Constraints on Determinism: Bell Versus Conway–Kochen

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Abstract Bell's Theorem from Physics 36:1–28 (1964) and the (Strong) Free Will Theorem of Conway and Kochen from Notices AMS 56:226–232 (2009) both exclude deterministic hidden variable theories (or, in modern parlance, 'ontological models') that are compatible with some small fragment of quantum mechanics, admit 'free' settings of the archetypal Alice and Bob experiment, and satisfy a locality condition akin to parameter independence. We clarify the relationship between these theorems by giving reformulations of both that exactly pinpoint their resemblance and their differences. Our reformulation imposes determinism in what we see as the only consistent way, in which the 'ontological state' initially determines both the settings and the outcome of the experiment. The usual status of the settings as 'free' parameters is subsequently recovered from independence assumptions on the pertinent (random) variables. Our reformulation also clarifies the role of the settings in Bell's later generalization of his theorem to stochastic hidden variable theories.

Keywords Free will theorem · Bell's theorem · Hidden variable theories · Determinism

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1 Introduction

Though not really new,[1] the (Strong) Free Will Theorem of Conway and Kochen [13,14] is one of the sharpest and most interesting results that give constraints on determinism. It does so by proving that determinism is incompatible with a number of a priori desirable assumptions, including a small fragment of quantum mechanics (viz. the theory of two EPR-correlated spin-one particles), the free choice of settings of an EPR-style bipartite experiment involving such particles, and a locality condition called MIN. The latter has a long pedigree, arguably going back to EPR, but it was first stated quite clearly by Bell:[2]

> 'The vital assumption is that the result B for particle 2 does not depend on the setting $\vec{a}$ of the magnet for particle 1, nor A on $\vec{b}$.' [3, p. 196].

Also a closer study shows that Bell's Theorem on deterministic hidden variable theories [3] and the (Strong) Free Will Theorem appear to achieve a very similar (if not identical) goal under strikingly similar assumptions, which prompts the question what exactly their mutual relationship is. Curiously, despite the stellar fame of Bell's paper [3] (which according to Google Scholar had about 8,500 citations as of May 2014) and the considerable attention that also the Free Will Theorem has received (e.g., [16,21]), as far as we are aware, there has been little research in this precise direction.[3]

Hence the main aim of this paper is to clarify the relationship between Bell's Theorem [3] and the Free Will Theorem. But in doing so, we will *en passant* attempt to resolve an issue that has troubled Bell as well as Conway and Kochen, namely the theoretical status of parameter settings. As pointed out by Conway and Kochen [14] themselves, it is odd to assume determinism for the physical system under considera-

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[1] Analogous earlier results were obtained, in chronological order, by Heywood and Redhead [20], Stairs [34], Brown and Svetlichny [7], and Clifton [11] (of which only [20] was cited by Conway and Kochen).

[2] Bell [3] even attributes it to Einstein. See [37] for a detailed analysis of the way this condition is actually used by Bell in [3,5], and of the way it has been (mis)perceived by others. In particular, one should distinguish it from the locality condition usually named after Bell [6]. The latter, also called *local causality*, is a conjunction of two (probabilistic) notions that are now generally called *Parameter Independence* (PI) and *Outcome Independence* (OI); see [8,22,23,27,31,33]. The latter is automatically satisfied in the type of deterministic theories studied in [3,13,14], upon which the former reduces to the condition stated in the main text above, but now *conditioned on certain values of the hidden variables*. Note that our definition of the term PI will be different from the literature so far, though in the same spirit.

[3] The only significant exception we could find is the small and otherwise interesting book by Hemmick and Shakur [17], whose scathing treatment of the Free Will Theorem is somewhat undermined by their claim (p. 90) that the assumption of determinism follows from the other assumptions in the Strong Free Will Theorem (notably PI and perfect correlation). This seems questionable [37]: either Bell's (later) locality condition (i.e., PI *plus* OI) in conjunction with perfect correlation implies determinism, or PI plus determinism implies OI (and hence Bell Locality). Perhaps our view (which is certainly shared by Conway and Kochen!) that the assumptions of the Strong Free Will Theorem have been chosen quite carefully is clearer from our reformulation below than from even their second paper [14] (not to speak of their first [13]). Indeed, if valid, the objection of Hemmick and Shakur could just as well be raised against Bell's Theorem [3], where it would be equally misguided if both results are construed as attempts to put constraints on determinism in the first place. Our treatment of parameter settings will also be different from [17].

tion but not for the experimenters, so that the contradiction that proves their theorem seems almost circular.[4]

In Bell's later work, there has been a similar tension between the idea that the hidden variables (in the pertinent causal past) should on the one hand include all ontological information relevant to the experiment, but on the other hand should leave Alice and Bob free to choose any settings they like; see especially [28,32] for a fine analysis of Bell's dilemma (by some of his greatest supporters).[5] We will show that in both contexts of Bells' Theorem (i.e. either deterministic or stochastic) this issue can be resolved in a straightforward way by initially including the settings among the random variables describing the experiment, after which they are 'liberated' by suitable independence assumptions.[6]

The plan of our paper is as follows. In Sect. 2 we present a version of Bell's original theorem [3] that addresses the above issues. As a warm-up for what is to come, in Sect. 3 we extend this version to the spin-one case, followed in Sect. 4 by a reformulation of the Strong Free Will Theorem [14] in the same spirit. Our final Sect. 5 goes beyond our primary goal of finding constraints on determinism, but has been included in order to show that our treatment of parameter settings through random variables also applies to Bell's later results on stochastic hidden variable theories [6,8,10,15,22,27,31,37].

Our conclusion is that the Strong Free Will Theorem uses fewer assumptions than Bell's Theorem [3], as no appeal to probability theory is made. This comes at a price, though. First, in the absence of an Aspect-type experiment using spin-one particles, the former so far lacks the experimental backing of the latter. Second, because of its dependence on the Kochen–Specker Theorem, the Strong Free Will Theorem might lack finite precision robustness, cf. [1,2,18], though this threat recently seems to have been obviated [19].

2 Bell's (1964) Theorem Revisited

The setting of Bell's Theorem in its simplest form is given by the usual EPR-Bohm experiment (with photons) [8], in which Alice and Bob each choose a setting $A = \alpha \in X_A$ and $B = \beta \in X_B$, respectively, where X_A and X_B are finite sets whose elements are angles in $[0, \pi)$. For the theorem, it is even enough to assume $X_A = \{\alpha_1, \alpha_2\}$ and $X_B = \{\beta_1, \beta_2\}$, for suitable α_i and β_j (see below). Alice and Bob each receive one photon from an EPR-correlated pair, and determine whether or not it passes through a polarizer whose principal axis is set at an angle α or β relative to some reference axis in the plane orthogonal to the direction of motion of the photon pair.[7] If Alice's photon passes through she writes down $F = 1|A = \alpha$, and if not she writes $F = 0|A = \alpha$. Likewise, Bob records his result as $G = 1|B = \beta$ or $G = 0|B = \beta$.

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[4] This even led them to their curious way of paraphrasing their theorem as showing that 'If we humans have free will, then elementary particles already have their own small share of this valuable commodity'.

[5] See also [6,10] and most recently [26] for the interpretation of hidden variables as ontological states.

[6] See also Colbeck and Renner [12] for at least the first step of this strategy in the context of stochastic hidden variable theories. Using settings as labels, on the other hand, is defended in e.g. [8,10,32].

[7] Equivalently, α and β could stand for the corresponding unit vectors $\vec{a}$ and $\vec{b}$, defined up to a sign.

Repeating this experiment, they determine empirical probabilities P_E for all possible outcomes through the frequency interpretation of probabilities, which they denote by $P_E(F = \lambda | A = \alpha)$ and $P_E(F = \mu | B = \beta)$, or, having got together and compared their results, by $P_E(F = \lambda, G = \mu | A = \alpha, B = \beta)$, where $\lambda, \mu \in \{0, 1\}$. If the photon pair is prepared in the EPR-correlated state $|\psi_{\text{EPR}}\rangle = (|0\rangle|0\rangle + |1\rangle|1\rangle)/\sqrt{2}$ (taking into account helicity only), they find (as confirmed by quantum mechanics):[8]

$$P_E(F \neq G | A = \alpha, B = \beta) = \sin^2(\alpha - \beta). \tag{2.1}$$

The question, then, is whether these probabilities are 'intrinsic' or 'irreducible', as claimed by mainstream quantum mechanics, or instead are just a consequence of our ignorance. To make this precise, we define the latter case, i.e. determinism, at least in our present context, adding the other assumptions of Bell's Theorem [3] along the way.

Definition 2.1 In the context of the EPR-Bohm experiment (with photons):

- **Determinism** means that there is a state space X with associated functions

$$A : X \to X_A, B : X \to X_B, F : X \to \{0, 1\}, G : X \to \{0, 1\}, \tag{2.2}$$

  which completely describe the experiment in the sense that some state $x \in X$ determines *both* its settings $\alpha = A(x)$, $\beta = B(x)$ *and* its outcome $\lambda = F(x)$, $\mu = G(x)$.
- **Probability Theory** means that the above set X can be upgraded to a probability space (X, Σ, P), carrying the above functions A, B, F, G as random variables,[9] so that the empirical probabilities are reproduced as conditional joint probabilities through[10]

$$P_E(F = \lambda, G = \mu | A = \alpha, B = \beta) = P(F = \lambda, G = \mu | A = \alpha, B = \beta). \tag{2.3}$$

  Furthermore, in terms of a postulated additional random variable $Z : X \to X_Z$:
- **Parameter Independence** means that $F = F(A, Z)$ and $G = G(B, Z)$, in that there are measurable functions $\hat{F} : X_A \times X_Z \to \{0, 1\}$ and $\hat{G} : X_B \times X_Z \to \{0, 1\}$ for which $F(x) = \hat{F}(A(x), Z(x))$ and $G(x) = \hat{G}(B(x), Z(x))$ (P-almost everywhere).
- **Freedom** means that (A, B, Z) are probabilistically independent relative to P.[11]

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[8] Here $P_E(F \neq G | A = \alpha, B = \beta) \equiv P_E(F = 0, G = 1 | A = \alpha, B = \beta) + P_E(F = 1, G = 0 | A = \alpha, B = \beta)$. The complete statistics are: $P_E(F = 1, G = 1 | A = \alpha, B = \beta) = P_E(F = 0, G = 0 | A = \alpha, B = \beta) = \frac{1}{2} \cos^2(\alpha - \beta)$ and $P_E(F = 0, G = 1 | A = \alpha, B = \beta) = P_E(F = 1, G = 0 | A = \alpha, B = \beta) = \frac{1}{2} \sin^2(\alpha - \beta)$.

[9] This formulation incorporates the assumption that P is independent of A, B, F, G, and *vice versa*.

[10] Here $P(F = \lambda, G = \mu | A = \alpha, B = \beta) \equiv P(F = \lambda, G = \mu, A = \alpha, B = \beta)/P(A = \alpha, B = \beta)$ and $P(F = \lambda, G = \mu, A = \alpha, B = \beta) \equiv P(\{x \in X \mid F(x) = \lambda, G(x) = \mu, A(x) = \alpha, B(x) = \beta\})$, etc.

[11] On the usual definition, this also implies that the pairs (A, B), (A, Z), and (B, Z) are independent.

Here Z is the traditional 'hidden variable' space that, in the spirit of Bell [6,28,32], carries exactly the 'ontological' information (including e.g. the photon variables) that is:

(i) Sufficiently complete for the outcome of the experiment to depend on (A, B, Z) alone;

(ii) Independent of the settings (A, B), in the pertinent probabilistic sense.

These conditions stand (or fall) together: without (ii), i.e., *Freedom*, one could take $X_Z = X$ and $Z = $ id, whereas without (i), X_Z could be a singleton. *Parameter Independence* in fact sharpens i), which *a priori* might have been $F = F(A, B, Z)$ and $G = G(A, B, Z)$, to the effect that Alice's outcome is independent of Bob's, given A and Z (and *vice versa*) [3].

Our reformulation of Bell's Theorem [3], then, is as follows.

Theorem 2.2 *Determinism, Probability Theory, Parameter Independence, Freedom, and Nature (i.e. the outcome* (2.1) *of the* EPR-*Bohm experiment) are contradictory.*

Proof Determinism, Probability Theory, and Parameter Independence imply[12]

$$P(F = \lambda, G = \mu | A = \alpha, B = \beta) = P_{ABZ}(\hat{F} = \lambda, \hat{G} = \mu | \hat{A} = \alpha, \hat{B} = \beta), \quad (2.4)$$

where the function $\hat{A} : X_A \times X_B \times X_Z \to X_A$ is just projection on the first coordinate, likewise the function $\hat{B} : X_A \times X_B \times X_Z \to X_B$ is projection on the second, and P_{ABZ} is the joint probability on $X_A \times X_B \times X_Z$ induced by the triple (A, B, Z) and the probability measure P. Similarly, let P_Z be the probability on X_Z defined by Z and P, and define the following random variables on the probability space (X_Z, Σ_Z, P_Z):

$$\hat{F}_\alpha(z) := \hat{F}(\alpha, z); \quad (2.5)$$
$$\hat{G}_\beta(z) := \hat{G}(\beta, z). \quad (2.6)$$

Freedom then implies (indeed, is equivalent to the fact) that P_{ABZ} is given by a product measure on $X_A \times X_B \times X_Z$ (cf. [24, Lemma 3.10]). A brief computation then yields

$$P_{ABZ}(\hat{F} = \lambda, \hat{G} = \mu | \hat{A} = \alpha, \hat{B} = \beta) = P_Z(\hat{F}_\alpha = \lambda, \hat{G}_\beta = \mu), \quad (2.7)$$

and hence, from (2.4),

$$P(F = \lambda, G = \mu | A = \alpha, B = \beta) = P_Z(\hat{F}_\alpha = \lambda, \hat{G}_\beta = \mu). \quad (2.8)$$

Adding the *Nature* assumption, i.e. (2.1), then gives the crucial result

$$P_Z(\hat{F}_\alpha \neq \hat{G}_\beta) = \sin^2(\alpha - \beta). \quad (2.9)$$

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[12] This is true even if $F = F(A, B, Z)$ and $G = G(A, B, Z)$ rather than $F = F(A, Z)$ and $G = G(B, Z)$.

However, any four {0, 1}-valued random variables must satisfy the ('Boole') inequality [30]

$$P_Z(\hat{F}_{\alpha_1} \neq \hat{G}_{\beta_1}) \leq P_Z(\hat{F}_{\alpha_1} \neq \hat{G}_{\beta_2}) + P_Z(\hat{F}_{\alpha_2} \neq \hat{G}_{\beta_1}) + P_Z(\hat{F}_{\alpha_2} \neq \hat{G}_{\beta_2}),$$
(2.10)

which can be proved directly from the axioms of (classical) probability theory. But for suitable values of $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ this inequality is violated by (2.9). Take, for example, $\alpha_2 = \beta_2 = 3\theta$, $\alpha_1 = 0$, and $\beta_1 = \theta$. The inequality (2.10) then assumes the form $f(\theta) \geq 0$ for $f(\theta) = \sin^2(3\theta) + \sin^2(2\theta) - \sin^2(\theta)$. But this is false for many values of $\theta \in [0, 2\pi]$. $\square$

As already mentioned, in the usual treatment of Bell's Theorem (either his deterministic version [3,8] or his stochastic version [6,8,10,15,22,27,31,36]), the hidden variable λ corresponds to our $z \in X_Z$ rather than $x \in X$. It is the distinction between X_Z and the 'super-deterministic' state space X that allowed us to give a consistent formulation of *Determinism* without jeopardizing *Freedom*. As shown above, this eventually enables one to treat the apparatus settings as parameters rather than as random variables.

3 Bell's (1964) Theorem for Spin-one

The Free Will Theorem relies on a variation of the EPR-Bohm experiment in which $\mathbb{C}^2$ is replaced by $\mathbb{C}^3$; specifically, photons with the helicity degree of freedom only (or electrons with spin only) are replaced by massive spin-one particles. Although such a 'Free Will Experiment' has never been performed (though it might be, one day), quantum mechanics gives unambiguous predictions that may be used *in lieu* of measurement outcomes. Compared to the set-up of the previous section, the following changes are to be made:

- The *settings* are now given by $A = \mathbf{a}$ and $B = \mathbf{b}$, where $\mathbf{a} = [\vec{a}_1, \vec{a}_2, \vec{a}_3]$ and $\mathbf{b} = [\vec{b}_1, \vec{b}_2, \vec{b}_3]$ are *frames* in $\mathbb{R}^3$, that is, orthonormal bases $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ etc. in which each unit vector is defined up to a minus sign so that, e.g., $[-\vec{a}_1, \vec{a}_2, -\vec{a}_3] = [\vec{a}_1, \vec{a}_2, \vec{a}_3]$.
- The *outcomes* are now given by $F = \lambda \in X_F$, $G = \mu \in X_G$, where

$$X_F = X_G = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}.$$
(3.1)

- If we write $F = (F_1, F_2, F_3)$ and $G = (G_1, G_2, G_3)$, so that e.g. $F = (1, 1, 0)$ corresponds to $F_1 = F_2 = 1$, $F_3 = 0$, the relevant outcome of the experiment in the EPR-state (defined in terms of the usual spin-1 basis $(|0\rangle, |\pm 1\rangle)$ of $\mathbb{C}^3$)

$$|\psi_{\mathrm{EPR}}\rangle = (|-1\rangle|-1\rangle + |0\rangle|0\rangle + |1\rangle|1\rangle)/\sqrt{3},$$
(3.2)

at least as predicted by quantum mechanics,[13] is given by[14]

$$P_{QM}(F_i \neq G_j | A = \mathbf{a}, B = \mathbf{b}) = \frac{2}{3} \sin^2 \theta_{\vec{a}_i, \vec{b}_j} \quad (i, j = 1, 2, 3). \qquad (3.3)$$

Here $\theta_{\vec{a}, \vec{b}}$ is the angle between $\vec{a}$ and $\vec{b}$, so that $\cos^2 \theta_{\vec{a}, \vec{b}} = \langle \vec{a}, \vec{b} \rangle^2$, cf. (2.1). Note that the right-hand side only depends on $(\vec{a}_i, \vec{b}_j)$ rather than on all six vectors $(\mathbf{a}, \mathbf{b})$.

Along the same lines as Theorem 2.2, and subject to analogous definitions,[15] one proves:

Theorem 3.1 *Determinism, Probability Theory, Parameter Independence, Freedom, and Nature (i.e. the outcome* (3.3) *of the Free Will Experiment) are contradictory.*

For future reference, we also record the following consequence of (3.3):[16]

$$P_{QM}(F_i = G_j | A_i = B_j) = 1. \qquad (3.4)$$

In other words, if the settings $(\mathbf{a}, \mathbf{b})$ have $\vec{a}_i = \pm \vec{b}_j$, then with probability one the measurements F_i and G_j have the same outcomes (i.e. either $F_i = G_j = 0$ or $F_i = G_j = 1$).

4 The Strong Free Will Theorem Revisited

The Strong Free Will Theorem [14] historically arose as a refinement of the Kochen–Specker Theorem [4,25], in which the assumption of *Non-contextuality* in a single-wing experiment on a (massive) spin-one particle was replaced by the assumption of *Parameter Independence* in the double-wing experiment described in the previous section. In turn, the Kochen–Specker Theorem (like Gleason's Theorem, from which it follows) freed von Neumann's no-go result for hidden variable theories [35] from its controversial linearity assumption (see [9] for a balanced discussion). Thus the Strong Free Will Theorem of 2009 may be seen as a finishing touch of the development started

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[13] What is being measured here by say Alice with setting $\mathbf{a}$ is the triple $(\langle \vec{a}_1, \vec{J} \rangle^2, \langle \vec{a}_2, \vec{J} \rangle^2, \langle \vec{a}_3, \vec{J} \rangle^2)$, where $\vec{J}$ is the angular momentum operator for spin one. Each operator $\langle \vec{a}_i, \vec{J} \rangle$ has spectrum $\{-1, 0, 1\}$, so each square $\langle \vec{a}_i, \vec{J} \rangle^2$ can be 0 or 1. Since $\vec{J}^2 = 2$, one has $\langle \vec{a}_1, \vec{J} \rangle^2 + \langle \vec{a}_2, \vec{J} \rangle^2 + \langle \vec{a}_3, \vec{J} \rangle^2 = 2$, which gives (3.1).

[14] The complete (theoretical) statistics are: $P_{QM}(F_i = 1, G_j = 1 | A = \mathbf{a}, B = \mathbf{b}) = \frac{1}{3}(1 + \langle \vec{a}_i, \vec{b}_j \rangle^2)$, $P_{QM}(F_i = 0, G_j = 0 | A = \mathbf{a}, B = \mathbf{b}) = \frac{1}{3} \langle \vec{a}_i, \vec{b}_j \rangle^2$, $P_{QM}(F_i = 1, G_j = 0 | A = \mathbf{a}, B = \mathbf{b}) = \frac{1}{3}(1 - \langle \vec{a}_i, \vec{b}_j \rangle^2)$, and $P_{QM}(F_i = 0, G_j = 1 | A = \mathbf{a}, B = \mathbf{b}) = \frac{1}{3}(1 - \langle \vec{a}_i, \vec{b}_j \rangle^2)$. See footnote 8 for notation like $P(F_i \neq G_j | \cdot)$.

[15] See Definition 4.1 below for Determinism, and Definition 2.1 for the others, *mutatis mutandis*.

[16] Here $P_{QM}(F_i = G_j | A_i = B_j)$ denotes $P_{QM}(F_i = 0, G_j = 0 | A_i = B_j) + P_{QM}(F_i = 1, G_j = 1 | A_i = B_j)$, where the setting $A_i = B_j$ stands for $(A = \mathbf{a}, B = \mathbf{b})$ subject to $\vec{a}_i = \pm \vec{b}_j$. It follows from (3.3) or the previous footnote that $P_{QM}(F_i = G_j | A_i = B_j) = \frac{1}{3}(1 + 2 \cos^2 \theta_{\vec{a}_i, \vec{b}_j})$, which for $\vec{a}_i = \pm \vec{b}_j$ equals unity.

by von Neumann in 1932. Ironically, we are now going to place the Strong Free Will Theorem in the Bell tradition, which emphatically arose in opposition (if not hostility) to the work of von Neumann!

Roughly speaking, the Strong Free Will Theorem removes the assumption of Probability Theory from Bell's Theorem [3] (in our spin-one version, i.e., Theorem 3.1), but in order to achieve this, some of the assumptions now acquire a somewhat different meaning.

Definition 4.1 In the context of the Free Will Experiment of the previous section:

- **Determinism** means that there is a state space X with associated functions

$$A : X \to X_A, B : X \to X_B, F : X \to X_F, G : X \to X_G,$$

  where $X_A = X_B$ is the set of all frames in $\mathbb{R}^3$, and $X_F = X_G$ is given by (3.1), which completely describe the experiment in the sense that each state $x \in X$ determines both its settings $\mathbf{a} = A(x), \mathbf{b} = B(x)$ and its outcome $\lambda = F(x), \mu = G(x)$.
  Furthermore, in terms of a postulated additional random variable $Z : X \to X_Z$:
- **Parameter Independence** means that $F = F(A, Z)$ and $G = G(B, Z)$, i.e., for all $x \in X$ one has $F(x) = \hat{F}(A(x), Z(x))$ and $G(x) = \hat{G}(B(x), Z(x))$ for certain functions $\hat{F} : X_A \times X_Z \to X_F, \hat{G} : X_B \times X_Z \to X_G$.
- **Freedom** means that (A, B, Z) are independent in the sense that for each $(\mathbf{a}, \mathbf{b}, z) \in X_A \times X_B \times X_Z$ there is an $x \in X$ for which $A(x) = \mathbf{a}, B(x) = \mathbf{b}$, and $Z(x) = z$.

Thus the main change lies in the Freedom assumption, which simply says that the function $A \times B \times Z : X \to X_A \times X_B \times X_Z, x \mapsto (A(x), B(x), Z(x))$, is surjective. The goal of this assumption is to remove any potential dependencies between (or constraints on) the variables $(\mathbf{a}, \mathbf{b}, z)$, and hence between the physical system Alice and Bob perform their measurements *on*, and the devices they perform their measurements *with*.

Also, rather than the probabilistic outcome (3.3) of the Free Will Experiment, we use its corollary (3.4), construed non-probabilistically (i.e., probability one is replaced by deterministic certainty): writing $\hat{F} = (\hat{F}_1, \hat{F}_2, \hat{F}_3)$ and $\hat{G} = (\hat{G}_1, \hat{G}_2, \hat{G}_3)$, analogous to F and G, so that $\hat{F}_i : X_A \times X_Z \to \{0, 1\}$ and $\hat{G}_j : X_B \times X_Z \to \{0, 1\}$, *Nature* reveals that:[17]

$$\vec{a}_i = \vec{b}_j \implies \hat{F}_i(\vec{a}_1, \vec{a}_2, \vec{a}_3, z) = \hat{G}_j(\vec{b}_1, \vec{b}_2, \vec{b}_3, z). \tag{4.1}$$

Our reformulation of the Strong Free Will Theorem [7,11,14,20,34], then, is as follows.

Theorem 4.2 *Determinism, Parameter Independence, Freedom, and Nature (here represented by the outcome (4.1) of the Free Will Experiment) are contradictory.*

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[17] To keep matters simple, we will not be bothered with the notational difference between frames $[\vec{a}_1, \vec{a}_2, \vec{a}_3]$ and orthonormal bases $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$, and similarly for $\mathbf{b}$, until the end of the proof.

Proof The Freedom assumption allows us to treat $(\mathbf{a}, \mathbf{b}, z)$ as free variables, a fact that will tacitly be used all the time. First, take $i = j$ in (4.1). This shows that $\hat{F}_i(\vec{a}_1, \vec{a}_2, \vec{a}_3, z)$ only depends on $(\vec{a}_i, z)$, whilst $\hat{G}_j(\vec{b}_1, \vec{b}_2, \vec{b}_3, z)$ only depends on $(\vec{b}_j, z)$. Hence we write $\hat{F}_i(\vec{a}_1, \vec{a}_2, \vec{a}_3, z) = \tilde{F}_i(\vec{a}_i, z)$, etc. Next, taking $i \neq j$ in (4.1) shows that $\tilde{F}_1(\vec{a}, z) = \tilde{F}_2(\vec{a}, z) = \tilde{F}_3(\vec{a}, z)$. Consequently, the function $\hat{F} : X_A \times X_Z \to X_F$ is given by

$$\hat{F}(\vec{a}_1, \vec{a}_2, \vec{a}_3, z) = (\tilde{F}(\vec{a}_1, z), \tilde{F}(\vec{a}_2, z), \tilde{F}(\vec{a}_3, z)), \tag{4.2}$$

Combined with its value set (3.1), this shows that for each fixed z, $\hat{F}$ is a *frame function*: to each frame $\mathbf{a}$ it assigns one of the triples in (3.1), in such a way that if two different frames $\mathbf{a}$ and $\mathbf{a}'$ overlap in that $\vec{a}'_i = \vec{a}_j$ for some i, j, then $\hat{F}_i(\vec{a}'_i, z) = \hat{F}_j(\vec{a}_j, z)$. However, such a function does not exist by the Kochen–Specker Theorem [25,29]. □

Through the proof of the Kochen–Specker Theorem, this proof shows that a suitable finite set of frames will do for $X_A = X_B$, a simplification that is not available in Theorem 3.1!

5 Bell's Theorem Revisited

To close, we show that what is usually called Bell's Theorem [6,8,10,15,22,27,31,36], in which *Determinism* is not assumed, may also be reformulated using our treatment of apparatus settings as random variables. We restrict ourselves to generalizing Theorem 2.2; Theorem 3.1 may be adapted to stochastic hidden variables in an analogous way.

Definition 5.1 In the context of the EPR-Bohm experiment (with photons):

- **Probability Theory** means that there is a probability space (X, Σ, P), carrying random variables (2.2), so that the empirical probabilities are reproduced as conditional joint probabilities through (2.3).
- **Bell-Locality** means that there is a fifth random variable $Z : X \to X_Z$ for which

$$P(F = \lambda, G = \mu | A = \alpha, B = \beta, Z = z) = \tag{5.1}$$
$$P(F = \lambda | A = \alpha, Z = z) \cdot P(G = \mu | B = \beta, Z = z). \tag{5.2}$$

- **Freedom** means that, for this fifth variable, $P(Z = z | A = \alpha, B = \beta) = P(Z = z)$.

Theorem 5.2 *Probability Theory, Bell-Locality, Freedom, and Nature are contradictory, where Nature is represented through the outcome* (2.1) *of the* EPR-*Bohm experiment.*

Proof Introduce a new probability space $\tilde{X}_Z = [0, 1] \times [0, 1] \times X_Z$, with elements (s, t, z), and probability measure $d\tilde{P}_Z(s, t, z) = ds \cdot dt \cdot dP_Z(z)$. On $\tilde{X}_Z$, define random variables

$$\tilde{F}_\alpha(s, t, z) = \chi_{[0, P(F=1|A=\alpha, Z=z)]}(s); \tag{5.3}$$
$$\tilde{G}_\beta(s, t, z) = \chi_{[0, P(G=1|B=\beta, Z=z)]}(t), \tag{5.4}$$

a move inspired by [36]. Using all assumptions of the theorem, one then finds

$$P(F = \lambda, G = \mu | A = \alpha, B = \beta) = \tilde{P}_Z(\tilde{F}_\alpha = \lambda, \tilde{G}_\beta = \mu), \qquad (5.5)$$

cf. (2.8), so that the proof may be completed exactly as in the case of Theorem 2.2. □

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