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Structures of Three Types of Local Quantum Channels Based on Quantum Correlations

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Abstract In a bipartite quantum system, quantum states are classified as classically correlated (CC) and quantum correlated (QC) states, the later are important resources of quantum information and computation protocols. Since correlations of quantum states may vary under a quantum channel, it is necessary to explore the influence of quantum channels on correlations of quantum states. In this paper, we discuss CC-preserving, QC-breaking and strongly CC-preserving local quantum channels of the form $\Phi_1 \otimes \Phi_2$ and obtain the structures of these three types of local quantum channels. Moreover, we obtain a necessary and sufficient condition for a quantum state to be transformed into a CC state by a specific local channel $\Phi_1 \otimes \Phi_2$ in terms of the structure of the input quantum state. Lastly, as applications of the obtained results, we present a classification of local quantum channels $\Phi_1 \otimes \Phi_2$ and describe the quantum states which are transformed as CC ones by the corresponding local quantum channel.

Keywords Structure · Local quantum channel · Classical correlation · Quantum correlation

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1 Introduction and Preliminaries

Luo introduced in [1] a quantum-classical dichotomy in order to classify correlations in bipartite states, in which a state ρ in a bipartite system $\mathbb{C}^n \otimes \mathbb{C}^m$ is said to be classical correlated (shortly, CC) if there exist von Neumann measurements $\{\Pi_i^a\}$ and $\{\Pi_j^b\}$ consisting of rank-one orthogonal projections on $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, such that

$$\Pi(\rho) := \sum_{i,j} (\Pi_i^a \otimes \Pi_j^b)\rho(\Pi_i^a \otimes \Pi_j^b) = \rho,$$

and ρ is said to be quantum correlated (QC) if it is not CC. It was proved in [1] that a state ρ is CC if and only if it can be represented as

$$\rho = \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}|e_i\rangle\langle e_i| \otimes |f_j\rangle\langle f_j|,$$

where $\{p_{ij}\}$ is a probability distribution, $\{|e_i\rangle\}$ and $\{|f_j\rangle\}$ are some orthonormal bases for $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively.

Representation above of a CC state shows that every CC state is separable and therefore every entangled state must be a QC state. Thus, quantum correlations are more general than entanglement and then become important resources for a number of quantum information applications without entanglement and then have drawn much attentions [2–11]. Quantum correlation is an intrinsic aspect of quantum theory that enables the manifestation of several interesting phenomena beyond the realms of the classical world. The practical realization of quantum information and computation protocols by using quantum systems is severely challenged due to decoherence caused by the interaction of the system with the environment. Such interactions create undesirable quantum correlations between the system and the environment leading to information being scattered in the intractable Hilbert space of the environment. Therefore, the dynamics of quantum correlations makes us understand generation, breaking and preservation of quantum correlations in a composite quantum system. From a theoretical point of view, in order to characterize the dynamics, one has to study the behavior of quantum correlations under noisy channels (described by trace-preserving completely positive maps). Equivalently, we have to discuss what quantum channels can preserve, create or break quantum correlations.

In this direction, Streltsov et al. proved in [12] that a quantum channel Λ acting on a single qubit in a two-qubit system can create quantum correlations from some initially classically correlated states if and only if Λ is neither semiclassical (i.e., measurement map) nor unital. In other words, there exist some classically correlated states in a two-qubit system which are transformed by $\Lambda \otimes 1$ into QC states if and only if Λ is neither semiclassical nor unital. Consequently, for the qubit case, $\Lambda \otimes 1$ is classical correlation-preserving (CC-preserving) if and only if Λ is either semiclassical or unital. Furthermore, for higher-dimensional systems, they claimed that even unital channels may increase the amount of quantum correlations, for example, a local decoherence channel can generate quantum correlations. Gessner et al. proved in [13] that the states of nonzero discord can be created from zero discord states only by a single local operation and the set of these states has measure zero. In [14], Hu et al. proved that a local quantum channel $1 \otimes \Lambda$ can create quantum correlations if and only if Λ is not a commutativity preserving channel. It was also proved in [14] that for a qubit system, a commutativity preserving channel is either a completely decohering channel or a mixing channel, and for a qutrit system, a commutativity preserving channel is either a completely decohering channel or an isotropic channel. Furthermore, for any finite dimensional system, Guo and Hou in [15] proposed an explicit form of a commutativity preserving channel and presented a necessary and sufficient condition for the local creation of quantum discord, which improves the result proposed by Streltsov et al. in [12] for the qubit case. Based on Luo's definition in [1], Guo and Cao in [16] considered general local quantum channels $\Phi_1 \otimes \Phi_2$ preserving classical correlations, and proved that $\Phi_1 \otimes \Phi_2$ is CC-preserving if and only if either one of Φ_1 and Φ_2 is trace-type (i.e., it maps any state to the same one), or they are commutativity preserving. Also, the specific structure of a CC-preserving channel $\Phi_1 \otimes \Phi_2$ was obtained in [16] for a two-qubit system. Thus, it is necessary to proceed with the structures of CC-preserving channels for higher-dimensional case. This leads to the following question:

Question 1 *Which types of local quantum channels $\Phi_1 \otimes \Phi_2$ preserve classical correlations?*

Furthermore, according to [17], a quantum channel Λ is said to be a *quantum-correlation breaking channel* (or a *QC-type channel*) if the quantum channel $1 \otimes \Lambda$ turns any bipartite state into a quantum-classical state, and it was proved that a quantum channel Λ is a QC-type channel if and only if its Choi-Jamiołkowski state is a quantum-classical state if and only if it is a quantum-to-classical measurement map. However, for a general local quantum channel $\Phi_1 \otimes \Phi_2$, we need to study whether it can fully break quantum correlations, i.e. it turns every bipartite state into a CC state. If so, then we say that $\Phi_1 \otimes \Phi_2$ is *QC-breaking*. Clearly, if $1 \otimes \Lambda$ is QC-breaking, then Λ is a QC-type channel. The inverse is not valid. This leads to the following question:

Question 2 *Which types of local quantum channels $\Phi_1 \otimes \Phi_2$ break quantum correlations?*

Besides, we say that a CC-preserving local quantum channel $\Phi_1 \otimes \Phi_2$ is *strongly CC-preserving* if the final state is a CC state implies that the initial state is a CC one. It is easy to see that a strongly CC-preserving local quantum channel is CC-preserving in both directions. Equivalently, a strongly CC-preserving local quantum channel can preserve quantum correlations in both directions. As our acknowledge, so far there does not exist any results on strongly CC-preserving local quantum channels. This leads to the following question:

Question 3 *Which types of local quantum channels $\Phi_1 \otimes \Phi_2$ preserve classical correlations in both directions?*

In addition, the authors in [17] discussed a characterization of quantum channels from a different perspective by defining a set $CC(\Lambda)$ of those bipartite states ρ which are mapped to a CC state by $1 \otimes \Lambda$, but it was not yet a complete answer there. This leads to the following question:

Question 4 *Which states can be transformed in the same way as CC ones by a given local quantum channel* $\Phi_1 \otimes \Phi_2$*?*

The goal of this paper is to give the answers to these questions by establishing the structures of classical correlation-preserving, quantum correlation-breaking and strongly classical correlation-preserving local quantum channels, from which we will completely characterize the behavior of quantum correlations under the influence of a general local noisy channel $\Phi_1 \otimes \Phi_2$.

To begin our discussion, let us recall some notations and concepts. As usual, the C^*-algebra of all $k \times k$ complex matrices is denoted by $\mathcal{M}_k$, which is identified with the C^*-algebra $\mathcal{B}(\mathbb{C}^k)$ of all bounded linear operators on the Hilbert space $\mathbb{C}^k$. A positive semi-definite matrix of trace 1 in $\mathcal{M}_k$ is called a *state* on the system $\mathbb{C}^k$. The set of all states on $\mathbb{C}^k$ is denoted by $\mathcal{D}(\mathbb{C}^k)$. Throughout this paper, we denote by I_n the $n \times n$ identity matrix. Moreover, a quantum channel Φ on $\mathcal{M}_n$ is said to be a *measurement map* [4] if $\Phi(A) = \sum_i \mathrm{tr}(M_i A)|i\rangle\langle i|$ for all $A \in \mathcal{M}_n$, where $\{M_i\}$ is a POVM and $\{|i\rangle\}$ is an orthonormal basis for $\mathbb{C}^n$. Note that a measurement map was also called a QC channel in [18], which is an entanglement-breaking channel. Such a channel is realized by complete decoherence, after which every density matrix becomes a diagonal matrix. If there exists a state $\sigma \in \mathcal{D}(\mathbb{C}^n)$ such that $\Phi(A) = \mathrm{tr}(A)\sigma$ for all $A \in \mathcal{M}_n$, then we say that Φ is *trace-type* [16]. It is easy to prove that a measurement map is trace-type if and only if its component POVM $\{M_i\}$ satisfies $M_i = \lambda_i I_n$ for some probability distribution $\{\lambda_i\}$. Moreover, Φ is called an *isotropic channel* if it has the form $\Phi(A) = t\Gamma(A) + (1-t)\mathrm{tr}(A)\frac{I_n}{n}, \forall A \in \mathcal{M}_n$, where Γ is either a unitary operation $A \mapsto UAU^\dagger$ and $-\frac{1}{n-1} \le t \le 1$ or a map which is unitarily equivalent to a transpose $A \mapsto UA^TU^\dagger$ and $-\frac{1}{n-1} \le t \le \frac{1}{n+1}$. An isotropic channel Φ is said to be *nontrivial* if the parameter t is not zero. A depolarizing channel [19] on a system $\mathcal{M}_n$ is a special isotropic channel, which is defined as a convex combination D_ε of the identity map on $\mathcal{M}_n$ and the totally depolarizing channel given by $\Phi(A) = \mathrm{tr}(A)\frac{I_n}{n}$:

$$D_\varepsilon(A) = (1-\varepsilon)\mathrm{tr}(A)\frac{I_n}{n} + \varepsilon A, \forall \rho \in \mathcal{M}_n,$$

where $\varepsilon \in [0, 1]$. Clearly, a depolarizing channel D_ε is an example of nontrivial isotropic channels when $\varepsilon \in (0, 1]$, and the totally depolarizing channel D_0 is an example of trace-type channels. Moreover, a channel Φ is called a *completely decohering channel* if $\Phi(\mathcal{M}_n)$ is commutative. A map Φ on $\mathcal{M}_n$ is said to be *commutativity preserving* if it satisfies $A, B \in \mathcal{M}_n$, $[A, B] := AB - BA = 0 \Rightarrow [\Phi(A), \Phi(B)] = 0$, and it is said to be *commutativity preserving in both directions* if it satisfies $[A, B] = 0 \Leftrightarrow [\Phi(A), \Phi(B)] = 0$.

The paper is organized as follows. In Sect. 2, we give the structures of quantum channels that can preserve commutativity and then obtain structures of CC-preserving local quantum channels. In Sect. 3, we obtain the structures of QC-breaking local channels. In Sect. 4, we establish the structures of strongly CC-preserving local channels. Sect. 5 is devoted to the characterization of the sets of all quantum states being mapped to classical correlated ones by a specific local quantum channel. Summary and conclusions are given in Sect. 6. Moreover, the proofs of lemmas are given in Appendix.

2 Structures of CC-Preserving Local Quantum Channels

In this section, we give structures of CC-preserving local quantum channels. The authors in [16] discussed local quantum channels that preserve classical correlations and proved that if one of Φ_1 and Φ_2 is trace-type, then $\Phi_1 \otimes \Phi_2$ is CC-preserving since it maps any state to a product state. When Φ_1 and Φ_2 are two quantum channels which are not trace-type, the following lemma gives a qualitative characterization of a CC-preserving channel.

Lemma 2.1 [16, Theorem 3.2] *Let Φ_1 and Φ_2 be two quantum channels which are not trace-type. Then $\Phi_1 \otimes \Phi_2$ is CC-preserving if and only if Φ_i is commutativity preserving on states for $i = 1, 2$.*

By Lemma 2.1, it is necessary to discuss firstly quantum channels that preserve commutativity but not trace-type. For a qubit system, the structure of a commutativity preserving quantum channel is characterized as follows.

Lemma 2.2 [16, Lemma 3.3] *Let Φ be a quantum channel on $\mathcal{M}_2$ but not trace-type. Then the following are equivalent.*

- (i) *Φ is commutativity preserving.*
- (ii) *Either Φ is unital, that is, $\Phi(I_2) = I_2$, or $\Phi(I_2) \neq I_2$ and there exist two †-linear functionals f, g on $\mathcal{M}_2$ such that $\Phi(A) = f(A)I_2 + g(A)\Phi(I_2),\ \forall A \in \mathcal{M}_2$.*
- (iii) *Φ is unital or a measurement map with $\Phi(I_2) \neq I_2$.*

Lemma 2.1 and 2.2 allow us to state the following, which gives the structure of a CC-preserving local quantum channel $\Phi_1 \otimes \Phi_2$ on $\mathcal{M}_2 \otimes \mathcal{M}_2$.

Theorem 2.1 *Let $\Phi_1, \Phi_2 : \mathcal{M}_2 \to \mathcal{M}_2$ be two quantum channels which are not trace-type. Then $\Phi_1 \otimes \Phi_2$ is CC-preserving if and only if Φ_i is unital or a measurement map with $\Phi_i(I_2) \neq I_2$ for $i = 1, 2$.*

Let us continue the above analysis and study the case where $n \geq 3$. Recently, Guo and Hou in [15, Theorem 1] proved that Φ on $\mathcal{M}_n (n \geq 3)$ preserves commutativity if and only if Φ is either a completely decohering channel or a nontrivial isotropic one. The following lemma shows that a completely decohering channel coincides with a measurement map, and then gives the structure of a completely decohering channel.

Lemma 2.3 *Let Φ be a quantum channel on $\mathcal{M}_n$. Then the following statements are equivalent.*

- (1) *Φ is a completely decohering channel.*
- (2) *Φ is a measurement map.*
- (3) *$[\Phi(\sigma_1), \Phi(\sigma_2)] = 0$ for all $\sigma_1, \sigma_2 \in \mathcal{D}(\mathbb{C}^n)$.*

From [15, Theorem 1] and Lemma 2.3, we obtain the following corollary, which gives the structure of a commutativity preserving quantum channel.

Corollary 2.1 *A quantum channel Φ on $\mathcal{M}_n (n \geq 3)$ is a commutativity preserving quantum channel if and only if it has one of the following forms:*

(a) $\Phi(A) = \sum_k \mathrm{tr}(M_k A)|k\rangle\langle k|$ *for all* $A \in \mathcal{M}_n$*, where* $\{M_k\}$ *is a POVM and* M_k *is not scalar multiplication for some* k*;*
(b) $\Phi(A) = tUAU^\dagger + \frac{1-t}{n}\mathrm{tr}(A)I_n$ *for all* $A \in \mathcal{M}_n$ *where* U *is unitary and* $-\frac{1}{n-1} \leq t \leq 1$ *and* $t \neq 0$*;*
(c) $\Phi(A) = tUA^TU^\dagger + \frac{1-t}{n}\mathrm{tr}(A)I_n$ *for all* $A \in \mathcal{M}_n$*, where* U *is unitary and* $-\frac{1}{n-1} \leq t \leq \frac{1}{n+1}$ *and* $t \neq 0$.
(d) $\Phi(A) = \mathrm{tr}(A)\sigma$ *for all* $A \in \mathcal{M}_n$ *and some* $\sigma \in \mathcal{D}(\mathbb{C}^n)$.

Combining Corollary 2.1 with Lemma 2.3 yields the following result, which gives the structure of a CC-preserving local quantum channel $\Phi_1 \otimes \Phi_2$ on $\mathcal{M}_n \otimes \mathcal{M}_m$ for the case where $n, m \geq 3$.

Theorem 2.2 *Let* $n, m \geq 3$*,* Φ_1 *and* Φ_2 *be quantum channels on* $\mathcal{M}_n$ *and* $\mathcal{M}_m$*, respectively, which are not trace-type. Then* $\Phi_1 \otimes \Phi_2$ *is CC-preserving if and only if* Φ_i *has one of the forms (a), (b) and (c) for each* $i = 1, 2$.

For example, when Φ_1 is a depolarizing channel and Φ_2 is a complete decoherence channel, $\Phi_1 \otimes \Phi_2$ is CC-preserving.

3 Structures of QC-Breaking Local Quantum channels

In this section, we give the structures of QC-breaking local quantum channels. We first present the following two lemmas, which will be used in the proof of Theorem 3.1.

Lemma 3.1 [7, Remark] *Let* $\rho = \rho_1 \otimes \rho_2$ *and* $\sigma = \sigma_1 \otimes \sigma_2$*. Then* $\rho_\lambda := \lambda\rho + (1-\lambda)\sigma (\lambda \in (0, 1))$ *is CC if and only if at least one of the following cases holds: (i)* $[\rho_1, \sigma_1] = 0$ *and* $[\rho_2, \sigma_2] = 0$*; (ii)* $\rho_1 = \sigma_1$*; (iii)* $\rho_2 = \sigma_2$.

Lemma 3.2 [16, Theorem 2.1] *A state* $\rho \in \mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^m)$ *is CC if and only if* ρ *admits a representation* $\rho = \sum_{i=1}^s A_i \otimes B_i$*, where* $\{A_i\}$*,* $\{B_i\}$ *are both commuting families of normal operators.*

Based on these lemmas, we obtain the following, which gives the structure of a QC-breaking local quantum channel $\Phi_1 \otimes \Phi_2$.

Theorem 3.1 *Let* Φ_1 *and* Φ_2 *be quantum channels on* $\mathcal{M}_n$ *and* $\mathcal{M}_m$*, respectively. Then* $\Phi_1 \otimes \Phi_2$ *is QC-breaking if and only if either one of* Φ_1 *and* Φ_2 *is trace-type or both* Φ_1 *and* Φ_2 *are measurement maps.*

Proof Necessity Suppose that $\Phi_1 \otimes \Phi_2$ is QC-breaking, then $(\Phi_1 \otimes \Phi_2)(\rho)$ is a CC state for every state ρ in $\mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^m)$. Assume that Φ_1 and Φ_2 are not trace-type. Then we have to prove that both Φ_1 and Φ_2 are measurement maps. Suppose that Φ_1 is not a measurement map. Then by Lemma 2.3 we can find two states $\sigma_1, \sigma_2 \in \mathcal{D}(\mathbb{C}^n)$ such that $[\Phi_1(\sigma_1), \Phi_1(\sigma_2)] \neq 0$. Take any states $\rho_1, \rho_2 \in \mathcal{D}(\mathbb{C}^m)$ and put

$$X = \frac{1}{2}\sigma_1 \otimes \rho_1 + \frac{1}{2}\sigma_2 \otimes \rho_2.$$

Then $X \in \mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^m)$ and

$$(\Phi_1 \otimes \Phi_2)(X) = \frac{1}{2}\Phi_1(\sigma_1) \otimes \Phi_2(\rho_1) + \frac{1}{2}\Phi_1(\sigma_2) \otimes \Phi_2(\rho_2),$$

which is CC since $\Phi_1 \otimes \Phi_2$ is QC-breaking. By Lemma 3.1, we see that $\Phi_2(\rho_1) = \Phi_2(\rho_2)$. This concludes that Φ_2 is trace-type, which contradicts the assumption. Hence, Φ_1 is a measurement map. Similarly, Φ_2 is also a measurement map.

Sufficiency Suppose that one of Φ_1 and Φ_2 is trace-type, then $(\Phi_1 \otimes \Phi_2)(X)$ is a product state for any state $X \in \mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^m)$ and therefore a CC state. Thus, $\Phi_1 \otimes \Phi_2$ is QC-breaking. Suppose that both Φ_1 and Φ_2 are measurement maps, then for any state $\rho = \sum_i A_i \otimes B_i \in \mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^m)$, $(\Phi_1 \otimes \Phi_2)(\rho) = \sum_i \Phi_1(A_i) \otimes \Phi_2(B_i)$. Lemma 2.3 yields that $\{\Phi_1(A_i)\}$ and $\{\Phi_2(B_i)\}$ are commuting families of normal operators since Φ_k is †-preserving. It follows from Lemma 3.2 that $(\Phi_1 \otimes \Phi_2)(\rho)$ is CC. This shows that $\Phi_1 \otimes \Phi_2$ is QC-breaking. □

For example, when Φ_1 and Φ_2 are complete decoherence channels or one of Φ_1 and Φ_2 is the totally depolarizing channel, $\Phi_1 \otimes \Phi_2$ is QC-breaking.

4 Structures of Strongly CC-Preserving Local Quantum Channels

To get the structure of a strongly CC-preserving local quantum channel, we need to prove the following lemmas.

Lemma 4.1 *A nontrivial isotropic channel Φ on $\mathcal{M}_n$ is a linear bijection.*

Lemma 4.2 *Let $\{X_i\}_{i=1}^k \subset \mathcal{M}_n$ be a linearly independent family and $\{Y_i\}_{i=1}^k \subset \mathcal{M}_m$. Then $\sum_{i=1}^k X_i \otimes Y_i = 0$ if and only if $Y_i = 0 (i = 1, 2, \ldots, k)$.*

Lemma 4.3 *If Φ_1 and Φ_2 are linear bijections on $\mathcal{M}_n$ and $\mathcal{M}_m$, respectively, then $\Phi_1 \otimes \Phi_2$ is a linear bijection on $\mathcal{M}_n \otimes \mathcal{M}_m$.*

Based on these lemmas, we can prove the following lemma, which gives a qualitative characterization of a strongly CC-preserving channel $\Phi_1 \otimes \Phi_2$ and will be used in the proof of Theorem 4.1.

Lemma 4.4 *Let Φ_1 and Φ_2 be quantum channels on $\mathcal{M}_n$ and $\mathcal{M}_m$, respectively. Then $\Phi_1 \otimes \Phi_2$ is strongly CC-preserving if and only if Φ_1 and Φ_2 are commutativity preserving in both directions.*

With these lemmas, we give the structure of a strongly CC-preserving local quantum channel as follows.

Theorem 4.1 *Let $n, m \geq 3$ and let Φ_1 and Φ_2 be quantum channels on $\mathcal{M}_n$ and $\mathcal{M}_m$, respectively. Then $\Phi_1 \otimes \Phi_2$ is strongly CC-preserving if and only if Φ_1 and Φ_2 are nontrivial isotropic channels.*

Proof Necessity Suppose that $\Phi_1 \otimes \Phi_2$ is strongly CC-preserving. Then we see from Lemma 4.4 that Φ_1 and Φ_2 are commutativity preserving in both directions. Suppose that Φ_1 is not an nontrivial isotropic channel, then we see from Corollary 2.1 that Φ_2 is a measurement map and so $\mathrm{ran}(\Phi_1)$ is commutative. Thus, Φ_1 is not commutativity preserving in both directions, a contradiction. Similarly, one can show that Φ_2 is an nontrivial isotropic channel. Therefore, both Φ_1 and Φ_2 are nontrivial isotropic channels.

Sufficiency Suppose that Φ_1 and Φ_2 are nontrivial isotropic channels on $\mathcal{M}_n$ and $\mathcal{M}_m$, respectively. It is easy to check that both Φ_1 and Φ_2 are commutativity preserving in both directions. It follows from Lemma 4.4 that $\Phi_1 \otimes \Phi_2$ is strongly CC-preserving. □

For example, when Φ_1 and Φ_2 are depolarizing channels but not totally depolarizing, $\Phi_1 \otimes \Phi_2$ is strongly CC-preserving.

5 Characterization of the CC-Set of a Local Quantum Channel

In this section, let us consider the properties of the CC-set $CC(\Phi_1 \otimes \Phi_2)$ of a given local quantum channel $\Phi_1 \otimes \Phi_2$, which is defined as the set of all states that are transformed into CC ones by $\Phi_1 \otimes \Phi_2$. By definition, we see that $\Phi_1 \otimes \Phi_2$ is CC-preserving if and only if $CC(\Phi_1 \otimes \Phi_2) \supset CC(\mathbb{C}^n \otimes \mathbb{C}^m)$, the set of all CC states on $\mathbb{C}^n \otimes \mathbb{C}^m$; it is strongly CC-preserving if and only if $CC(\Phi_1 \otimes \Phi_2) = CC(\mathbb{C}^n \otimes \mathbb{C}^m)$ and it is QC-breaking if and only if $CC(\Phi_1 \otimes \Phi_2) = \mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^m)$. Now, we discuss the case where $\Phi_1 \otimes \Phi_2$ is CC-preserving but neither strongly CC-preserving nor QC-breaking. Combining Theorem 2.2, 3.1 and 4.1, we only need consider the case that one of Φ_1 and Φ_2 is a nontrivial isotropic channel and the other is a measurement map.

Obviously, for any channels Φ_1, Φ_2 and any state $\rho \in \mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^m)$, we have $(\Phi_1 \otimes \Phi_2)(\rho) = (\Phi_1 \otimes 1_m)((1_n \otimes \Phi_2)(\rho))$. When Φ_1 is a nontrivial isotropic channel on $\mathcal{M}_n$, we see that 1_m and Φ_1 are commutativity preserving in both directions and it follows from Lemma 4.4 that $(\Phi_1 \otimes \Phi_2)(\rho)$ is CC if and only if $(1_n \otimes \Phi_2)(\rho)$ is CC. Therefore, $CC(\Phi_1 \otimes \Phi_2) = CC(1_n \otimes \Phi_2)$ provided that Φ_1 is a nontrivial isotropic channel. Similarly, $CC(\Phi_1 \otimes \Phi_2) = CC(\Phi_1 \otimes 1_m)$ provided that Φ_2 is a nontrivial isotropic channel.

With these observations, suppose that Φ_1 is a nontrivial isotropic channel and Φ_2 is a measurement map, we will give a characterization of $CC(\Phi_1 \otimes \Phi_2)$. Firstly, we introduce some notations. For any $\rho \in \mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^m)$, and any orthonormal bases $e := \{|e_i\rangle\}$, $f := \{|f_k\rangle\}$ for $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, we have

$$\rho = \sum_{k\ell} A_{k\ell}(\rho) \otimes |f_k\rangle\langle f_\ell| = \sum_{ij} |e_i\rangle\langle e_j| \otimes B_{ij}(\rho),$$

where

$$A_{k\ell}(\rho) = \sum_{ij} \langle e_i|\langle f_k|\rho|e_j\rangle|f_\ell\rangle \cdot |e_i\rangle\langle e_j|, \quad B_{ij}(\rho) = \sum_{k\ell} \langle e_i|\langle f_k|\rho|e_j\rangle|f_\ell\rangle \cdot |f_k\rangle\langle f_\ell|,$$

called the *component operators* of ρ.

With these notations, it was proved in [7, Corollary 2.1] that $\rho \in \mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^m)$ is CC if and only if $\{A_{k\ell}(\rho)\}$ and $\{B_{ij}(\rho)\}$ are commuting families of normal operators. Since ρ is hermitian, we observe that $\overline{\langle e_i|\langle f_k|\rho|e_j\rangle|f_\ell\rangle} = \langle e_j|\langle f_\ell|\rho|e_i\rangle|f_k\rangle$ and so

$$\begin{aligned}(A_{k\ell}(\rho))^\dagger &= \sum_{ij} \overline{\langle e_i|\langle f_k|\rho|e_j\rangle|f_\ell\rangle} \cdot |e_j\rangle\langle e_i| \\ &= \sum_{ij} \langle e_j|\langle f_\ell|\rho|e_i\rangle|f_k\rangle \cdot |e_j\rangle\langle e_i| = A_{\ell k}(\rho),\end{aligned}$$

and $\left(B_{ij}(\rho)\right)^\dagger = B_{ji}(\rho)$, similarly. Thus, when $\{A_{k\ell}(\rho)\}$ is a commuting family, the operators $A_{k\ell}(\rho)$ are all normal and when $\{B_{ij}(\rho)\}$ is a commuting family, the operators $B_{ij}(\rho)$ are normal. Hence, $\{A_{k\ell}(\rho)\}$ and $\{B_{ij}(\rho)\}$ are commuting families if and only if $\{A_{k\ell}(\rho)\}$ and $\{B_{ij}(\rho)\}$ are commuting families of normal operators if and only if ρ is CC. From this observation, we obtain the following lemma, which will be used in the proof of Theorem 5.1 below.

Lemma 5.1 *Let $e = \{|e_i\rangle\}$ and $f = \{|f_k\rangle\}$ be any orthonormal bases for $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Then $\rho \in \mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^m)$ is CC if and only if $\{A_{k\ell}(\rho)\}$ and $\{B_{ij}(\rho)\}$ are commuting families.*

With this lemma, we have the following theorem, which gives a necessary and sufficient condition for a state ρ and P^+ to be transformed into a CC state under a local quantum channel $\Phi_1 \otimes \Phi_2$, respectively.

Theorem 5.1 *Let Φ_1 be a nontrivial isotropic channel on $\mathcal{M}_n$ and Φ_2 a measurement map on $\mathcal{M}_m$ with $\Phi_2(X) = \sum_k \mathrm{tr}(M_k X) \cdot |e_k\rangle\langle e_k|$, $\forall X \in \mathcal{M}_m$.*

- *(i) Let $\rho = \sum_{ij} D_{ij}(\rho) \otimes E_{ij} \in \mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^n)$ with $E_{ij} = |e_i\rangle\langle e_j|$. Then $\rho \in CC(\Phi_1 \otimes \Phi_2)$ if and only if $\{A_k(\rho)\}$ is a commuting family, where $A_k(\rho) = \sum_{i,j} \mathrm{tr}(M_k E_{ij}) \cdot A_{ij}$.*
- *(ii) When $m = n$, let $|\beta\rangle = \frac{1}{\sqrt{n}} \sum_i |e_i\rangle|e_i\rangle$ be a maximally entangled state in $\mathbb{C}^n \otimes \mathbb{C}^n$ and $P_+ = |\beta\rangle\langle\beta| = \frac{1}{n}\sum_{i,j} E_{ij} \otimes E_{ij}$. Then $P_+ \in CC(\Phi_1 \otimes \Phi_2)$ if and only if $P_+ \in CC(\Phi_2 \otimes \Phi_1)$ if and only if $\{M_k\}$ is a commuting family.*

Proof (i) Directly computing shows that $\rho' := (1_n \otimes \Phi_2)(\rho) = \sum_k A_k(\rho) \otimes |e_k\rangle\langle e_k|$. So, the component operators $A_{k\ell}(\rho')$ and $B_{ij}(\rho')$ of ρ' satisfy

$$\begin{aligned}A_{k\ell}(\rho') &= \delta_{k,\ell} A_k(\rho), \\ B_{ij}(\rho') &= \sum_{k,\ell} \langle e_i|\langle e_k|\rho'|e_j\rangle|e_\ell\rangle \cdot |e_k\rangle\langle e_\ell| \\ &= \sum_k \langle e_i|\langle e_k|\rho'|e_j\rangle|e_k\rangle \cdot |e_k\rangle\langle e_k|,\end{aligned}$$

this implies that $\{B_{ij}(\rho')\}$ is clearly a commuting family. Thus, we see from Lemma 5.1 that ρ' is a CC state if and only if $\{A_{k\ell}(\rho')\}$ is a commuting family if and only if $\{A_k(\rho)\}$ is a commuting family.

(ii) First, by using the fact that the swap operation $\Phi : X \otimes Y \mapsto Y \otimes X$ is CC-preserving in both directions with $\Phi(P_+) = P_+$, we see that $(\Phi_1 \otimes \Phi_2)(P_+)$ is CC if and only if $(\Phi_2 \otimes \Phi_1)(P_+)$ is CC. Next, let us use conclusion (i) to complete the proof of (ii). To do this, by an easy computation, we have

$$(1_n \otimes \Phi_2)(P_+) = \frac{1}{n}\sum_k \left(\sum_{i,j} \mathrm{tr}(M_k E_{ij})E_{ij}\right) \otimes |e_k\rangle\langle e_k| = \frac{1}{n}\sum_k M_k^T \otimes |e_k\rangle\langle e_k|.$$

Therefore, $(1_n \otimes \Phi_2)(P_+)$ is CC if and only if $\{M_k\}$ is a commuting family. □

Remark 1 Let Φ_1 be any quantum channel on $\mathcal{M}_n$ and Φ_2 a measurement map on $\mathcal{M}_m$ with $\Phi_2(X) = \sum_k \mathrm{tr}(M_k X)|e_k\rangle\langle e_k|$. For any state $\rho = \sum_{ij} D_{ij}(\rho) \otimes E_{ij} \in \mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^n)$ with $E_{ij} = |e_i\rangle\langle e_j|$, we define $A_k(\rho) = \sum_{i,j} \mathrm{tr}(M_k E_{ij})D_{ij}(\rho)$ and let $G(\Phi_1 \otimes \Phi_2)$ be the set of all states $\rho = \sum_{ij} D_{ij}(\rho) \otimes E_{ij}$ such that $[A_k(\rho), A_j(\rho)] = 0$ for all k, j. Then Theorem 5.1 (i) tells us that when Φ_1 is a nontrivial isotropic channel on $\mathcal{M}_n$ and Φ_2 is a measurement map on $\mathcal{M}_m$, we have $CC(\Phi_1 \otimes \Phi_2) = G(\Phi_1 \otimes \Phi_2)$.

Remark 2 From Theorem 5.1 (ii), $P_+ \in CC(\Phi_1 \otimes \Phi_2)$ if and only if $\{M_k\}$ is a commuting family. However, the commutativity of the family $\{M_i\}$ does not imply that $\Phi_1 \otimes \Phi_2$ is QC-breaking. For example, let $\{|0\rangle, |1\rangle\}$ be the canonical orthonormal basis for $\mathbb{C}^2$ and $M_0 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1|$, $M_1 = \frac{1}{2}|0\rangle\langle 0| + \frac{2}{3}|1\rangle\langle 1|$. Define $\Phi_2(X) = \sum_{i=0}^1 \mathrm{tr}(M_i X)|i\rangle\langle i|, \forall X \in \mathcal{M}_2$, then we get a measurement map Φ_2 on $\mathcal{M}_2$. Put

$$\rho = \frac{1}{2}A \otimes |0\rangle\langle 0| + \frac{1}{2}B \otimes |1\rangle\langle 1| \in \mathcal{D}(\mathbb{C}^3 \otimes \mathbb{C}^2)$$

where $A, B \in \mathcal{D}(\mathbb{C}^3)$ with $[A, B] \neq 0$. We know from Lemma 3.1 that ρ is not CC. We compute that the matrices $D_{ij}(\rho)$ and $A_k(\rho)$ in Theorem 5.1 (i) are as follows:

$$D_{00}(\rho) = \frac{1}{2}A,\ D_{11}(\rho) = \frac{1}{2}B,\ D_{01}(\rho) = D_{10}(\rho) = 0,$$
$$A_0(\rho) = \frac{1}{4}A + \frac{1}{6}B,\ A_1(\rho) = \frac{1}{4}A + \frac{1}{3}B.$$

Since $[A_0(\rho), A_1(\rho)] = \frac{1}{24}[A, B] \neq 0$, it follows from Theorem 5.1 (i) that $\rho \notin CC(1_3 \otimes \Phi_2)$. This shows that $\mathrm{id}_3 \otimes \Phi_2$ is not QC-breaking, while $\{M_0, M_1\}$ is a commuting family and so $P_+ \in G(1_3 \otimes \Phi_2)$.

Combining the results in Sects. 2–4 with Theorem 5.1 and Remark 5.1, we get a classification (Case 1 and Case 2 below) of local quantum channels and find out the corresponding CC-set $CC(\Phi_1 \otimes \Phi_2)$ of $\Phi_1 \otimes \Phi_2$ for the case where $m, n \geq 3$ as follows.

Case 1. Both of Φ_1 and Φ_2 are commutativity preserving (CP). In this case, $\Phi_1 \otimes \Phi_2$ has just the following sixteen types: $(x) \otimes (y)$ where $x, y \in \{a, b, c, d\}$ (please refer to Corollary 2.1). See Table 1 below.

Case 2. One of Φ_1 and Φ_2 is not commutativity preserving (NCP). In this case, $\Phi_1 \otimes \Phi_2$ has just the following five types: NCP $\otimes$ NCP, NCP $\otimes$ (d), NCP $\otimes$ Not(d), $(d) \otimes$ NCP and Not$(d) \otimes$ NCP. See Table 2 below.

Table 1 The types of $\Phi_1 \otimes \Phi_2$ and the corresponding CC-set $CC(\Phi_1 \otimes \Phi_2)$

$CC(\Phi_1 \otimes \Phi_2)$		Φ_1			
		(a)	(b)	(c)	(d)
Φ_2	(a)	$\mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^m)$	$G(\Phi_1 \otimes \Phi_2)$	$G(\Phi_1 \otimes \Phi_2)$	$\mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^m)$
	(b)	$G(\Phi_1 \otimes \Phi_2)$	$CC(\mathbb{C}^n \otimes \mathbb{C}^m)$	$CC(\mathbb{C}^n \otimes \mathbb{C}^m)$	$\mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^m)$
	(c)	$G(\Phi_1 \otimes \Phi_2)$	$CC(\mathbb{C}^n \otimes \mathbb{C}^m)$	$CC(\mathbb{C}^n \otimes \mathbb{C}^m)$	$\mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^m)$
	(d)	$\mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^m)$	$\mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^m)$	$\mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^m)$	$\mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^m)$

Table 2 The types of $\Phi_1 \otimes \Phi_2$ and the corresponding CC-set $CC(\Phi_1 \otimes \Phi_2)$

$CC(\Phi_1 \otimes \Phi_2)$			Φ_1		
			NCP	CP	
				(d)	Not(d)
Φ_2	NCP		Unknown	$\mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^m)$	Unknown
	CP	(d)	$\mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^m)$	Impossible	Impossible
		Not(d)	Unknown	Impossible	Impossible

6 Summary and Conclusions

Motivated by the fact that correlations of quantum states may change under local quantum channels, depending on the type of channels and the type of input states, we have considered three types of general local quantum channels in the form of $\Phi_1 \otimes \Phi_2$-(i) the *CC-preserving* channels, which preserve classical correlations by turning a classically correlated state into a classically correlated one, (ii) the *QC-breaking* channels, which fully break quantum correlations by turning any state into a classically correlated one and (iii) the *strongly CC-preserving* channels, which preserve classical correlations in both directions. For any $n \otimes m$ systems, we have shown that when $n, m \geq 3$, $\Phi_1 \otimes \Phi_2$ is CC-preserving if and only if Φ_i is a nontrivial isotropic channel or measurement map for each $i = 1, 2$ (e.g., Φ_1 is a depolarizing channel and Φ_2 is a complete decoherence channel); equivalently, it can create quantum correlations from an input CC state if and only if one of Φ_1 and Φ_2 is neither a nontrivial isotropic channel nor measurement map. We have also proved that $\Phi_1 \otimes \Phi_2$ is QC-breaking if and only if either one of Φ_1 and Φ_2 is trace-type (*i.e.*, mapping any state to the same one), or both Φ_1 and Φ_2 are measurement maps, in that case, $\Phi_1 \otimes \Phi_2$ can not create quantum correlations from any initial state (e.g., Φ_1 and Φ_2 are complete decoherence channels or one of Φ_1 and Φ_2 is the totally depolarizing channel). We have further proved that $\Phi_1 \otimes \Phi_2$ is strongly CC-preserving if and only if both Φ_1 and Φ_2 are nontrivial isotropic channels, in that case, $\Phi_1 \otimes \Phi_2$ preserves quantum correlations in both directions, and so $\Phi_1 \otimes \Phi_2$ can create quantum correlations from only a quantum correlated state (e.g., Φ_1 and Φ_2 are depolarizing channels but not totally depolarizing). According to these results, we have presented that a classification of local quantum channels based on the influence on commutativity of Φ_1 and Φ_2, and obtained the corresponding set of bipartite states that are mapped into the classically correlated form by $\Phi_1 \otimes \Phi_2$.

It is remarkable to point out that our findings also apply to the situation where one wants to perform local operations on a composite quantum system with the aim of creating or preserving quantum (classical) correlations. We believe that our results are useful for the storage, preparation and generation of quantum correlations in practical applications.

Lastly, our discussion in Sect. 5 leads to an interesting question for further study: when one of Φ_1 and Φ_2 is not commutativity preserving and the other is not trace-type, how to characterize the set of all states which are transformed into CC states by $\Phi_1 \otimes \Phi_2$?

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Appendix: Proofs of Lemmas

Proof of Lemma 2.3 (1) $\Rightarrow$ (2) : Suppose that $\Phi(\mathcal{M}_n)$ is commutative, then the operators in $\Phi(\mathcal{M}_n)$ are all normal and pairwise commute. Assume that $\Phi(\mathcal{M}_n) = \text{span}\{A_1, A_2, \ldots, A_m\}$, $m = \dim(\Phi(\mathcal{M}_n))$, where $[A_i, A_j] = 0$ for $i \neq j$, then there exists an orthonormal basis $\{|k\rangle\}_{k=1}^n$ for $\mathbb{C}^n$ such that $A_j = \sum_{k=1}^n \lambda_{kj}|k\rangle\langle k|$ for all $j = 1, 2, \ldots, m$. For any $A \in \mathcal{M}_n$, there exists a unique sequence $\{c_j(A)\}_{j=1}^m$ of complex numbers such that

$$\Phi(A) = \sum_{j=1}^{m} c_j(A)A_j = \sum_{k=1}^{n}\sum_{j=1}^{m} c_j(A)\lambda_{kj}|k\rangle\langle k| = \sum_{k=1}^{n} f_k(A)|k\rangle\langle k|,$$

where $f_k(A) = \sum_{j=1}^m c_j(A)\lambda_{kj}$ for all k. Clearly, $f_k(A) = \langle k|\Phi(A)|k\rangle = \text{tr}(\Phi^\dagger(|k\rangle\langle k|)A)$, where $\Phi^\dagger$ denotes the dual map of Φ with respect to the Hilbert-Schmidt inner product on $\mathcal{M}_n$. Thus,

$$\Phi(A) = \sum_{k=1}^{n} \text{tr}(\Phi^\dagger(|k\rangle\langle k|)A)|k\rangle\langle k|, \quad \forall A \in \mathcal{M}_n.$$

It is easy to see that $\{\Phi^\dagger(|k\rangle\langle k|)\}$ is a POVM for $\mathbb{C}^n$. This shows that Φ is a measurement map.

(2) $\Rightarrow$ (1) and (1) $\Rightarrow$ (3): Evidently.

(3) $\Rightarrow$ (1) : Let (3) hold. Then $[\Phi(\sigma_1), \Phi(\sigma_2)] = 0$ for all positive operators σ_1, σ_2 on $\mathbb{C}^n$. By using the spectral theorem we see that $[\Phi(\sigma_1), \Phi(\sigma_2)] = 0$ for all Hermitian operators σ_1, σ_2 on $\mathbb{C}^n$. So, $[\Phi(\sigma_1), \Phi(\sigma_2)] = 0$ for all operators σ_1, σ_2 on $\mathbb{C}^n$. □

Proof of Lemma 4.1 To show this, without loss of generality, we may assume that $\Phi(A) = tUAU^\dagger + \frac{1-t}{n}\mathrm{tr}(A)I_n$ for all $A \in \mathcal{M}_n$ where U is a unitary matrix, $-\frac{1}{n-1} \leq t \leq 1$ and $t \neq 0$. Clearly, for any $Y \in \mathcal{M}_n$, the matrix $X = \frac{1}{t}(U^\dagger YU - \frac{1-t}{n}\mathrm{tr}(Y)I_n)$ satisfies $\Phi(X) = Y$. This implies that Φ is surjective. On the other hand, let $\Phi(A) = \Phi(B)$, that is, $tU(A-B)U^\dagger + \frac{1-t}{n}\mathrm{tr}(A-B)I_n = 0$. Then we have $A - B = \frac{t-1}{tn}\mathrm{tr}(A-B)I_n$ since $t \neq 0$. Now we take trace operation and get that $(t-1)\mathrm{tr}(A-B) = t \cdot \mathrm{tr}(A-B)$. This shows that $\mathrm{tr}(A-B) = 0$ and thus $\mathrm{tr}(A) = \mathrm{tr}(B)$. Since $\Phi(A) = \Phi(B)$, we get that $tUAU^\dagger = tUBU^\dagger$ and so $A = B$. Hence, Φ is injective. □

Proof of Lemma 4.2 The "if part" is clear. We only need to prove the "only if part". Let $\sum_{i=1}^k X_i \otimes Y_i = 0$. Then for any pure states $|c\rangle, |d\rangle \in \mathbb{C}^m$ we have

$$\mathrm{tr}_2\left(\left(\sum_{i=1}^k X_i \otimes Y_i\right)(I_n \otimes |d\rangle\langle c|)\right) = \sum_{i=1}^k \langle c|Y_i|d\rangle X_i = 0.$$

Since $\{X_i\}_{i=1}^k$ is a linearly independent family, $\langle c|Y_i|d\rangle = 0$ for all i and all $|c\rangle, |d\rangle \in \mathbb{C}^m$. That is, $Y_i = 0$ for all $i \in \{1, 2, \ldots, k\}$. □

Proof of Lemma 4.3 For any $\rho = \sum_i A_i \otimes B_i$, there exist C_i, D_i such that $\Phi_1(C_i) = A_i$ and $\Phi_2(D_i) = B_i$ for all i. Put $\sigma = \sum_i C_i \otimes D_i$, then $(\Phi_1 \otimes \Phi_2)(\sigma) = \rho$ and thus $\Phi_1 \otimes \Phi_2$ is surjective. On the other hand, for any σ_1, σ_2 in $\mathcal{M}_n \otimes \mathcal{M}_m$, we write $\sigma_1 = \sum_{i,j} E_{ij} \otimes A_{ij}$, $\sigma_2 = \sum_{i,j} E_{ij} \otimes B_{ij}$, where $E_{ij} = |i\rangle\langle j|$ and $\{|i\rangle\}$ is an orthonormal basis for $\mathbb{C}^n$. Suppose that $(\Phi_1 \otimes \Phi_2)(\sigma_1) = (\Phi_1 \otimes \Phi_2)(\sigma_2)$, then

$$\sum_{i,j} \Phi_1(E_{ij}) \otimes [\Phi_2(A_{ij}) - \Phi_2(B_{ij})] = 0.$$

Since $\{E_{ij}\}$ is a Hamel basis for $\mathcal{M}_n$ and Φ_1 is a linear isomorphism, $\{\Phi_1(E_{ij})\}$ is a linearly independent set. By Lemma 4.2, we have $\Phi_2(A_{ij}) - \Phi_2(B_{ij}) = 0$ for all i, j, so $A_{ij} = B_{ij}$ for all i, j since Φ_2 is also a bijection. Therefore, $\sigma_1 = \sigma_2$. Thus, $\Phi_1 \otimes \Phi_2$ is a linear bijection. □

Proof of Lemma 4.4 *Necessity* Suppose that $\Phi_1 \otimes \Phi_2$ is strongly CC-preserving. Then Φ_1 and Φ_2 are commutativity preserving on states and then commutativity preserving. Assume that Φ_2 is not commutativity preserving in both directions. Then there exist $\rho_2, \sigma_2 \in \mathcal{D}(\mathbb{C}^m)$ such that $[\rho_2, \sigma_2] \neq 0$ but $[\Phi_2(\rho_2), \Phi_2(\sigma_2)] = 0$. Choose $\rho_1, \sigma_1 \in \mathcal{D}(\mathbb{C}^n)$ such that $\rho_1 \neq \sigma_1$ and $[\rho_1, \sigma_1] = 0$, and let $\rho = \frac{1}{2}\rho_1 \otimes \rho_2 + \frac{1}{2}\sigma_1 \otimes \sigma_2$. Then ρ is not a CC state (Lemma 3.1) and

$$(\Phi_1 \otimes \Phi_2)(\rho) = \frac{1}{2}\Phi_1(\rho_1) \otimes \Phi_2(\rho_2) + \frac{1}{2}\Phi_1(\sigma_1) \otimes \Phi_2(\sigma_2).$$

Since $[\Phi_1(\rho_1), \Phi_1(\sigma_1)] = 0$ and $[\Phi_2(\rho_2), \Phi_2(\sigma_2)] = 0$, it follows from Lemma 3.1 that $(\Phi_1 \otimes \Phi_2)(\rho)$ is a CC state, while ρ is not a CC state, a contradiction. Thus,

Φ_2 is commutativity preserving in both directions. Similarly, one can show that Φ_1 is commutativity preserving in both directions.

Sufficiency. Suppose that Φ_1 and Φ_2 are commutativity preserving in both directions. First, let us check that both Φ_1 and Φ_2 are bijective. To do this, we assume that $\Phi_1(X) = 0$. Then $[\Phi_1(X), \Phi_1(A)] = 0$ for all $A \in \mathcal{M}_n$. Since Φ_1 is commutativity preserving in both directions, we conclude that $X = cI_n$. Because that Φ_1 is trace-preserving, we see that $c = 0$ and then $X = 0$. This shows that Φ_1 is injective and so bijective since $\dim \mathcal{M}_n = n^2 < \infty$. Similarly, Φ_2 is bijective. Let $\rho \in CC(\mathbb{C}^n \otimes \mathbb{C}^m)$. Then by using Lemma 3.2, we can find two commuting families of normal operators $\{C_i\}$ and $\{D_i\}$ such that $\rho = \sum_i C_i \otimes D_i$. Thus, $(\Phi_1 \otimes \Phi_2)(\rho) = \sum_i \Phi_1(C_i) \otimes \Phi_2(D_i)$. Since Φ_1 and Φ_2 are commutativity preserving and $\dagger$-preserving, we see that $\{\Phi_1(C_i)\}$ and $\{\Phi_2(D_i)\}$ are commuting families of normal operators. By using Lemma 3.2 again, we conclude that $(\Phi_1 \otimes \Phi_2)(\rho)$ is a CC state. Let $\rho \in \mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^m)$ and $(\Phi_1 \otimes \Phi_2)(\rho)$ be a CC state. It follows from Lemma 3.2 that $(\Phi_1 \otimes \Phi_2)(\rho) = \sum_i A_i \otimes B_i$ for some commuting families $\{A_i\}$ and $\{B_i\}$ of normal operators. Since Φ_1 and Φ_2 are bijective (Lemma 4.1) and commutativity preserving in both directions, we can find two commuting families of normal operators $\{C_i\}$ and $\{D_i\}$ such that $\Phi_1(C_i) = A_i$ and $\Phi_2(D_i) = B_i$ for all i. Therefore, $(\Phi_1 \otimes \Phi_2)(\sum_i C_i \otimes D_i) = \sum_i A_i \otimes B_i$ and thus $\rho = \sum_i C_i \otimes D_i$ since $\Phi_1 \otimes \Phi_2$ are injective. Hence, $\rho \in CC(\mathbb{C}^n \otimes \mathbb{C}^m)$ (Lemma 3.2). Thus, $\Phi_1 \otimes \Phi_2$ is strongly CC-preserving. □

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