear maps of the probability four-vector $\vec{p} = (p_1, p_2, p_3, p_4)$ onto two probability two-vectors, which are

$$\vec{p} \to \vec{P}^A(1) = \frac{1}{p_1 + p_3}\begin{pmatrix} p_1 \\ p_3 \end{pmatrix}, \qquad \vec{p} \to \vec{P}^A(2) = \frac{1}{p_2 + p_4}\begin{pmatrix} p_2 \\ p_4 \end{pmatrix}. \tag{12}$$

The Shannon entropies associated with the probability vectors (12) read

$$H^A(1) = -\frac{p_1}{p_1 + p_3}\ln\frac{p_1}{p_1 + p_3} - \frac{p_3}{p_1 + p_3}\ln\frac{p_3}{p_1 + p_3}, \tag{13}$$

$$H^A(2) = -\frac{p_2}{p_2 + p_4}\ln\frac{p_2}{p_2 + p_4} - \frac{p_4}{p_2 + p_4}\ln\frac{p_4}{p_2 + p_4}, \tag{14}$$

and the Shannon entropy associated with the four-vector $\vec{p}$ provides the known chain relation for the joint probability distribution, e.g., (6), where we use the standard notation

$$H(A, B) = -p_1 \ln p_1 - p_2 \ln p_2 - p_3 \ln p_3 - p_4 \ln p_4,$$

and the conditional entropy $H(A|B)$ reads

$$H(A|B) = (p_1 + p_3)H^A(1) + (p_2 + p_4)H^B(2); \tag{15}$$

also

$$H(B) = -\mathcal{P}_2(1)\ln\mathcal{P}_2(1) - \mathcal{P}_2(2)\ln\mathcal{P}_2(2). \tag{16}$$

The Tsallis q-entropy of a bipartite system defined as

$$H_q(A, B) = -\sum_{j=1}^{n}\sum_{k=1}^{m} P^q(j,k)\frac{P^{1-q}(j,k) - 1}{1 - q}, \tag{17}$$

for $q \to 1$, has the limit $H_1(A, B) = H(A, B)$.

The Tsallis q-entropy $H_q(B)$ is defined as

$$H_q(B) = \frac{1}{q-1}\sum_{j=1}^{m}\left\{\left(\frac{P(j,k)}{\sum_{k'=1}^{m} P(j,k')}\right)^q\left[\left(\frac{P(j,k)}{\sum_{k''=1}^{m} P(j,k'')}\right)^{1-q} - 1\right]\right\}. \tag{18}$$

The conditional q-entropy $H_q(A|B)$ is the difference

$$H_q(A|B) = H_q(A, B) - H_q(B). \tag{19}$$

Thus, we arrive at the chain relation [7,8,41]

$$H_q(A,B) = H_q(A|B) + H_q(B). \tag{20}$$

One can write analogous relations using all permutations of numbers p_s.

The Renyi q-entropy is the function of Tsallis q-entropy, i.e.

$$H_R(A,B) = \frac{1}{1-q}\ln\left[(1-q)H_q(A,B)+1\right].$$

Thus the relations obtained for the Tsallis entropy can be expressed in terms of relations for Renyi q-entropy. From the consideration of the conditional entropies for the probability distribution of one random variable $P(s)$ follows that the deformed chain relation is valid for constructed 'artificial' subsystems A and B, e.g., described by the probability distributions given by (8) and (9).

3 Entropic Relations for Qudit Tomograms

Now we consider the qudit state tomograms. The tomograms are fair probability distributions, which determine the density matrices of quantum states. In view of this fact, the entropic relations for the tomograms correspond to quantum properties of the qudits, in particular, to the quantum correlations in multipartite system states but also to the quantum correlations in noncomposite system states. The tomograms can be introduced for an arbitrary Hermitian nonnegative matrix ρ with Tr $\rho = 1$.

The tomogram associated with the matrix ρ reads

$$w(s,u) = (u\rho u^+)_{ss}. \tag{21}$$

Here $s = 1, 2, 3, \ldots, N = nm$ is an index characterising the basis in the linear space where the density matrix is given. The tomogram is the standard probability distribution depending on the unitary matrix u and it satisfies the normalization condition

$$\sum_{s=1}^{N} w(s,u) = 1. \tag{22}$$

One can introduce the deformed Shannon entropy for the tomogram, which is the tomographic Tsallis entropy [3]

$$H_q(u) = -\sum_{s=1}^{N} w(s,u)\frac{w^{q-1}(s,u)-1}{q-1}. \tag{23}$$

For the joint probability distribution $P(j,k)$, there exists the deformed subadditivity condition [8], which we apply to the tomogram.

The deformed inequality which is a characteristics of the classical probability N-vector $\vec{w}(u)$ with components $w(s,u)$ can be written in the form $(q > 1)$

$$-\sum_{s=1}^{N} w(s,u)\frac{w^{q-1}(s,u)-1}{q-1} \leq -\sum_{j=1}^{n} w_1(j,u)\frac{w_1^{q-1}(j,u)-1}{q-1} \\ -\sum_{k=1}^{m} w_2(k,u)\frac{w_2^{q-1}(k,u)-1}{q-1}, \qquad (24)$$

where we have two probability vectors $\vec{w}_1(u)$ and $\vec{w}_2(u)$. The components of these probability vectors are given as marginal probabilities obtained from the table $P(j,k)$, where $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, m$. In this case, the table $P(j,k)$ is constructed from the tomographic-probability distribution $w(s,u), s = 1, 2, \ldots, N$, using the same tool, which was used in the second section while considering the probability distribution of one random variable $P(s)$ as the joint probability distribution $P(j,k)$ of two artificial subsystems A and B. This means that instead of the probability distribution $P(s)$ we use the tomographic-probability distribution $w(s,u)$, where the probabilities depend on unitary $N \times N$ - matrix u. The new quantum inequality (24) is valid for different systems.

We present the examples with two qubits and qudit with $j = 3/2$.

The density matrix for two qubits is written in the basis $\mid m_1 m_2\rangle$, where $m_1, m_2 = \pm 1/2$ in the Hilbert space $H = H_1 \otimes H_2$, which is the tensor product of two Hilbert spaces H_1 and H_2 corresponding to the states of the qubits. The matrix elements $\rho_{m_1 m_2, m_1' m_2'}$ provide the tomogram, which is the joint probability distribution $w(m_1, m_2, u)$; thus, the index $s = 1, 2, 3, 4$ in the probability vector $w(s,u)$ is mapped onto pairs $1/2\,1/2$, $1/2\ -1/2$, $-1/2\,1/2$, $-1/2\ -1/2$.

Then the quantum inequality (24) for Tsallis q-entropy of the two-qubit state reads

$$-\sum_{m_1 m_2=-1/2}^{1/2} w(m_1,m_2,u)\frac{w^{q-1}(m_1,m_2,u)-1}{q-1} \\ \leq -\sum_{m_1=-1/2}^{1/2} w_1(m_1,u)\frac{w_1^{q-1}(m_1,u)-1}{q-1} - \sum_{m_2=-1/2}^{1/2} w_2(m_2,u)\frac{w_2^{q-1}(m_2,u)-1}{q-1}, \qquad (25)$$

where marginals $w_1(m_1,u) = -\sum_{m_2=-1/2}^{1/2} w(m_1,m_2,u)$ and $w_2(m_2,u) = -\sum_{m_1=-1/2}^{1/2} w(m_1,m_2,u)$ are the tomograms for qubits, if the unitary 4×4-matrix u is taken as the tensor product $u = u_1 \times u_2$ of local unitary transforms.

In the case of two qubits, one can get the chain rule for the entropies of two subsystems given by Eq. (6), where the entropy reads

$$H(A,B) = -\sum_{m_1 m_2=-1/2}^{1/2} w(m_1,m_2,u) \ln w(m_1,m_2,u), \qquad (26)$$

and the entropy for the second qubit $H(B)$ is

$$H(B) = -\sum_{m_2=-1/2}^{1/2} w_2(m_2, u) \ln w_2(m_2, u). \tag{27}$$

The conditional tomographic entropy $H(A|B)$ is given by Eq. (5).

The second example under consideration is qudit with $j = 3/2$; it provides the same entropic inequalities, which are new for this system. We employ the map of indices in the tomogram $w(s, u)$ interpreting index $s = 1, 2, 3, 4$ as the spin projection $m = -3/2, -1/2, 1/2, 3/2$. This means that the tomogram $w(s, u) \equiv w(m, u)$ satisfies the inequality

$$\begin{aligned} &-\sum_{m=-3/2}^{3/2} w(m, u) \frac{w^{q-1}(m, u) - 1}{q - 1} \\ &\quad \leq -\sum_{j=1}^{2} \Omega_1(j, u) \frac{\Omega_1^{q-1}(j, u) - 1}{q - 1} - \sum_{k=1}^{2} \Omega_2(k, u) \frac{\Omega_2^{q-1}(k, u) - 1}{q - 1}, \end{aligned} \tag{28}$$

where the probability distributions $\Omega_1(j, u)$ and $\Omega_2(k, u)$, with $j, k = 1, 2$, are expressed in terms of the qudit tomograms according to Eqs. (8) and (9) as

$$\Omega_1(1, u) = w(-3/2, u) + w(-1/2, u), \quad \Omega_1(2, u) = w(1/2, u) + w(3/2, u), \tag{29}$$

$$\Omega_2(1, u) = w(-3/2, u) + w(1/2, u), \quad \Omega_2(2, u) = w(-1/2, u) + w(3/2, u). \tag{30}$$

Inequality (28) is a new entropic inequality for the single qudit state with $j = 3/2$; it can be checked experimentally.

The realization of the qudit state can be provided either by the four-level atomic state or by the Josephson-junction state in the quantum-circuit experiments.

In the limit $q \to 1$, inequalities (25) and (28) become the subadditivity conditions for Shannon entropies determined by the tomograms. Inequality (25) provides the standard subadditivity condition for bipartite system, and inequality (28) determines the new subadditivity condition for a single random variable.

Now we introduce the conditional entropy for the tomogram of the qudit state with $j = 3/2$. To do this, we write the q-entropy for the qudit state with $j = 3/2$ determined by the state tomogram as follows:

$$\begin{aligned} H_q^{(3/2)}(u) = -\Bigg\{ & w^q(3/2, u) \frac{w^{1-q}(3/2, u) - 1}{1 - q} + w^q(1/2, u) \frac{w^{1-q}(1/2, u) - 1}{1 - q} \\ & + w^q(-1/2, u) \frac{w^{1-q}(-1/2, u) - 1}{1 - q} + w^q(-3/2, u) \frac{w^{1-q}(-3/2, u) - 1}{1 - q} \Bigg\}. \end{aligned} \tag{31}$$

This expression for the q-entropy is equivalent to the left-hand side of inequality (28).

The q-entropy related to the probability to obtain positive and negative spin projections $\Omega_1(+,u) = w(1/2,u)+w(3/2,u)$ and $\Omega_2(-,u) = w(-3/2,u)+w(-1/2,u)$ reads

$$H_q^B = -\Omega_1^q(+,u)\frac{\Omega_1^{1-q}(+,u)-1}{1-q} - \Omega_1^q(-,u)\frac{\Omega_1^{1-q}(-,u)-1}{1-q}. \qquad (32)$$

Thus, we interpret an "artificial" subsystem B as a set of events where one has either only positive or only negative values of the spin projections for the system with spin $j = 3/2$ (qudit). The other "artificial" subsystem A is considered as a set of events where the modulus of the sum of the spin projections is equal to unity. Quantum correlations of these two subsystems correspond to the correlations of the different spin projections, which play the role of different qubits in the qubit bipartite system.

We introduce the conditional entropy and the chain relation for the tomogram of the qudit state with $j = 3/2$ taking

$$\begin{aligned} H_q(A|B) = {} & \Omega_1^q(+,u)\frac{\Omega_1^{1-q}(+,u)-1}{1-q} + \Omega_1^q(-,u)\frac{\Omega_1^{1-q}(-,u)-1}{1-q} \\ & - \left\{ w^q(3/2,u)\frac{w^{1-q}(3/2,u)-1}{1-q} + w^q(1/2,u)\frac{w^{1-q}(1/2,u)-1}{1-q} \right. \\ & \left. + w^q(-1/2,u)\frac{w^{1-q}(-1/2,u)-1}{1-q} + w^q(-3/2,u)\frac{w^{1-q}(-3/2,u)-1}{1-q} \right\}. \end{aligned} \qquad (33)$$

Thus, one has the chain relation (20) for the single qudit state with $j = 3/2$.

Analogous chain relations can be constructed for the other single qudits.

The physical meaning of ambiguity in the partition of the noncomposite systems like qudit with $j = \frac{N-1}{2}$ reflects the different kinds of correlations of degrees of freedom in the systems.

In the limit $q \to 1$, the chain relations become the entropic relations for conditional tomographic Shannon entropies for the systems without subsystems.

One can obtain some relations replacing the tomograms by other quasidistributions because the quasidistributions are expressed in terms of the tomograms. It is straightforward to do for nonnegative and normalized quasidistributions like Husimi function since it has formal properties of the classical probability distribution.

4 Araki–Lieb Inequality for the Single Qudit State

The subadditivity condition for the von Neumann entropy of the two-qudit state with the density matrix $\rho(1,2)$ and the entropies for each qudit states with the density matrices $\rho(1) = \mathrm{Tr}_2\,\rho(1,2)$ and $\rho(2) = \mathrm{Tr}_1\,\rho(1,2)$, respectively, can be written in a form of the matrix inequality [11]. In fact, the first qudit state with $j = (n-1)/2$ and the second qudit state with $j = (m-1)/2$ are described by the density matrix $\rho(1,2)$ of a block form

$$\rho(1,2)=\begin{pmatrix} R_{11} & R_{12} & \dots & R_{1n} \\ R_{21} & R_{22} & \dots & R_{2n} \\ \dots & \dots & \dots & \dots \\ R_{n1} & R_{n2} & \dots & R_{nn} \end{pmatrix}, \tag{34}$$

where blocks R_{kl} $(k,l=1,2,\dots,n)$ are $m\times m$-matrices and $\rho(1,2)$ is the $N\times N$-matrix with $N=nm$. Then the density $n\times n$-matrix of the first qudit state $\rho(1)$ reads

$$\rho(1)=\begin{pmatrix} \mathrm{Tr}R_{11} & \mathrm{Tr}R_{12} & \dots & \mathrm{Tr}R_{1n} \\ \mathrm{Tr}R_{21} & \mathrm{Tr}R_{22} & \dots & \mathrm{Tr}R_{2n} \\ \dots & \dots & \dots & \dots \\ \mathrm{Tr}R_{n1} & \mathrm{Tr}R_{n2} & \dots & \mathrm{Tr}R_{nn} \end{pmatrix}, \tag{35}$$

and the density $m\times m$ matrix $\rho(2)$ is expressed in terms of blocks R_{kl} as

$$\rho(2)=\sum_{k=1}^{n} R_{kk}. \tag{36}$$

The subadditivity condition means that

$$\begin{aligned} &-\mathrm{Tr}\begin{pmatrix} R_{11} & R_{12} & \dots & R_{1n} \\ R_{21} & R_{22} & \dots & R_{2n} \\ \dots & \dots & \dots & \dots \\ R_{n1} & R_{n2} & \dots & R_{nn} \end{pmatrix}\ln\begin{pmatrix} R_{11} & R_{12} & \dots & R_{1n} \\ R_{21} & R_{22} & \dots & R_{2n} \\ \dots & \dots & \dots & \dots \\ R_{n1} & R_{n2} & \dots & R_{nn} \end{pmatrix} \\ &\leq -\mathrm{Tr}\begin{pmatrix} \mathrm{Tr}R_{11} & \mathrm{Tr}R_{12} & \dots & \mathrm{Tr}R_{1n} \\ \mathrm{Tr}R_{21} & \mathrm{Tr}R_{22} & \dots & \mathrm{Tr}R_{2n} \\ \dots & \dots & \dots & \dots \\ \mathrm{Tr}R_{n1} & \mathrm{Tr}R_{n2} & \dots & \mathrm{Tr}R_{nn} \end{pmatrix}\ln\begin{pmatrix} \mathrm{Tr}R_{11} & \mathrm{Tr}R_{12} & \dots & \mathrm{Tr}R_{1n} \\ \mathrm{Tr}R_{21} & \mathrm{Tr}R_{22} & \dots & \mathrm{Tr}R_{2n} \\ \dots & \dots & \dots & \dots \\ \mathrm{Tr}R_{n1} & \mathrm{Tr}R_{n2} & \dots & \mathrm{Tr}R_{nn} \end{pmatrix} \\ &\quad -\mathrm{Tr}\left(\sum_{k=1}^{n} R_{kk}\right)\ln\left(\sum_{k=1}^{n} R_{kk}\right). \end{aligned} \tag{37}$$

For the bipartite system state, the Araki–Lieb inequality provides a bound for the difference of two subsystem quantum entropies; it reads

$$-\mathrm{Tr}\,[\rho(1,2)\ln\rho(1,2)]\geq|-\mathrm{Tr}\,[\rho(1)\ln\rho(1)]+\mathrm{Tr}\,[\rho(2)\ln\rho(2)]\,|. \tag{38}$$

The inequality can be rewritten in the matrix form as follows:

$$\begin{aligned} &-\mathrm{Tr}\begin{pmatrix} R_{11} & R_{12} & \dots & R_{1n} \\ R_{21} & R_{22} & \dots & R_{2n} \\ \dots & \dots & \dots & \dots \\ R_{n1} & R_{n2} & \dots & R_{nn} \end{pmatrix}\ln\begin{pmatrix} R_{11} & R_{12} & \dots & R_{1n} \\ R_{21} & R_{22} & \dots & R_{2n} \\ \dots & \dots & \dots & \dots \\ R_{n1} & R_{n2} & \dots & R_{nn} \end{pmatrix} \\ &\geq \left| -\mathrm{Tr}\begin{pmatrix} \mathrm{Tr}R_{11} & \mathrm{Tr}R_{12} & \dots & \mathrm{Tr}R_{1n} \\ \mathrm{Tr}R_{21} & \mathrm{Tr}R_{22} & \dots & \mathrm{Tr}R_{2n} \\ \dots & \dots & \dots & \dots \\ \mathrm{Tr}R_{n1} & \mathrm{Tr}R_{n2} & \dots & \mathrm{Tr}R_{nn} \end{pmatrix}\ln\begin{pmatrix} \mathrm{Tr}R_{11} & \mathrm{Tr}R_{12} & \dots & \mathrm{Tr}R_{1n} \\ \mathrm{Tr}R_{21} & \mathrm{Tr}R_{22} & \dots & \mathrm{Tr}R_{2n} \\ \dots & \dots & \dots & \dots \\ \mathrm{Tr}R_{n1} & \mathrm{Tr}R_{n2} & \dots & \mathrm{Tr}R_{nn} \end{pmatrix} \right. \end{aligned}$$

$$+\operatorname{Tr}\left(\sum_{k=1}^{n} R_{kk}\right)\ln\left(\sum_{k=1}^{n} R_{kk}\right)\Bigg|. \tag{39}$$

The Araki–Lieb entropic inequality written in the matrix form (39) is valid for an arbitrary matrix ρ given in the form (34), which is a nonnegative Hermitian matrix with the unit trace. In view of this fact, we obtain an analog of the Araki–Lieb inequality for an arbitrary matrix ρ of the form (34), including the matrix of the single qudit state.

For example, for a qutrit state (or the spin state with $j = 1$) with the density matrix

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{10} & \rho_{1-1} \\ \rho_{01} & \rho_{00} & \rho_{0-1} \\ \rho_{-11} & \rho_{-10} & \rho_{-1-1} \end{pmatrix}, \tag{40}$$

the subadditivity condition reads [36]

$$\begin{aligned} -\operatorname{Tr}&\begin{pmatrix} \rho_{11} & \rho_{10} & \rho_{1-1} \\ \rho_{01} & \rho_{00} & \rho_{0-1} \\ \rho_{-11} & \rho_{-10} & \rho_{-1-1} \end{pmatrix} \ln \begin{pmatrix} \rho_{11} & \rho_{10} & \rho_{1-1} \\ \rho_{01} & \rho_{00} & \rho_{0-1} \\ \rho_{-11} & \rho_{-10} & \rho_{-1-1} \end{pmatrix} \\ &\leq -\operatorname{Tr}\begin{pmatrix} \rho_{11}+\rho_{-1-1} & \rho_{10} \\ \rho_{01} & \rho_{00} \end{pmatrix} \ln \begin{pmatrix} \rho_{11}+\rho_{-1-1} & \rho_{10} \\ \rho_{01} & \rho_{00} \end{pmatrix} \\ &\quad - \operatorname{Tr}\begin{pmatrix} \rho_{11}+\rho_{00} & \rho_{1-1} \\ \rho_{-11} & \rho_{-1-1} \end{pmatrix} \ln \begin{pmatrix} \rho_{11}+\rho_{00} & \rho_{1-1} \\ \rho_{-11} & \rho_{-1-1} \end{pmatrix}. \end{aligned} \tag{41}$$

But for the qutrit state, one has the Araki–Lieb inequality

$$\begin{aligned} -\operatorname{Tr}&\begin{pmatrix} \rho_{11} & \rho_{10} & \rho_{1-1} \\ \rho_{01} & \rho_{00} & \rho_{0-1} \\ \rho_{-11} & \rho_{-10} & \rho_{-1-1} \end{pmatrix} \ln \begin{pmatrix} \rho_{11} & \rho_{10} & \rho_{1-1} \\ \rho_{01} & \rho_{00} & \rho_{0-1} \\ \rho_{-11} & \rho_{-10} & \rho_{-1-1} \end{pmatrix} \\ &\geq \Bigg| -\operatorname{Tr}\begin{pmatrix} \rho_{11}+\rho_{-1-1} & \rho_{10} \\ \rho_{01} & \rho_{00} \end{pmatrix} \ln \begin{pmatrix} \rho_{11}+\rho_{-1-1} & \rho_{10} \\ \rho_{01} & \rho_{00} \end{pmatrix} \\ &\quad + \operatorname{Tr}\begin{pmatrix} \rho_{11}+\rho_{00} & \rho_{1-1} \\ \rho_{-11} & \rho_{-1-1} \end{pmatrix} \ln \begin{pmatrix} \rho_{11}+\rho_{00} & \rho_{1-1} \\ \rho_{-11} & \rho_{-1-1} \end{pmatrix} \Bigg|. \end{aligned} \tag{42}$$

A qutrit is a system without subsystems. The Araki–Lieb inequality was known for a system with two subsystems. Thus, we obtained a new entropic inequality of a form of the Araki–Lieb inequality, which can be checked in the experiments where the density matrix of qutrit is measured.

5 Deformed Subadditivity Condition Classical and Quantum

For any density matrix ρ, the quantum deformed entropy $S_q(\rho)$ reads (see, e.g. [8])

$$S_q(\rho) = -\operatorname{Tr}\left[\rho\left(\frac{\rho^{q-1}-1}{q-1}\right)\right]. \tag{43}$$

In the limit $q \to 1$, the deformed entropy is equal to the von Neumann entropy

$$\lim_{q\to 1} S_q(\rho) = -\mathrm{Tr}\,[\rho \ln \rho]\,. \tag{44}$$

For a bipartite system with subsystems 1 and 2 and the density matrix $\rho(1,2)$, one has the inequality, which is deformed subadditivity condition; it reads

$$-\,\mathrm{Tr}\left[\rho(1,2)\,\frac{\rho^{q-1}(1,2)-1}{q-1}\right] \le -\mathrm{Tr}\left[\rho(1)\,\frac{\rho^{q-1}(1)-1}{q-1}\right] - \mathrm{Tr}\left[\rho(2)\,\frac{\rho^{q-1}(2)-1}{q-1}\right], \tag{45}$$

where the density matrices of the subsystem states $\rho(1)$ and $\rho(2)$ are

$$\rho(1) = \mathrm{Tr}_2\rho(1,2), \qquad \rho(2) = \mathrm{Tr}_1\rho(1,2). \tag{46}$$

For $q \to 1$, inequality (45) is the subadditivity conditions for the von Neumann entropy

$$-\,\mathrm{Tr}\,[\rho(1,2)\ln\rho(1,2)] \le -\mathrm{Tr}\,[\rho(1)\ln\rho(1)] - \mathrm{Tr}\,[\rho(2)\ln\rho(2)]\,. \tag{47}$$

For any nonnegative $N \times N$-matrix ρ with $N = nm$ given in block form (34), where R_{jk} are $m \times m$-matrices, one has the inequality

$$-\,\mathrm{Tr}\left[\rho\,\frac{\rho^{q-1}-1}{q-1}\right] \le -\mathrm{Tr}\left[R_1\,\frac{R_1^{q-1}-1}{q-1}\right] - \mathrm{Tr}\left[R_2\,\frac{R_2^{q-1}-1}{q-1}\right], \tag{48}$$

where the $n \times n$-matrix R_1 has the form (35), and the $m \times m$-matrix R_2 has the form (36).

Thus one has an analog of the deformed subadditivity condition for the single qudit state with $j = (N-1)/2$. The $N \times N$-matrix ρ can be considered as a part of the $\tilde N \times \tilde N$-matrix $\tilde\rho$, if one uses the appropriate number of extra zero columns and rows, i.e., $\tilde\rho = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}$. In view of this, one has the general inequality for the matrix elements of the matrix ρ considering different product forms of the integer $\tilde N = \tilde n \tilde m$. This means that we can derive several different entropic inequalities starting from the $\tilde N \times \tilde N$-matrix $\tilde\rho$ and considering different matrices $\tilde R_1$ and $\tilde R_2$ based on the block form of the matrix $\tilde\rho$.

The density matrix ρ can be transformed, using unitary matrix u, to become

$$\rho \to \rho_u = u\rho u^+. \tag{49}$$

The matrix ρ_u which has the form (34) also satisfies the subadditivity condition (48), where the matrices R_1 and R_2 are replaced by the matrices $R_1(u)$ and $R_2(u)$. The matrices $R_1(u)$ and $R_2(u)$ are given by formulae (35) and (36), where the blocks

R_{jk} are replaced by the blocks $R_{jk}(u)$ obtained from the matrix ρ_u. Then one has a transformed inequality (48)

$$-\operatorname{Tr}\left[\rho\,\frac{\rho^{q-1}-1}{q-1}\right] \leq -\operatorname{Tr}\left[R_1(u)\,\frac{R_1^{q-1}(u)-1}{q-1}\right] - \operatorname{Tr}\left[R_2(u)\,\frac{R_2^{q-1}(u)-1}{q-1}\right], \tag{50}$$

where the left-hand side of the inequality contains the matrix ρ, but in right-hand side the matrices $R_1(u)$ and $R_2(u)$ depend on the unitary matrix u.

In the limit $q = 1$, one has the inequality

$$-\operatorname{Tr}[\rho \ln \rho] \leq -\operatorname{Tr}[R_1(u) \ln R_1(u)] - \operatorname{Tr}[R_2(u) \ln R_2(u)]. \tag{51}$$

This inequality is valid for an arbitrary unitary matrix u. One can introduce the quantum information, which depends on global unitary transform

$$I(u) = \operatorname{Tr}[\rho \ln \rho] - \operatorname{Tr}[R_1(u) \ln R_1(u)] - \operatorname{Tr}[R_2(u) \ln R_2(u)] \geq 0. \tag{52}$$

The minimum value of the sum of entropies

$$\Sigma(u_0) = -\operatorname{Tr}[R_1(u_0) \ln R_1(u_0)] - \operatorname{Tr}[R_2(u_0) \ln R_2(u_0)] \tag{53}$$

provides the minimum value of information

$$I(u_0) = \Sigma(u_0) - S, \tag{54}$$

where $S = -\operatorname{Tr}[\rho \ln \rho]$.

If the matrix ρ is the density matrix of a bipartite system $\rho(1,2)$, and R_1 and R_2 are the density matrices of the first and second subsystems, respectively, the quantum information is

$$I_q = -\operatorname{Tr}[R_1(u_{10}) \ln R_1(u_{10})] - \operatorname{Tr}[R_2(u_{20}) \ln R_2(u_{20})] + \operatorname{Tr}[\rho \ln \rho], \tag{55}$$

where $R_1(u_{10})$ and $R_2(u_{20})$ are the diagonalized density matrices R_1 and R_2, and u_{10} and u_{20} are local transforms such that $u = u_{10} \times u_{20}$. Thus, the difference of information $\Sigma(u_0) - I_q$ provides a characteristic of the correlations related to global and local transforms u_0 and $u_{10} \times u_{20}$. Analogous characteristics can be introduced using deformed information and the deformed subadditivity condition.

6 Conclusions

Concluding, we list our main results obtained in this paper.

We obtained new classical and quantum entropic inequalities for the systems without subsystems. The new inequalities have the same form as known inequalities for composite systems. For example, the Araki–Lieb inequality provides the relation of the von Neumann entropy of the quantum state of a bipartite system to the difference of the entropies of the subsystem states. We found an analogous inequality given by Eq. (42) for the single qudit state, e.g., we found the inequality for the qutrit state; this inequality can be checked experimentally. Other new inequalities are given by Eqs. (37), (39) and (52).

We formulated new entropic inequalities for quantum system tomograms, which use the properties of tomograms to be fair probability distributions. In view of the facts mentioned above, the information and entropic relations, which are known for classical probability distributions are also valid for quantum system states described by the tomographic probability distributions, including the q-entropic inequalities.

We obtained new q-entropic inequalities for tomograms of single qudit states, e.g., for $j = 3/2$. One can do the partition of the single system into more that two parts and get also new entropic inequalities, e. g. to obtain strong subadditivity condition for single qudit state [28]. The physical meaning of the entropic inequalities we obtained is related to the fact that they describe the properties of quantum correlations in the single qudit state connected with quantum fluctuations of the degrees of freedom like different spin projections in the same system in contrast to bipartite systems where the inequalities are related to the properties of quantum correlations of the degrees of freedom of different subsystems.

The existence of quantum correlations in a single system formally is completely analogous to the correlations in multipartite systems, that is demonstrated by the existence of the entropic inequalities found here. The inequalities can be used to elaborate the quantum resource for applications in quantum technologies. Such a resource is usually considered as the resource of quantum correlations in multipartite systems. In our study, we showed that a resource is available in the systems without subsystems due to quantum correlations of their degrees of freedom. This possibility was mentioned in [13]; it will be studied in a future publication.

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