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Quantum-Like Model for Decision Making Process in Two Players Game

A Non-Kolmogorovian Model

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Abstract In experiments of games, players frequently make choices which are regarded as irrational in game theory. In papers of Khrennikov (Information Dynamics in Cognitive, Psychological and Anomalous Phenomena. Fundamental Theories of Physics, Kluwer Academic, Norwell, 2004; Fuzzy Sets Syst. 155:4–17, 2005; Biosystems 84:225–241, 2006; Found. Phys. 35(10):1655–1693, 2005; in QP-PQ Quantum Probability and White Noise Analysis, vol. XXIV, pp. 105–117, 2009), it was pointed out that statistics collected in such the experiments have "quantum-like" properties, which can not be explained in classical probability theory. In this paper, we design a simple quantum-like model describing a decision-making process in a two-players game and try to explain a mechanism of the irrational behavior of players. Finally we discuss a mathematical frame of non-Kolmogorovian system in terms of liftings (Accardi and Ohya, in Appl. Math. Optim. 39:33–59, 1999).

Keywords Game theory · Decision-making · Non-Kolmogorovian probability · Quantum-like model

1 Introduction

Game theory is mathematics to explain behavior in various strategic situations. In the game, each player has several choices of action, and a payoff for his choice is

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decided depending on the actions of other players. Many studies of game theory assume a player acts rationally in the following means. A rational player examines which choice is appropriate to maximize his own payoff, and if an appropriate choice is decided uniquely, he chooses it without any concern for actions of other players. On the other hand, a irrational player does not choose his action with 1 probability. We can easily find several experimental examples in games that show this irrational behavior. For example, Shafir and Tversky gave the example based on the experiments of Prisoner's Dilemma (PD) game where two players can cooperate with or defect the other player [1]. The rational choice is to play defect, nevertheless, the experiments estimate the probability of the rational choice is 0.63.

Statistical data of experiments shows real players often behave irrationally [1, 2]. We believe, in order to make the game theory more useful, it is important to discuss decision-making process that makes the irrational choice. This will not be so easy because the decision-making process involves psychological factors. Some thought-provoking arguments exist for this problem, and these arguments refer to mathematical application of quantum mechanics in psychology [3, 4]. Khrennikov showed in his recent study, quantum statistics violates the law of total probability, which is widely used in classical statistics. The violation is found not only in microscopic phenomena of physics but also in experimental statistics of the game. He pointed out that the decision-making process will be described in "quantum-like model" where quantum superposition, quantum interference and quantum measurement are crucial [5–8].

In this paper, we design a quantum-like model of decision-making process on the two-players game, where each player has alternatives of actions and total four consequences of pay-off are prepared (see Sect. 2). In principle, a player has no idea about the choice of the other player and the player just predict it, so he will decide his own action with comparing all possible consequences. Our model describes this psychological process in the form of quantum superposition called "mental state". As seen in Sect. 3, the mental state is defined in 4-dimensional Hilbert space, and written as a superposition of four orthogonal basis corresponding to the four consequences. The form of mental sate is basically decided from the player's prediction of the other's choice. Furthermore, we assume that through the comparison of consequences, the mental state becomes changed and it approaches to an equilibrium. The form of equilibrium is discussed in Sect. 4, where we can see effects of quantum interference in the dynamics. The approach to equilibrium does not means a decision of action. The decision is described by a quantum measurement to the mental state in equilibrium. Then, two quantum states to be measured correspond to alternatives the player chooses. These states are called the alternative states, which are formed depending on uncertainty for choice of the other player (see Sect. 3). Generally, a measurement is probabilistic. It means the player in our model behave irrationally.

Our model is designed simply, so it is applicable to a more general and complicated game, or we may be able to reform it with a mathematical formalism beyond quantum mechanics. Through our discussion, the process of decision-making is described as a kind of quantum dynamics on a compound system in products space $\mathbb{C}^2 \otimes \mathbb{C}^2$, and we use the mathematical term called lifting [10]. Lifting is a map proposed by Accardi and Ohya. It is useful to discuss various dynamics on a compound system generally, and has potential to mathematically represent a dynamics that is

Table 1 Pay-off table of two-players game. 0 and 1 denote actions of the player A or B chooses. $C_{ij}(x, y)$ denote consequences of the game, and $x, y = a, b, c, d$ represent values of pay-off. For example, in the dilemma game, the relation of $c > a > d > b$ is assumed

A\B	0	1
0	$C_{00}(a, a)$	$C_{01}(b, c)$
1	$C_{10}(c, b)$	$C_{11}(d, d)$

difficult to be studied in conventional quantum mechanics. (In Appendix, we explain briefly mathematical properties of liftings.)

2 Two-Players Game and Rational Behavior

In this section, we explain two-players game and the rational behavior that is conventionally discussed in game theory. Two players A and B have two alternatives for actions denoted by 0 and 1. These players are under a same condition on their pay-offs as seen in the pay-off table (Table 1), where $C_{ij}(x, y)$ denote consequences of the game, and $x, y = a, b, c, d$ represent values of the pay-off.

Here, we explain the rational behavior of player in an example so-called dilemma game. The dilemma game has the relation of $c > a > d > b$ on the values of the pay-offs. In the game, a player who chooses the action of 1 can avoid the possibility of the minimum pay-off of d at least, and furthermore, he can gain the maximal pay-off of c if the other player chooses 0. In this mean, to choose the action 1 is reasonable for a player who wants to maximize his own pay-off. Game theory considers that a rational player chooses 1 without any concern for the action of the other player, and the consequence of the game is always C_{11} if both players are rational. (Note that in this case, the sum of pay-offs in C_{11} is less than the one in the consequence C_{00} where both players do irrational choices of 0. This is an interesting property of dilemma game.)

3 Construction of Mental State

In principle in the game, a player has no idea about other player's action. We assume this uncertainness gives some psychological influence to the decision-making of each player, that is, yields irrational choices. The decision-making in our model is mathematically described in the form of quantum state called mental state. In this section, we discuss the definition and construction of the mental state.

3.1 Prediction State

We consider two players named A and B. The player B must choose between two alternatives of actions denoted by "0" and "1". He is not informed of which action the player A chose, in principle. A player placed in such the situation only can make a prediction, where he thinks of two potentials about another player's action. However, he cannot deny the either potentials. It will be impossible for him to make a definitive judgment about another player's action until he decides his own action and sees a result of a game. We assume the player B holds two conflicting tendencies, one of which judges another player's action as 0, and the other of which judges it as 1. These two conflicting aspects affect simultaneously his decision-making process. In order to describe the indeterminacy the player B has in judgment of the A's action, our model uses a quantum mechanical superposition state with the following form.

$$\sigma = |\phi\rangle\langle\phi|, \quad |\phi\rangle = \alpha|0_A\rangle + \beta|1_A\rangle \in \mathcal{H} = \mathbb{C}^2. \tag{1}$$

Here, we interpret a state of $|0_A\rangle\langle 0_A|$ or $|1_A\rangle\langle 1_A|$ as a situation where the player B can make a definitive judgment of 0 or 1. The above state is different essentially from a form of mixed state like $p|0_A\rangle\langle 0_A| + (1-p)|1_A\rangle\langle 1_A|$ which is interpreted as a situation that the B can make a definitive judgment of 0 or 1 probabilistically. Such a situation given by a mixed state is not assumed in our model. A superposition of (1) represents two conflicting tendencies of judgments, which work simultaneously in the player's mind. The coefficients of α and β in the above equation are given as complex numbers and satisfy $|\alpha|^2 + |\beta|^2 = 1$, in accordance with formalism of quantum mechanics. Magnitudes of these coefficients mean degrees of the tendencies of judgment. Our assumption mentioned above does not give a clear interpretation about the relative phases. It is allowed for α and β to be real numbers, however, we believe they will correspond to some psychological factors related to the tendencies. Actually, in our model, effects similar to quantum interferences play crucial roles, and magnitudes and phases of α and β are parameters characterizing the effects. Hereafter, we call the state of (1) a prediction state.

3.2 Alternative State

When the player B decides his action 0, he thinks the consequence of the game will be C_{00} or C_{10}. In a sense, he chooses a consequence described in the super position of C_{00} and C_{10} as

$$\alpha|C_{00}\rangle + \beta|C_{10}\rangle.$$

Now, when we put $|C_{00}\rangle = |0_A\rangle \otimes |0_B\rangle$ and $|C_{10}\rangle = |1_A\rangle \otimes |0_B\rangle$ where $|0_B\rangle$ and $|1_B\rangle$ are orthogonal basis in $\mathcal{K} = \mathbb{C}^2$, the state vector;

$$\begin{aligned} |\Phi_0\rangle &= \alpha|C_{00}\rangle + \beta|C_{10}\rangle \\ &= |\phi\rangle \otimes |0_B\rangle. \end{aligned} \tag{2}$$

In a similar way, we gives another vector as

$$\begin{aligned} |\Phi_1\rangle &= \alpha|C_{01}\rangle + \beta|C_{11}\rangle \\ &= |\phi\rangle \otimes |1_B\rangle. \end{aligned} \tag{3}$$

We call the states of $|\Phi_{0,1}\rangle\langle\Phi_{0,1}|$, alternative states.

3.3 Mental State

In the process of decision-making, the player images the four consequences, and he decides his own action probabilistically. We describe such the mental situation by using the following quantum state on $\mathcal{H}\otimes\mathcal{K}$;

$$\theta=|\Psi\rangle\langle\Psi|,\quad |\Psi\rangle=x|\Phi_0\rangle+y|\Phi_1\rangle. \tag{4}$$

This description means the player chooses 0 with probability $\lambda=|x|^2$ and 1 with probability $\mu=1-\lambda=|y|^2$.

4 Process of Decision-Making

In this section, we discuss a process of decision-making that consists of three stages; prediction, comparison and decision.

4.1 Making Initial Mental State

The decision-making starts from the prediction of the other player's choice. Before the prediction, the player's mental is given by

$$\rho_0=|\xi\rangle\langle\xi|,\quad |\xi\rangle=x_0|0_B+y_0|1_B\rangle\in\mathcal{K}, \tag{5}$$

where $|x_0|=|y_0|=\frac{1}{\sqrt{2}}$. With using this ρ_0 and the prediction state σ in the subsection 3.1, we define the initial mental state $\theta_I=|\Psi_I\rangle\langle\Psi_I|$;

$$\theta_I=\sigma\otimes\rho_0. \tag{6}$$

4.2 Comparison of Consequences

Main part of the decision-making process is comparing the consequences of the game. The coefficients x,y of the mental $|\Psi\rangle=x|\Phi_0\rangle+y|\Phi_1\rangle$ become changed by the comparison. We assume its dynamics obeys the following differential equations, and approach to a state of equilibrium;

$$\begin{aligned}\frac{d\lambda}{dt}&=-k\lambda+\tilde{k}\mu,\\ \frac{d\mu}{dt}&=-\tilde{k}\mu+k\lambda,\end{aligned} \tag{7}$$

where $\lambda=|x|^2$ and $\mu=|y|^2$. These equations are similar to a chemical equilibration in a reaction system written as

$$\Phi_0 \underset{\tilde{k}}{\overset{k}{\rightleftharpoons}} \Phi_1, \tag{8}$$

where k ($\tilde{k}$) $\in \mathbb{R}$ means a rate of reaction from Φ_0 to Φ_1 (from Φ_1 to Φ_0). Using k and $\tilde{k}$, we define the mental state in equilibrium;

$$\theta_E = |\Psi_E\rangle\langle\Psi_E|, \quad |\Psi_E\rangle = x_E|\Phi_0\rangle + y_E|\Phi_1\rangle,$$
$$|x_E| = \sqrt{\frac{\tilde{k}}{k+\tilde{k}}}, \qquad |y_E| = \sqrt{\frac{k}{k+\tilde{k}}}. \tag{9}$$

The probabilities $\lambda_E = |x_E|^2$ and $\mu_E = |y_E|^2$ satisfy $\frac{d\lambda}{dt} = \frac{d\mu}{dt} = 0$ in the above differential equations. Here note that this equilibrium state θ_E is described as $\theta_E = \sigma \otimes \rho_E$. We call ρ_E a mental state after comparison.

In our concept, the reaction rate k or $\tilde{k}$ is related with the degree of player's tendency of choosing 1 or that of choosing 0. To discuss these parameters simply, we consider the case that the prediction state vector is $|\phi\rangle = |0\rangle$, that is, $\alpha = 1$ and $\beta = 0$ in (1). In this case, the alternative state vectors in (2) and (3) are $|\Phi_0\rangle = |C_{00}\rangle$ and $|\Phi_1\rangle = |C_{01}\rangle$. Therefore the equilibrium state $|\Psi_E\rangle$ is decided in the reaction system of

$$C_{00} \underset{\tilde{k}}{\overset{k}{\rightleftharpoons}} C_{01}, \tag{10}$$

and the value of $\frac{\tilde{k}}{k}$ means a degree of the worth of C_{00} relative to C_{01} for the player. On the contrary, if $|\phi\rangle = |1\rangle$, k and $\tilde{k}$ decide $|\Psi_E\rangle$ in the comparison of

$$C_{10} \underset{\tilde{k}}{\overset{k}{\rightleftharpoons}} C_{11}. \tag{11}$$

Here we denote k and $\tilde{k}$ of the above two comparisons with indices $k_i, \tilde{k}_i$ ($i = 1, 2$). In general, the alternative vectors are given by

$$|\Phi_0\rangle = \alpha|C_{00}\rangle + \beta|C_{10}\rangle, \qquad |\Phi_1\rangle = \alpha|C_{01}\rangle + \beta|C_{11}\rangle.$$

In this quantum mechanical representation, not only the two comparisons of $(k_1, \tilde{k}_1)$ and $(k_2, \tilde{k}_2)$, the player is allowed to make the following comparisons.

$$C_{00} \underset{\tilde{k}_3}{\overset{k_3}{\rightleftharpoons}} C_{11}, \qquad C_{10} \underset{\tilde{k}_4}{\overset{k_4}{\rightleftharpoons}} C_{01}. \tag{12}$$

Generally, the rates of $(k, \tilde{k})$ in (8) are decided by these $(k_1, \tilde{k}_1)$, $(k_2, \tilde{k}_2)$, $(k_3, \tilde{k}_3)$ and $(k_4, \tilde{k}_4)$. If the four comparisons are made independently, we can write the rates of k and $\tilde{k}$ of (8) as a classical expectation value given by

$$k = |\alpha|^4 k_1 + |\beta|^4 k_2 + |\alpha|^2|\beta|^2 k_3 + |\alpha|^2|\beta|^2 k_4,$$
$$\tilde{k} = |\alpha|^4 \tilde{k}_1 + |\beta|^4 \tilde{k}_2 + |\alpha|^2|\beta|^2 \tilde{k}_3 + |\alpha|^2|\beta|^2 \tilde{k}_4. \tag{13}$$

However, we consider the four comparisons of (10), (11) and (12) are not made probabilistically in such the classical means. In the mental state, these comparisons are made with influencing each other, in a sense, simultaneously. We gives k and $\tilde{k}$ in the following forms.

$$\begin{aligned} &k = |t|^2, \qquad \tilde{k} = |\tilde{t}|^2 \quad (t, \tilde{t} \in \mathbb{C}), \\ &t = |\alpha|^2 k_1^{\frac{1}{2}} + |\beta|^2 k_2^{\frac{1}{2}} + \alpha^* \beta k_3^{\frac{1}{2}} + \alpha\beta^* k_4^{\frac{1}{2}}, \\ &\tilde{t} = |\alpha|^2 \tilde{k}_1^{\frac{1}{2}} + |\beta|^2 \tilde{k}_2^{\frac{1}{2}} + \alpha\beta^* \tilde{k}_3^{\frac{1}{2}} + \alpha^* \beta \tilde{k}_4^{\frac{1}{2}}. \end{aligned} \tag{14}$$

The values of k and $\tilde{k}$ in this definition are different from that in the classical form of (13). The differences come from the effect of quantum interferences. In our quantum mechanical model, these mean the influences from simultaneous comparisons.

Here, we introduce a quantum channel Λ^* mapping from the prediction state σ to the mental state ρ_E;

$$\begin{aligned} &\Lambda^* : S(\mathcal{H}) \longrightarrow S(\mathcal{K}), \\ &\Lambda^* \sigma = \rho_E. \end{aligned}$$

Since the comparison of consequences is made on a compound system in $\mathcal{H} \otimes \mathcal{K}$, the channel Λ^* is defined with using a lifting $\mathcal{E}^* : S(\mathcal{H}) \longrightarrow S(\mathcal{H} \otimes \mathcal{K})$. This lifting is defined by

$$\mathcal{E}^* \sigma = \frac{T\sigma \otimes \rho_0 T^*}{\mathrm{tr}(T\sigma \otimes \rho T^*)}, \tag{15}$$

where,

$$T = \begin{pmatrix} 0 & 0 & \tilde{k}_1^{\frac{1}{2}} & \tilde{k}_3^{\frac{1}{2}} \\ 0 & 0 & \tilde{k}_4^{\frac{1}{2}} & \tilde{k}_2^{\frac{1}{2}} \\ k_1^{\frac{1}{2}} & k_4^{\frac{1}{2}} & 0 & 0 \\ k_3^{\frac{1}{2}} & k_2^{\frac{1}{2}} & 0 & 0 \end{pmatrix}. \tag{16}$$

In general, the state $\mathcal{E}^* \sigma$ is a quantum entangled state in $\mathcal{H} \otimes \mathcal{K}$. Note that t and $\tilde{t}$ of (14) is rewritten as

$$t = \langle \Phi_1 | T | \Phi_0 \rangle, \qquad \tilde{t} = \langle \Phi_0 | T | \Phi_1 \rangle, \tag{17}$$

and it is easily checked that the equilibrium state θ_E is described in the form of

$$\theta_E = \frac{M \mathcal{E}^* \sigma M^*}{\mathrm{tr}(M \mathcal{E}^* \sigma M^*)}, \tag{18}$$

where M is a projection defined by $M = \sigma \otimes I$. From $\theta_E = \sigma \otimes \rho_E$, we obtain

$$\Lambda^* \sigma = \mathrm{tr}_{\mathcal{H}} \left(\frac{M \mathcal{E}^* \sigma M^*}{\mathrm{tr}(M \mathcal{E}^* \sigma M^*)} \right). \tag{19}$$

4.3 Decision of Action

After the comparison, the player decides his own action. We assume the decision corresponds to a quantum measurement of the state $|0_B\rangle\langle 0_B|$ or $|1_B\rangle\langle 1_B|$ to the mental state ρ_E. Therefore, his decision is made probabilistically in general. The probabilities of decisions of 0 and 1 are given by

$$\begin{aligned}\lambda_E &= \mathrm{tr}(|0_B\rangle\langle 0_B|\rho_E|0_B\rangle\langle 0_B|),\\ \mu_E &= \mathrm{tr}(|1_B\rangle\langle 1_B|\rho_E|1_B\rangle\langle 1_B|).\end{aligned} \tag{20}$$

5 Example: Decision-Making in PD

In this section, we apply our model to Prisoner's Dilemma (PD) game, and show the model makes irrational choices in this example.

In our model, characters of game are reflected in the contents of the operator T of (16), that is, the pay-offs of the game as seen in the Table.1 can be factors deciding the values of contents k_i and $\tilde{k}_i$. Here, we give the operator T for the example of PD game, in the following form.

$$T = \begin{pmatrix} 0 & 0 & 0 & \tilde{k}_3^{\frac{1}{2}} \\ 0 & 0 & 0 & 0 \\ k_1^{\frac{1}{2}} & k_4^{\frac{1}{2}} & 0 & 0 \\ 0 & k_2^{\frac{1}{2}} & 0 & 0 \end{pmatrix}. \tag{21}$$

In this form, we assume $k_3 = 0$ and $\tilde{k}_1 = \tilde{k}_2 = \tilde{k}_4 = 0$, because the pay-off d for the consequence C_{10} is smaller than these of other consequences and the pay-off a for C_{00} is smaller than c for C_{01} but larger than d for C_{11}. This setting of the parameters provides the probability,

$$\begin{aligned}\lambda_E &= \frac{|\alpha|^2|\beta|^2\tilde{k}_3}{|\alpha|^2|\beta|^2\tilde{k}_3 + ||\alpha|^2 k_1^{\frac{1}{2}} + |\beta|^2 k_2^{\frac{1}{2}} + \alpha\beta^* k_4^{\frac{1}{2}}|^2},\\ \mu_E &= \frac{||\alpha|^2 k_1^{\frac{1}{2}} + |\beta|^2 k_2^{\frac{1}{2}} + \alpha\beta^* k_4^{\frac{1}{2}}|^2}{|\alpha|^2|\beta|^2\tilde{k}_3 + ||\alpha|^2 k_1^{\frac{1}{2}} + |\beta|^2 k_2^{\frac{1}{2}} + \alpha\beta^* k_4^{\frac{1}{2}}|^2}.\end{aligned} \tag{22}$$

As mentioned in Sect. 2, in the game theory for PD, a rational player always chooses his action 1, that is, $\mu_E = 1$. The player in our model becomes rational in special two cases; the case of $\tilde{k}_3 = 0$ and the case of $\alpha = 0$ or $\beta = 0$. Since the parameters of α and β decide the player's prediction for the other player's choice, the case of $\alpha = 0$ or $\beta = 0$ is unnatural to the principle of game that the player is uncertain about the other player's choice. It may correspond to a special situation where the player obtains some information and can judge another player's choice before his own choice.

The experiment in [1] tried to make this special mental condition for real players, and it showed that most of players choose the rational action "1". Khrennikov pointed out that the experimental data has non Kolmogorovian property and represented it as violation of the *law of total probability* [5–8];

$$P(1_B|C) \neq P(1_B|C_{0_A})P(0_A|C) + P(1_B|C_{1_A})P(1_A|C). \tag{23}$$

Here, $P(1_B|C)$ means a probability for the action "1" of B-player under the mental condition C where B has no idea about A-player, $P(1_B|C_{i_A})$ $(i = 0, 1)$ are the probabilities for the actions of B under the condition C_{i_A} where B can judge A-player's choice, and $P(i_A|C)$ $(i = 0, 1)$ are the probabilities of B's judgment under the condition of C.

6 Conclusion

In this study, we designed the quantum-like model describing the irrational behavior of real players. In our model, the irrational choice comes from the comparisons of consequences given in game. These comparisons give a dynamics in a mental system, which is described by a superposition of consequences. The dynamics is written in terms of the mathematics of lifting, and it can not be discussed in conventional quantum mechanics. Moreover, the violation of the law of total probability as (23) appears and it is treated beyond quantum physics generally. Further details of study on a relation of our model and the non-Kolmogorovian property will be described in elsewhere.

Appendix: Channels and Liftings

In quantum information theory, a certain map is important for describing an information transition such as a measurement process or a signal transmission. This map is called a channel [11, 12] $\Lambda^* : \mathcal{S}(\mathcal{A}) \mapsto \mathcal{S}(\mathcal{B})$. Here $\mathcal{S}(\mathcal{A})$ $(\mathcal{S}(\mathcal{B}))$ are state spaces of C^*-algebras $\mathcal{A}$ $(\mathcal{B})$. For example, a set of all bounded linear operators $\mathcal{B}(\mathcal{H})$ on Hilbert space $\mathcal{H}$ realizes a C^*-algebra $\mathcal{A}$. If a channel Λ^* is affine, i.e., $\Lambda^*(\sum_n \lambda_n \rho_n) = \sum_n \lambda_n \Lambda^*(\rho_n), \forall \rho_n \in \mathcal{S}(\mathcal{A}), \forall \lambda_n \in [0, 1], \sum \lambda_n = 1$, it is called a linear channel. A completely positive (CP) channel is a linear channel Λ^* that its dual $\Lambda : \mathcal{B} \mapsto \mathcal{A}$ (i.e. $\mathrm{tr}(\Lambda^*(\rho)A) = \mathrm{tr}(\rho\Lambda(A))$ for any $A \in \mathcal{A}$) satisfies

$$\sum_{i,j=1}^{n} A_i^* \Lambda(B_i^* B_j) A_j \geq 0,$$

for any $\{A_j\} \subset \mathcal{A}$, $\{B_j\} \subset \mathcal{B}$ and $n \in \mathbb{N}$.

Liftings are a class of channels from $\mathcal{S}(\mathcal{A})$ to $\mathcal{S}(\mathcal{A} \otimes \mathcal{B})$;

$$\mathcal{E}^* : \mathcal{S}(\mathcal{A}) \mapsto \mathcal{S}(\mathcal{A} \otimes B). \tag{24}$$

The following liftings are often used in physics.

1. *Linear Lifting*: A linear lifting is affine and its dual is a completely positive map.
2. *Pure Lifting*: A pure lifting maps pure states into pure states.
3. *Nondemolition Lifting*: A lifting is nondemolition for a state $\rho \in \mathcal{S}(\mathcal{A})$ if ρ is invariant for any $A \in \mathcal{A}$ in the sense of

$$(\mathcal{E}^* \rho)(A \otimes 1) = \rho(A).$$

   Here, $\rho(A) \equiv \mathrm{tr}(\rho A)$, $A \in \mathcal{A}$.
4. *Compound State Lifting*: A compound state lifting [13–15] is a non-linear and non-demolition lifting such that for a density matrix $\rho = \sum_k \lambda_k E_k$, $E_k \in \mathcal{S}(\mathcal{A})$,

$$\mathcal{E}^*(\rho) = \sum_k \lambda_k E_k \otimes \Lambda^* E_k.$$

5. *Transition Lifting*: A transition expectation is a completely positive linear map given by $\mathcal{E} : \mathcal{A} \otimes \mathcal{B} \mapsto \mathcal{A}$, and it satisfies

$$\mathcal{E}(1 \otimes 1) = 1.$$

   Transition expectations play a crucial role in the construction of quantum Markov chains and they appear in the framework of measurement theory. The dual of a transition expectation is an example of liftings.
6. *Isometric Lifting*: A lifting is a channel from a subsystem to a compound system, and it is useful to describe open system dynamics. Let us consider a situation; a system interacts with another (environment) system, and a correlated state is generated in a compound system. We assume that the system is independent from another system before the interaction and initial states of the two systems are given by $\rho \in \mathcal{S}(\mathcal{A})$ and $\sigma \in \mathcal{S}(\mathcal{B})$. The compound system after the interaction is represented by $\mathcal{E}^* \rho \in \mathcal{S}(\mathcal{A} \otimes \mathcal{B})$ in general. Now let the lifting $\mathcal{E}^*$ be an *isometric lifting* defined as

$$\mathcal{E}^* \rho = V \rho V^*,$$

   where the operator $V : \mathcal{H}_{\mathcal{A}} \mapsto \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$ satisfies $V^* V = 1_{\mathcal{H}_{\mathcal{A}}}$. Then for an observable $B \in \mathcal{B}(\mathcal{B})$, the expectation value $\mathrm{tr}_{\mathcal{A}} \mathcal{E}^* \rho(B)$ is written as

$$\mathrm{tr}_{\mathcal{A}} \mathcal{E}^* \rho(B) = \mathrm{tr}_{\mathcal{A}}(\rho V^*(1 \otimes B) V) = \mathrm{tr}_{\mathcal{A}}(\rho \mathcal{E}(1 \otimes B)) = \rho(\mathcal{E}(1 \otimes B)).$$

   Here, we introduce the map $\mathcal{E}(1 \otimes B) = V^*(1 \otimes B) V$. This map is a transition expectation and its dual is the isometric $\mathcal{E}^*$. We can easily check $\mathrm{tr}_{\mathcal{A}} \mathcal{E}^* \rho(I) = 1$.

When the $\mathcal{E}^*$ represents a time evolution; $\mathcal{E}^* \rho = U \rho \otimes \sigma U^*$ (U is a unitary operator), the corresponding transition expectation is written by

$$\mathcal{E}(1 \otimes B) = \mathrm{tr}_{\mathcal{B}}(U(1 \otimes \sigma) U^*(1 \otimes B)).$$

If B is written as $B = \sum_k \lambda_k P_k$ with projections $\{P_k\}$,

$$\mathrm{tr}_{\mathcal{A}} \mathcal{E}^* \rho(B) = \sum_k \lambda_k \,\mathrm{tr}_{\mathcal{A}} \mathcal{E}^* \rho(P_k) = \sum_k \lambda_k \rho(\mathcal{E}(1 \otimes P_k)) \equiv \sum_k \lambda_k \Lambda_k^* \rho(P_k).$$

In this form, values of $\Lambda_k^* \rho(P_k)$ mean probabilities $p(\lambda_k)$, and $\mathcal{E}(1 \otimes P_k)$ mean POVM (positive operator valued measure) Q_k that satisfy $Q_k \geq 0$, $\sum_k Q_k = I$.

References

1. Shafir, E., Tversky, A.: Thinking through uncertainty: nonconsequential reasoning and choice. Cogn. Psychol. **24**, 449–474 (1992)
2. Tversky, A., Shafir, E.: The disjunction effect in choice under uncertainty. Psychol. Sci. **3**, 305–309 (1992)
3. Bohr, N.: The Philosophical Writings of Niels Bohr, 3 vols. Ox Bow Press, Woodbridge (1987)
4. Jung, C.G., Pauli, W.: In: Meier, C.A. (ed.) Ein Briefweksel, pp. 679–702. Springer, Berlin (1992)
5. Khrennikov, A.: Information Dynamics in Cognitive, Psychological and Anomalous Phenomena. Fundamental Theories of Physics. Kluwer Academic, Norwell (2004)
6. Khrennikov, A.: Interference in the classical probabilistic framework. Fuzzy Sets Syst. **155**, 4–17 (2005)
7. Khrennikov, A.: Quantum-like brain: interference of minds. Biosystems **84**, 225–241 (2006)
8. Khrennikov, A.: The principle of supplementarity: a contextual probabilistic viewpoint to complementarity, the interference of probabilities and incompatibility of variables in quantum mechanics. Found. Phys. **35**(10), 1655–1693 (2005)
9. Khrennikov, A.: Quantum-like models for decision making in psychology and cognitive science. In: QP-PQ Quantum Probability and White Noise Analysis, vol. XXIV, pp. 105–117 (2009)
10. Accardi, L., Ohya, M.: Compound channels, transition expectations and liftings. Appl. Math. Optim. **39**, 33–59 (1999)
11. Ohya, M.: Quantum ergodic channels in operator algebras. J. Math. Anal. Appl. **84**(2), 318–327 (1981)
12. Ohya, M.: Some aspects of quantum information theory and their applications to irreversible processes. Rep. Math. Phys. **27**, 19–47 (1989)
13. Ohya, M.: On compound state and mutual information in quantum information theory. IEEE Trans. Inf. Theory **29**(5), 770–774 (1983)
14. Ohya, M.: Note on quantum probability. L. Nuovo Cimento **38**(11), 402–406 (1983)
15. Ohya, M.: State change and entropies in quantum dynamical systems. In: Quantum Probability and Application II. Lecture Notes in Math., vol. 1136, pp. 397–408. Springer, Heidelberg (1985)