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Inducing the Cosmological Constant from Five-Dimensional Weyl Space

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Abstract We investigate the possibility of inducing the cosmological constant from extra dimensions by embedding our four-dimensional Riemannian space-time into a five-dimensional Weyl integrable space. Following the approach of the space-time-matter theory we show that when we go down from five to four dimensions, the Weyl field may contribute both to the induced energy-tensor as well as to the cosmological constant Λ, or more generally, it may generate a time-dependent cosmological parameter $\Lambda(t)$. As an application, we construct a simple cosmological model in which $\Lambda(t)$ has some interesting properties.

Keywords Five-dimensional vacuum · Integrable Weyl theory of gravity · Induced-matter theory

1 Introduction

In a very recent past it appeared that the role played by the cosmological constant in cosmology was merely historical, mainly connected with Einstein's attempt to build a cosmological scenario in which the Universe was static and finite [1]. However, since the recent discovery of cosmic acceleration there has been a renewed interest in the role the cosmological constant could play to explain the new data. The evidence which appears to call for dark energy is perfectly consistent with a cosmological constant. Moreover, the present most popular model of cosmology, the Lambda-CDM model, tacitly assumes the existence of the cosmological constant [2]. On the other

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hand, some physicists have always argued in favour of the existence of the cosmological constant as a consequence of the energy density of the vacuum [3].

From the standpoint of cosmological theory it seems then desirable to have a justification of the cosmological constant on theoretical grounds. This quest has led some theoreticians to modify Einstein's gravitational theory, these attempts going back to the works of Eddington and Schrödinger [4].

Our aim in the present article is to introduce a new approach to this old question. We argue that the appearance of the cosmological constant in the equations of four-dimensional (4D) general relativity might be related to the assumption, made by some modern spacetime theories, that our Universe may have extra dimensions. Among such theories are the so-called braneworld scenario [5–8] and the non-compact Kaluza-Klein theories [9–12], both sharing some basic assumptions concerning the geometry of the fundamental higher-dimensional embedding space. For example, in these proposals our ordinary spacetime is viewed as a hypersurface embedded in a five-dimensional (5D) manifold (*the bulk*). On the other hand, mathematical theorems regulate these embeddings, in particular, the Campbell-Magaard theorem [13, 14] and its extensions specify the conditions under which the embeddings are possible [15–21].

The possibility of generating matter and fields from a higher-dimensional vacuum was first realized by Kaluza and Klein, an idea that has been a source of inspiration for practically all higher-dimensional theories [22–24]. In particular, a mathematical formalism has been developed by Wesson and Ponce de Leon [25] that permits to "induce" an energy-momentum tensor from the vacuum Einstein field equations in five dimensions, a schema now known as the induced-matter or space-time-matter (STM) theory. It has later been shown that from a geometrical point of view such generated energy-momentum tensor is also related to the extrinsic curvature of the spacetime hypersurface [26]. Besides, an interesting feature of this mechanism, apart from this "geometrization" of matter, is that it is also powerful enough to generate a cosmological constant in four dimensions out of a pure five-dimensional vacuum [27]. It should be noted that this has already been done in the context of the usual space-time-matter theory by considering a Riemannian bulk, and, in fact, it has been shown that it is possible to generate simultaneously the cosmological constant and an induced energy-momentum tensor $T_{\alpha\beta}^{(IM)}$ that describes macroscopic matter [28–31]. However, it seems interesting to investigate the subject when the geometry of the embedding space has more degrees of freedom, as in the case of Weyl geometry, perhaps the simplest generalization of Riemannian geometry [32, 33]. It turns out then that when we go down from 5D to 4D, as in the space-time-matter theory, the Weyl field may contribute both to the induced energy-tensor as well as to the cosmological constant Λ, or more generally, to a time-dependent cosmological function $\Lambda(t)$ [34].

In this paper we shall consider a particular case of Weyl geometry. Conceived by Weyl in 1918, as an attempt to unify gravity with electromagnetism, in its original form Weyl's theory [32, 33] turned out to be inadequate as a physical theory as was firstly pointed by Einstein soon after the appearance of the theory [35–38]. As is well known, Einstein's argument was that in a non-integrable Weyl geometry it would not be possible the existence of sharp spectral lines in the presence of an electromagnetic field since atomic clocks would depend on their past history. However, a variant of

Weyl geometry, namely, the one in which the Weyl field is integrable, does not suffer from the flaw pointed out by Einstein, and for this reason it has attracted the attention of some cosmologists [39–42], particularly in the context of higher-dimensional theories [43]. A Weylian Kaluza-Klein "hybrid" theory has also been studied in connection with Chern-Simons [44, 45] modified gravity and perhaps is of interest to mathematical physicists due to its rich geometrical structure.

It turns out that in this case if the Weyl field depends only on the extra dimension, then the embedded spacetime is Riemannian and general relativity holds in the hypersurface [42, 46], although the non-Riemannian character of the whole bulk propagates into the hypersurface in the form of induced matter and a cosmological function.

The paper is organized as follows. We start in Sect. 1 with a brief review of some fundamental concepts that underlie Weyl geometry. We proceed in Sect. 2 to develop a five-dimensional Weylian theory of gravity in vacuum, the dynamics of which is given by a certain action chosen among the simplest ones [39–42]. In the same section we set the equations for a particular choice of the five-dimensional metric and examine the case of a simple cosmological model. In Sect. 3 we illustrate the theory by going through a simple example taken from Cosmology. Our final remarks are contained in Sect. 4.

2 Weyl Geometry

The starting point of the new geometry created by Weyl is the assumption that the covariant derivative of the metric tensor g is not zero, but, instead, is given by

$$\nabla_a g_{bc} = \sigma_a g_{bc}, \tag{1}$$

where σ_a denotes the components of a one-form field σ with respect to a local coordinate basis. This is, of course, a generalization of the idea of Riemannian compatibility between the connection ∇ and g, which is equivalent to require the length of a vector to remain unaltered by parallel transport [35–38]. If $\sigma = d\phi$, where ϕ is a scalar field, then we have an integrable Weyl geometry. A differentiable manifold M endowed with a metric g and a Weyl field σ is usually referred to as a *Weyl frame*. It is interesting to see that the Weyl condition (1) remains unchanged when we go to another Weyl frame $(M, \overline{g}, \overline{\sigma})$ by performing the following simultaneous transformations in g and σ:

$$\overline{g} = e^{-f} g, \tag{2}$$

$$\overline{\sigma} = \sigma - df, \tag{3}$$

where f is a scalar function defined on M.

A clear geometrical insight on the properties of Weyl parallel transport is given by the following proposition: Let M be a differentiable manifold with an affine connection ∇, a metric g and a Weyl field of one-forms σ. If ∇ is compatible with g in the Weyl sense, i.e. if (1) holds, then for any smooth curve $\alpha = \alpha(\lambda)$ and any pair of two parallel vector fields V and U along α, we have

$$\frac{d}{d\lambda}g(V,U)=\sigma\left(\frac{d}{d\lambda}\right)g(V,U), \tag{4}$$

where $\frac{d}{d\lambda}$ denotes the vector tangent to α.

If we integrate the above equation along the curve α, starting from a point $P_0 = \alpha(\lambda_0)$, then we obtain

$$g(V(\lambda),U(\lambda))=g(V(\lambda_0),U(\lambda_0))e^{\int_{\lambda_0}^{\lambda}\sigma(\frac{d}{d\rho})d\rho}. \tag{5}$$

Putting $U=V$ and denoting by $L(\lambda)$ the length of the vector $V(\lambda)$ at an arbitrary point $P=\alpha(\lambda)$ of the curve, then it is easy to see that in a local coordinate system $\{x^a\}$ (4) reduces to

$$\frac{dL}{d\lambda}=\frac{\sigma_a}{2}\frac{dx^a}{d\lambda}L.$$

Consider the set of all closed curves $\alpha : [a,b] \in R \to M$, i.e., with $\alpha(a)=\alpha(b)$. Then, we have

$$g(V(b),U(b))=g(V(a),U(a))e^{\int_a^b\sigma(\frac{d}{d\lambda})d\lambda}.$$

Now, it is the integral $\int_a^b\sigma(\frac{d}{d\lambda})d\lambda$ that is responsible for the difference between the readings of two identical atomic clocks following different paths.

It follows from Stokes' theorem that if σ is an exact form, that is, if there exists a scalar function ϕ, such that $\sigma = d\phi$, then

$$\oint\sigma\left(\frac{d}{d\lambda}\right)d\lambda=0$$

for any loop. In other words, in this case the integral $e^{\int_{\lambda_0}^{\lambda}\sigma(\frac{d}{d\rho})d\rho}$ does not depend on the path. Since it is this integral that regulates the way atomic clocks run this variant of Weyl geometry does not suffer from the flaw pointed out by Einstein, and we have what is often called in the literature a *Weyl integrable manifold.*

Our next step is to consider a Weylian theory of gravity. If we consider a Weyl spacetime the simplest action that gives the dynamics of the gravitational field in the absence of matter is given by

$$\mathcal{S}=\int d^4x\sqrt{g}[\mathcal{R}+\xi\phi^a{}_{;a}], \tag{6}$$

where ξ is an arbitrary coupling constant, $\phi_a \equiv \phi_{,a}$ is the Weyl field, $\mathcal{R}$ is the Weylian Ricci scalar and the semicolon $(;)$ denotes covariant derivative with respect to the Weyl connection [39–42]. The extension of this formulation to a higher-dimensional space is straightforward. In the next section we shall consider a five-dimensional Weyl integrable space.

3 A Weyl Integrable Dynamics in Five Dimensions

Let us consider a five-dimensional space M^5 endowed with a metric tensor ${}^{(5)}g$ and an integrable Weyl scalar field ϕ. In local coordinates $\{y^a\}$ the five-dimensional line element of will be denoted by

$$dS^2 = g_{ab}(y)\,dy^a dy^b, \tag{7}$$

where g_{ab} are the components of ${}^{(5)}g$. As we have already mentioned the simplest action that can be constructed for a Weylian theory of gravity in a five-dimensional vacuum is given by

$${}^{(5)}\mathcal{S} = \int d^5y\sqrt{|{}^{(5)}g|}\big[{}^{(5)}\mathcal{R} + \xi\phi^a{}_{;a}\big], \tag{8}$$

where ξ is an arbitrary coupling constant, $\phi_a \equiv \phi_{,a}$ is the gauge vector associated with the Weyl field, $|{}^{(5)}g|$ is the absolute value of the determinant of the metric ${}^{(5)}g_{ab}$, ${}^{(5)}\mathcal{R}$ is the Weylian Ricci scalar. One can easily check that the variation of the action (8) with respect to the tensor metric and with respect to the Weyl scalar field yields

$${}^{(5)}\mathcal{G}_{ab} + \phi_{a;b} - (2\xi - 1)\phi_a\phi_b + \xi g_{ab}\phi_c\phi^c = 0, \tag{9}$$

$$\phi^a{}_{;a} + 2\phi_a\phi^a = 0, \tag{10}$$

where ${}^{(5)}\mathcal{G}_{ab}$ denotes the Einstein tensor calculated with the Weyl connection ${}^{(5)}\Gamma^a_{bc} ={}^{(5)} \{{}^{\,a}_{bc}\} - (1/2)[\phi_b\delta^a_c + \phi_c\delta^a_b - g_{bc}\phi^a]$ and $\{{}^{\,a}_{bc}\}$ are the Christoffel symbols of Riemannian geometry. Equations (9) and (10) are the field equations of the five-dimensional Weyl gravitational theory and describes the dynamics of a five-dimensional bulk in vacuum. A better insight may be gained if we recast the field equations (9) and (10) into its Riemannian part plus the contribution of the Weyl scalar field. Thus after excluding total derivatives of the scalar field ϕ the action (8) can be written as [39–42]

$${}^{(5)}\mathcal{S} = \int d^5y\sqrt{g_5}\left[{}^{(5)}\tilde{R} + \frac{1}{2}(5\xi - 6)\phi_a\phi^a\right]. \tag{11}$$

The field equations are obtained by taking a variation of the above action with respect to the pair (g_{ab}, ϕ). We are then led to

$${}^{(5)}\tilde{G}_{ab} - \frac{1}{2}(6 - 5\xi)\left[\phi_a\phi_b - \frac{1}{2}g_{ab}\phi_c\phi^c\right] = 0, \tag{12}$$

$${}^{(5)}\tilde{\Box}\phi = 0, \tag{13}$$

where the tilde ($\sim$) is used to denote quantities calculated with the Riemannian part of the Weyl connection and ${}^{(5)}\tilde{\Box}$ denotes the 5D d'Alembertian operator in the Riemannian sense.

At this point let us express the local coordinates $\{y^a\}$ as $\{x^\alpha, l\}$, denoting by l the fifth (spacelike) coordinate, and choose for simplicity the line element (7) in the form[1]

$$dS^2 = g_{\alpha\beta}(x,l)dx^\alpha dx^\beta - \Phi^2(x,l)dl^2, \tag{14}$$

where the function $\Phi^2(x,l)$ is the 5D analogue of the lapse function used in canonical general relativity [49].

As in the space-time-matter theory [25], with respect to this foliation the field equations (12) can be split into

$$^{(5)}\tilde{G}_{\alpha\beta} - \frac{1}{2}(6-5\xi)\left[\phi_\alpha\phi_\beta - \frac{1}{2}g_{\alpha\beta}\left(\phi_\gamma\phi^\gamma - \Phi^{-2}\phi_l^2\right)\right] = 0, \tag{15}$$

$$^{(5)}\tilde{G}_{\alpha l} - \frac{1}{2}(6-5\xi)\phi_\alpha\phi_l = 0, \tag{16}$$

$$^{(5)}\tilde{G}_{ll} - \frac{1}{4}(6-5\xi)\left[\phi_l^2 + \Phi^2\phi_\gamma\phi^\gamma\right] = 0. \tag{17}$$

The five-dimensional Weylian equations we have now obtained assuming the geometry given by (14) supposes that in principle the Weyl scalar field ϕ depends on all coordinates, that is, $\phi = \phi(x,l)$. Some solutions of the above field equations have been worked out in detail by Novello and collaborators considering different geometric settings [39–42]. However our interest here is mainly to study a particular case of these field equations when the four-dimensional spacetime can be embedded in a five-dimensional ambient space whose dynamics comes from an integrable Weyl theory of gravity. In fact this is strongly motivated by a recently result concerning the existence of necessary and sufficient conditions for a Riemannian manifold to be embedded in a Weyl space [42, 46]. According to these, if the Weyl scalar field depends only on the extra coordinate l, then each leaf of the foliation $l =$ const has a Riemannian character and can be locally and isometrically embedded in a five-dimensional Weylian space whose metrical properties are given by (14). Therefore, this "anti-cylinder" condition on the Weyl field guarantees that even if the five-dimensional bulk is Weylian, the geometry of the hypersurface $l =$ const geometry is strictly Riemannian. (In passing, we should note that an analogous anti-cylinder condition has been used in five-dimensional models in which quantum confinement of fermions on hypersurfaces are driven by a scalar field depending only on the extra dimension [50].) Since we regard the spacetime as one of the leaves of the foliation and given that such embedding preserves the Riemannian character of the spacetime we proceed to investigate the four-dimensional field dynamics induced by the five-dimensional space. In much the same way as in induced-matter theory [25] one would interpret the extra contributions coming from the extra dimension as macrosocopic matter in 4D.

In view of the above let us assume that $\phi = \phi(l)$, i.e. the Weyl scalar field depends only on the extra coordinate l. In this case the field equations (15), (16), (17) and (13) become

$$^{(5)}\tilde{G}_{\alpha\beta} + \frac{1}{4}(5\xi-6)\Phi^{-2}g_{\alpha\beta}\phi_l^2 = 0, \tag{18}$$

[1] We shall adopt the convention diag$(+---)$ for the signature of $g_{\alpha\beta}$.

$$^{(5)}\tilde{G}_{\alpha l} = 0, \tag{19}$$

$$^{(5)}\tilde{G}_{ll} - \frac{1}{4}(6-5\xi)\phi_l^2 = 0, \tag{20}$$

$$\frac{\partial}{\partial l}\left[\sqrt{|g_5|}\,\Phi^{-2}\phi_l^2\right] = 0. \tag{21}$$

To illustrate with an example let the five-dimensional space M^5 correspond to a five-dimensional cosmological model in the form [47, 48] with metric given by

$$dS^2 = dt^2 - a^2(t)dr^2 - e^{2F(t)}dl^2, \tag{22}$$

where $dr^2 = \delta_{ij}dx^i dx^j$ is the three-dimensional Euclidian line element, t represents the cosmic time for co-moving observers, $F(t)$ is a well-behaved real function and $a(t)$ is the cosmological scale factor. Inserting the metric (22) in (21) it can be easily seen that the Weyl scalar field in this case is given by

$$\phi(l) = C_1 l + C_2, \tag{23}$$

where C_1 and C_2 are integration constants. The field equations (18), (19) and (20) now give

$$3H^2 + 3\dot{F}H = \frac{1}{4}(6-5\xi)C_1^2 e^{-2F}, \tag{24}$$

$$2\frac{\ddot{a}}{a} + H^2 + 2\dot{F}H + \ddot{F} + \dot{F}^2 = \frac{1}{4}(6-5\xi)C_1^2 e^{-2F}, \tag{25}$$

$$3\left(\frac{\ddot{a}}{a} + H^2\right) = -\frac{1}{4}(6-5\xi)C_1^2 e^{-2F}, \tag{26}$$

where $H(t) = \dot{a}/a$ is the Hubble "constant". From (25) and (26) we obtain the equation

$$\ddot{F} + \dot{F}^2 + 2H\dot{F} + 5\frac{\ddot{a}}{a} + 4H^2 = 0. \tag{27}$$

In order to simplify the structure of this equation we introduce a new function $u(t)$ defined by $u(t) = a(t)e^{F(t)}$. In this way (27) becomes

$$\ddot{u} + 4\left(\frac{\ddot{a}}{a} + H^2\right)u = 0. \tag{28}$$

The above equation relates $u(t)$ with $a(t)$ in such a way that the solutions of (28) can be substituted in (24) yielding a differential equation for $a(t)$, which in principle can be solved.

4 The Dynamics Induced on the Four-Dimensional Riemannian Hypersurface

As we have mentioned in the previous section one of our aims is to explore the possibility of interpreting the extra contributions of the five-dimensional Weylian bulk to the four-dimensional Riemannian hypersurface as four-dimensional matter induced geometrically. In this section we shall study the four-dimensional dynamics geometrically induced on a generic hypersurface. We recall that we are assuming that the five-dimensional space is foliated by a family of hypersurfaces $\{\Sigma\}$ defined by the equation $l = \text{const}$. Clearly, on a particular hypersurface Σ_0 the induced line element will be given by

$$dS^2_{\Sigma_0} = h_{\alpha\beta}(x)dx^\alpha dx^\beta, \tag{29}$$

where $h_{\alpha\beta}(x) = g_{\alpha\beta}(x, l_0)$ is the induced metric on Σ_0. From the Gauss-Codazzi equations it is easy to show (see, for instance [43]) that the induced dynamics on the hypersurface Σ_0 is governed by the four-dimensional field equations

$$^{(4)}\tilde{G}_{\alpha\beta} = T^{(IM)}_{\alpha\beta} + \Lambda(x)h_{\alpha\beta}, \tag{30}$$

where $T^{(IM)}_{\alpha\beta}$ is the usual energy momentum tensor obtained in the space-time-matter theory, which has the form [25]

$$\begin{aligned} T^{(IM)}_{\alpha\beta} &= \frac{\Phi_{\alpha||\beta}}{\Phi} + \frac{1}{2\Phi^2}\left\{\frac{\overset{\star}{\Phi}}{\Phi}\overset{\star}{g}_{\alpha\beta} - \overset{\star\star}{g}_{\alpha\beta} + g^{\lambda\mu}\overset{\star}{g}_{\alpha\lambda}\overset{\star}{g}_{\beta\mu} - \frac{1}{2}g^{\mu\nu}\overset{\star}{g}_{\mu\nu}\overset{\star}{g}_{\alpha\beta}\right. \\ &\quad \left. + \frac{1}{4}g_{\alpha\beta}\left[\overset{\star}{g}^{\mu\nu}\overset{\star}{g}_{\mu\nu} + \left(g^{\mu\nu}\overset{\star}{g}_{\mu\nu}\right)^2\right]\right\}, \end{aligned} \tag{31}$$

with the bars ($||$) denoting covariant derivative in a Riemannian sense and the star ($\star$) denoting derivative with respect to the fifth coordinate l, and the function $\Lambda(x)$ is given by

$$\Lambda(x) = \frac{1}{4}(6 - 5\xi)\Phi^{-2}\, \phi_l^2\Big|_{l=l_0}. \tag{32}$$

(We recall that we are assuming that the extra coordinate has a spacelike character.) Clearly, both terms $T^{(IM)}_{\alpha\beta}$ and $\Lambda(x)$ comes from the Weylian bulk. The induced energy-momentum tensor $T^{(IM)}_{\alpha\beta}$ can be obtained even if the bulk is Riemannian, but the interesting fact here is that the function $\Lambda(x)$ is a new contribution depending directly on the Weyl scalar field. It is worth mentioning that when the lapse function Φ depends only on the time, then $\Lambda(t)$ can be interpreted as an induced cosmological function, whereas if Φ is constant then (32) reduces to an induced cosmological constant.

5 A Simple Application to Cosmology

As a simple application of the ideas developed in the previous section let us have a quick look into the cosmological scenario that takes place in the four-dimensional hypersurface Σ_0, whose geometry is induced by the line element (22). In this case the induced line element (29) becomes

$$dS^2_{\Sigma_0} = dt^2 - a^2(t)dr^2, \tag{33}$$

which is nothing more than the line element of a Friedmann-Robertson-Walker model. The induced energy-momentum tensor (31) reduces to

$$T^{(IM)}_{\alpha\beta} = F_{,\alpha,\beta} + F_{,\alpha}F_{,\beta} - {}^{(4)}\{^{\gamma}_{\alpha\beta}\}F_{,\gamma}, \tag{34}$$

where the ${}^{(4)}\{^{\gamma}_{\alpha\beta}\}$ denote the four-dimensional Christoffel symbols calculated with the induced metric $h_{\mu\nu}$ in (33). Assuming that the induced matter configuration given by (34) is that of a perfect fluid, as viewed by four-dimensional comoving observers located at the hypersurface Σ_0, we can define the energy density $\rho_{(IM)}$ and pressure $P_{(IM)}$ for the induced matter by $\rho_{(IM)} = T^{(IM)\,t}{}_t$ and $P_{(IM)} = -T^{(IM)r}{}_r$ respectively. Thus using (33) the 4D field equations (30) become

$$3H^2 = \rho_{(IM)} + \Lambda(t), \tag{35}$$

$$2\frac{\ddot{a}}{a} + H^2 = -(P_{(IM)} - \Lambda(t)), \tag{36}$$

where according to (32) and (23) the induced varying cosmological "constant" $\Lambda(t)$ is given by

$$\Lambda(t) = \left(\frac{C_1}{2}\right)^2 (6 - 5\xi)e^{-2F(t)}. \tag{37}$$

Introducing the effective energy density $\rho_{eff} = \rho_{(IM)} + \Lambda(t)$ and the effective pressure $P_{eff} = P_{(IM)} - \Lambda(t)$ we define a parameter ω_{eff} associated with the effective equation of state, which is given by

$$\omega_{eff} \equiv \frac{P_{eff}}{\rho_{eff}} = -\left[1 - \frac{\dot{F}^2 + \ddot{F} - H\dot{F}}{\ddot{F} + \dot{F}^2 + \Lambda(t)}\right]. \tag{38}$$

By simple inspection it can easily be seen that ω_{eff} depends entirely on the metric function $F(t)$, which, in turn, can be determined by the bulk dynamics, i.e. by finding solutions of the system (24)–(26). A particular solution $F(t)$ for a given scale factor $a(t)$ can be obtained by solving (28). Thus if we look for solutions $F(t)$ in the case of a power-law expanding universe with the scale factor given by $a(t) = a_0(t/t_0)^p$, (28) becomes

$$\ddot{u} + \frac{4p(2p-1)}{t^2}u = 0, \tag{39}$$

whose general solution is given by

$$u(t) = A_1 t^{1/2+(1/2)\sqrt{1-32p^2+16p}} + A_2 t^{1/2-(1/2)\sqrt{1-32p^2+16p}}, \tag{40}$$

where A_1 and A_2 are integration constants. Moreover, choosing $A_2 = 0$, the corresponding particular solution for $F(t)$ can be written as

$$F(t) = \ln(B_1 t^{\gamma}), \tag{41}$$

where $B_1 = (A_1 t_0^p/a_0)$ and $\gamma = (1/2 - p) + (1/2)\sqrt{1 - 32p^2 + 16p}$. Note that if we want to have real values for the power γ that are compatible with an expanding universe ($p > 0$), the values of p must range in the interval $0 < p \leq (1/4) + \sqrt{6}/8$. On the other hand, if we insert (41) into (37) and (38), then the induced variable cosmological "constant" $\Lambda(t)$ and the effective parameter ω_{eff} are given, respectively, by

$$\Lambda(t) = \left(\frac{C_1}{2}\right)^2 (6 - 5\xi) B_1^{-2} t^{-2\gamma}, \tag{42}$$

$$\omega_{eff} = -\left[1 - \frac{\gamma^2 - \gamma - p\gamma}{\gamma^2 - \gamma + (C_1/2)^2 (6 - 5\xi) B_1^{-2} t^{2-2\gamma}}\right]. \tag{43}$$

If we want to have ω_{eff} decreasing with time we must require $2 - 2\gamma > 0$, and this condition restricts the range of variation of the parameter p to $p > 1/3$. Finally, if the former inequality is to be compatible with an expanding universe p must range in the interval $1/3 < p \leq (1/4) + \sqrt{6}/8$. One reason for restricting the parameter p to this interval is that with a suitable choice of p the effective parameter ω_{eff} will tend asymptotically to -1. We conclude that, in its final state, our model would tend to a de Sitter universe. Note that in the case when $p = 5/9$ the induced $\Lambda(t)$ given by (42) becomes a constant, while the value of ω_{eff} is exactly -1. This means that, in such models, $p = 5/9$ corresponds to a de Sitter universe. The fact that ω_{eff} is decreasing with time can perhaps be interpreted as if this model were effectively mimicking freezing quintessential models [51].

Finally, in order to have an energy density associated to $\Lambda(t)$, here denoted by $\rho_{\Lambda(t)}$, which can not be steeper than the energy densities of radiation $\rho_r \sim t^{-2}$ and matter ($\rho_m \sim t^{-2}$), during the epochs dominated by radiation and matter respectively, (42) requires the condition $\gamma < 1$ to be valid, in and accordance with the condition $2 - 2\gamma > 0$.

6 Final Remarks

In this paper we have considered the idea of generating a cosmological constant, or rather, a cosmological function, from extra dimensions. Although this has already been investigated in the context of space-time-matter theory, the novelty of our approach is to regard the same problem in a more general setting, i.e. by assuming the geometry of the embedding space to have a Weylian character. Being one of the simplest generalizations of Riemannian geometry, the theory developed by Weyl, in the opinion of some authors, "contains a suggestive formalism and may still have the germs of a future fruitful theory" [38]. Two comments are in order here: Firstly, the embedding space has a prescribed dynamics; secondly, the embedding does not affect the Riemannian geometry of the spacetime. These features depend on the fact that the Weyl field is assumed to be integrable and depending only on the extra dimension.

Finally, we think that it would be interesting to compare the expression we have obtained for $\Lambda(t)$ with similar ones found in other theories [52] and also to discuss some observational consequences of our model. However, we think this would take us far beyond our main goal, namely, to set up a simple "toy model" to call attention to the richness of non-Riemannian geometries, in particular to the Weyl integrable manifolds, as a way of providing new degrees of freedom that might play a role in the theoretical framework of higher-dimensional embedding theories of spacetime. We believe that in this context issues such as the nature of the cosmological constant, dark energy and other important questions may be investigated from an entirely new point of view. We leave these subjects for future work.

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