Time Translation of Quantum Properties

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Abstract Based on the notion of time translation, we develop a formalism to deal with the logic of quantum properties at different times. In our formalism it is possible to enlarge the usual notion of context to include composed properties involving properties at different times. We compare our results with the theory of consistent histories.

1 Introduction

The absence of determinism is one of the main differences of the quantum theory with the classical one. Nevertheless, it is necessary in quantum theory to deal in a consistent way with expressions involving different properties of the system at different times. For example, it is necessary to relate a property of a microscopic system at a given time, previous to a measurement, with a property of an instrument when the measurement is finished. Moreover, in the famous double slit experiment it is argued about the impossibility to say which slit a particle passed before producing a spot on a photographic plate [1].

In a series of papers starting in 1984, an approach to quantum interpretation known as consistent histories, or decoherent histories, has been introduced by R. Griffiths

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[2], R. Omnès [3], M. Gell-Mann and J. Hartle [4]. In their approach, the notion of history is defined as a sequence of properties at different times. The probability for a history is also defined through a formula motivated by the path integral formalism, but with no direct relation with the Born rule. A consistency condition should be satisfied by the possible different histories which can be included in a legitimate description of the system, therefore the name of consistent histories. For a given physical system the possible sets of consistent histories depend on the state. This is not entirely satisfactory, because in axiomatic theories of quantum mechanics the state is usually considered as a functional on the space of observables, and it appears after these observables in a somehow subordinate position. The importance of the notion of state functionals acting on a previously defined space of observables was stressed by one of us in references [5] and [6].

In this paper we explore a different approach to define the probability of a conjunction of properties at different times, and to discriminate which are the properties that can be simultaneously considered in a description of the system. Essentially, we consider the time translation of a property, already at our disposal in the dynamic generated by the Schrödinger equation. When properties corresponding to different times are translated to a common single time, we can apply to them the usual rules of compatibility between different observables, and compute the probabilities using the Born rule.

In Sect. 2 we present a brief summary of the theory of consistent histories, and we discuss its application to a spin system. In Sect. 3 we give a short review of the notions of quantum logic, contexts and probabilities. The notion of time translation of quantum properties is introduced in Sect. 4, where we also obtain the non distributive lattice of time dependent properties. This definitions are used in Sect. 5 to implement expressions involving the conjunction of properties at different times, and to define the compatibility of these type of expressions. Distributive lattices of time dependent properties, called generalized contexts, are also obtained in this section. The generalized contexts are compared with the theory of consistent histories in Sect. 6. The conclusions are given in Sect. 7.

2 The Theory of Consistent Histories

In what follows we present the main features of the theory of consistent histories, following essentially the approach given by R. Omnès [7–9], and we also discuss the application of this theory to the case of a spin system.

Let us consider a state of a system at time t_0, represented by the state operator $\widehat{\rho}_{t_0}$. An observable represented by an operator $\widehat{A}_j$ with spectrum σ_j is considered for each time t_j $(j = 1, \ldots, n)$ of a sequence verifying $t_0 < t_1 < \cdots < t_n$. Each spectrum σ_j is partitioned by a family $\{\Delta_j^{(\mu)}\}$ of mutually exclusive sets $(\bigcup_\mu \Delta_j^{(\mu)} = \sigma_j, \Delta_j^{(\mu)} \cap \Delta_j^{(\mu')} = \varnothing \ (\mu \neq \mu'))$.

The operator $\widehat{E}_j^{(\mu)}$ is the projector onto the subspace of the Hilbert space corresponding to the subset $\Delta_j^{(\mu)}$ of the spectrum σ_j, and it represents the property "the

value of the observable A_j is in the set $\Delta_j^{(\mu)}$ at time t_j". These projectors satisfy the equations $\widehat{E}_j^{(\mu)} \widehat{E}_j^{(\mu')} = \delta_{\mu\mu'} \widehat{E}_j^{(\mu')}$ and $\sum_\mu \widehat{E}_j^{(\mu)} = \widehat{I}$.

A *history a* is defined by the property "(the value of A_1 is in $\Delta_1^{(k_1)}$ at t_1) and (the value of A_2 is in $\Delta_2^{(k_2)}$ at t_2) and … and (the value of A_n is in $\Delta_n^{(k_n)}$ at t_n)". It is represented by the *history operator*

$$\widehat{C}_a = \widehat{E}_n^{(k_n)}(t_n) \ldots \widehat{E}_2^{(k_2)}(t_2) \, \widehat{E}_1^{(k_1)}(t_1), \quad \widehat{E}_j^{(k_j)}(t_j) = e^{i\widehat{H}(t_j - t_0)/\hbar} \widehat{E}_j^{(k_j)} e^{-i\widehat{H}(t_j - t_0)/\hbar},$$
(1)

where $\widehat{H}$ is the Hamiltonian operator of the system.

The probability of the history *a* is defined by the expression

$$\Pr(a) = \mathrm{Tr}(\widehat{C}_a \, \widehat{\rho}_{t_0} \, \widehat{C}_a^\dagger).$$
(2)

As the probability should verify positivity, normalization and additivity, the possible histories to be included in a valid description of the system should verify some *consistency conditions*. Sufficient conditions, given by Gell-Mann and Hartle, are

$$\mathrm{Tr}(\widehat{C}_a \, \widehat{\rho}_{t_0} \, \widehat{C}_b^\dagger) = 0, \quad a \neq b.$$
(3)

The theory of consistent histories is a framework suitable to include properties of a system at different times in the language of quantum theory. Moreover, these properties at different times are given a well defined probability by (2), provided we consider properties within a consistent family of histories. Each family of consistent histories generate a possible *universe of discourse* about a quantum system. In general, it is not possible to include two families in a single larger one. Different sets of consistent histories are considered *complementary descriptions* of the system.

For simplicity we are going to consider the case $n = 2$, involving histories with only two different times t_1 and t_2. For the time t_1 the spectrum of the observable $\widehat{A}_1$ is partitioned by two complementary sets Δ_1 and $\overline{\Delta}_1$ with projectors $\widehat{E}_1$ and $\widehat{\overline{E}}_1$, while for the time t_2 the spectrum of the corresponding observable $\widehat{A}_2$ is partitioned by the sets Δ_2 and $\overline{\Delta}_2$ with projectors $\widehat{E}_2$ and $\widehat{\overline{E}}_2$.

For this special case the necessary and sufficient consistency condition to obtain well defined probabilities is

$$\mathrm{Re} \, \mathrm{Tr}(\widehat{E}_1(t_1) \, \widehat{\rho}_{t_0} \, \widehat{\overline{E}}_1(t_1) \, \widehat{E}_2(t_2)) = 0,$$
(4)

which is called the Griffiths condition.

In this case, the sufficient Gell-Mann and Hartle condition is

$$\mathrm{Tr}(\widehat{E}_1(t_1) \, \widehat{\rho}_{t_0} \, \widehat{\overline{E}}_1(t_1) \, \widehat{E}_2(t_2)) = 0.$$
(5)

Logical operations and relations are well defined on a family of consistent histories. If $\Sigma_1 = \Delta_1 \cup \overline{\Delta}_1$ is the spectrum of $\widehat{A}_1$ and $\Sigma_2 = \Delta_2 \cup \overline{\Delta}_2$ is the spectrum of $\widehat{A}_2$, the elementary histories are represented in $\Sigma_1 \times \Sigma_2$ by the sets $\Delta_1 \times \Delta_2$, $\Delta_1 \times \overline{\Delta}_2$, $\overline{\Delta}_1 \times \Delta_2$ and $\overline{\Delta}_1 \times \overline{\Delta}_2$. All the properties of the family are represented by the unions of these four sets. In this way, the notions of *conjunction*, *disjunction* and *negation*

are obtained. According to the approach of R. Omnès [7, 8], a property a is said to *imply* another property b of the same family if $\Pr(b|a) = \Pr(b \text{ and } a)/\Pr(a) = 1$. The conventional axioms of formal logic are satisfied with these definitions [8]. We notice that in this theory the implication relation is defined trough a *previous* definition of probability. This makes a big difference with the usual approaches to quantum mechanics, where the logical relations and operators on the properties have their own definition independent of the probability, which is later on given by the Born rule. Our own construction of the logical operators and relations, to be developed in Sects. 4 and 5, will be established in a way which do not depend on the definition of probability or on the state of the system.

R. Omnès [8] applied the theory of consistent histories to the case of a $\frac{1}{2}$ spin system, prepared at the time t_0 in a pure state with $S_x = +\frac{1}{2}\hbar$, represented by the vector $|x+\rangle$ ($\widehat{\rho}_{t_0} = |x+\rangle\langle x+|$). The description of the system include the two possible values of the spin along the z axis direction for the time t_2, which may be obtained by an Stern-Gerlach experiment. With the simplifying assumption of the vanishing of the Hamiltonian, he searched for the possibility to include in the description the two possible values of the spin along a direction $\overline{n}_1$ ($|\overline{n}_1| = 1$) for a time t_1 after the preparation of the spin along the positive direction of the x axis, and before the time t_2 corresponding to the measurement along the z axis ($t_0 < t_1 < t_2$). The state vectors $|\overline{n}_1+\rangle$ and $|\overline{n}_1-\rangle$ correspond to the values $+\frac{1}{2}\hbar$ and $-\frac{1}{2}\hbar$ along the direction $\overline{n}_1$. If the sufficient Gell-Mann and Hartle condition (5) is applied with $\widehat{E}_1(t_1) = |\overline{n}_1+\rangle\langle\overline{n}_1+|$, $\widehat{\overline{E}}_1(t_1) = |\overline{n}_1-\rangle\langle\overline{n}_1-|$, $\widehat{\rho}_0 = |x+\rangle\langle x+|$ and $\widehat{E}_2(t_2) = |z+\rangle\langle z+|$, two possibilities are obtained:

(i) A set of histories including the two possible spin values along the x axis at time t_1, represented by the projectors $\widehat{E}_1 = |x+\rangle\langle x+|$ and $\widehat{\overline{E}}_1 = |x-\rangle\langle x-|$, *together with* the two possible spin values along the z axis at time t_2, represented by the projectors $\widehat{E}_2 = |z+\rangle\langle z+|$ and $\widehat{\overline{E}}_2 = |z-\rangle\langle z-|$. Therefore in this case $\overline{n}_1 = (1, 0, 0)$.

(ii) A set of histories including the two possible spin values along the z axis at time t_1, represented by the projectors $\widehat{E}_1 = |z+\rangle\langle z+|$ and $\widehat{\overline{E}}_1 = |z-\rangle\langle z-|$, *together with* the two possible spin values along the z axis at time t_2, represented by the projectors $\widehat{E}_2 = |z+\rangle\langle z+|$ and $\widehat{\overline{E}}_2 = |z-\rangle\langle z-|$. In this case $\overline{n}_1 = (0, 0, 1)$.

More possible consistent histories are obtained using the necessary and sufficient Griffiths condition given in (4). If $\overline{n}_0$ is the preparation spin direction at the time t_0, and if $\overline{n}_2$ is the spin direction at the time t_2, the equation $(\overline{n}_0 \times \overline{n}_1) \cdot (\overline{n}_1 \times \overline{n}_2) = 0$ is obtained for the possible $\overline{n}_1$ spin directions at the time t_1 (see reference [9], page 161). For $\overline{n}_0 = (1, 0, 0)$ and $\overline{n}_2 = (0, 0, 1)$, representing the x and the z directions, the direction $\overline{n}_1$ could be any vector in the planes xy or yz.

Up to now, we have considered well known facts of the theory of consistent histories. We now analyze in more detail the consistent family obtained in the case (i). Well defined probabilities can be obtained for all the members of the family applying (2). Let us consider the probability of the spin to be $+\frac{1}{2}\hbar$ along the x axis at time

t_1 *and* to be $+\frac{1}{2}\hbar$ along the z axis at time t_2,

$$\Pr((x+, t_1); (z+, t_2)) = \text{Tr}(\widehat{E}_2\widehat{E}_1\widehat{\rho}_{t_0}\widehat{E}_1\widehat{E}_2) = |\langle x+|z+\rangle|^2 = \frac{1}{2}.$$

Provided that $t_0 < t_1 < t_2$, any value of t_1 gives a consistent family of histories, and therefore a valid description of the spin system. As the time t_1 can be chosen very close to the time t_2 we have

$$\lim_{t_1 \longrightarrow t_2} \Pr((x+, t_1); (z+, t_2)) = \frac{1}{2}. \tag{6}$$

The property giving simultaneously well defined values to different components of the spin is forbidden in the universe of discourse of ordinary quantum mechanics, due to the uncertainty principle (operators representing two different components of the spin do not commute). Therefore the limit of (6) cannot be interpreted as the probability for the conjunction of the spin values $x+$ and $z+$ at the single time t_2. The theory of consistent histories is discontinuous in its property ascriptions for different times.

In the following sections we present our own approach to the description of time dependent properties of a quantum system. We are going to prove that in our formalism only the case (ii) survives as a valid description for the two times properties of the spin system. Only for this case, the limit $t_1 \longrightarrow t_2$ of (6) can be interpreted as a single time probability.

3 Quantum Logic, Contexts and Probabilities

We shall give in this section a brief summary of the logical structure of quantum mechanics according to G. Birkhoff and J. von Neumann [10], which is one of the standard approaches to quantum logic [11]. This summary is a preparation for our own approach to the problem of time dependent properties, to be developed in Sect. 4.

A Hilbert space $\mathcal{H}$ is associated with each isolated physical system. Every quantum property p is represented by a subspace $\mathcal{H}_p$ of the Hilbert space $\mathcal{H}$. For each subspace $\mathcal{H}_p$ there exists a projection operator $\widehat{\Pi}_p$ such that $\mathcal{H}_p = \widehat{\Pi}_p\mathcal{H}$, and therefore a property p can also be represented by the projector $\widehat{\Pi}_p$.

The *implication relation* between two properties is *defined* by the inclusion of the corresponding subspaces ($p_1 \Rightarrow p_2$ iff $\mathcal{H}_{p_1} \subseteq \mathcal{H}_{p_2}$). The *conjunction* of two properties p and p' is represented by the greatest lower bound of the two corresponding subspaces $\mathcal{H}_p$ and $\mathcal{H}_{p'}$ ($\mathcal{H}_{p \wedge p'} = \text{Inf}(\mathcal{H}_p, \mathcal{H}_{p'}) = \mathcal{H}_p \cap \mathcal{H}_{p'}$), while the *disjunction* is the least upper bound ($\mathcal{H}_{p \vee p'} = \text{Sup}(\mathcal{H}_p, \mathcal{H}_{p'})$). Moreover, the *negation* of a property p is represented by the orthogonal complement of the corresponding subspace ($\mathcal{H}_{-p} = \mathcal{H}_p^{\perp}$).

Endowed with these implication, conjunction, disjunction and negation, the set of all properties of a physical system is an orthocomplemented nondistributive lattice. According to quantum theory, not all the properties can simultaneously be considered in a description of a physical system. Different descriptions involve different sets of properties.

Each possible description is called a *context*, and it is defined through a set of *atomic or elementary properties* p_j (where j belongs to a set σ of indexes). The properties p_j, represented by projectors $\widehat{\Pi}_j$, are mutually exclusive and complete, i.e.

$$\widehat{\Pi}_i \widehat{\Pi}_j = \delta_{ij} \widehat{\Pi}_i, \quad \sum_{i \in \sigma} \widehat{\Pi}_i = \widehat{I}.$$

By disjunction of these elementary properties it is possible to generate all the properties of the context, obtaining a *distributive lattice*. Each property p of the context can be represented by a projector which is a sum of *some* of the projectors $\widehat{\Pi}_i$,

$$\widehat{\Pi}_p = \sum_{i \in \sigma_p} \widehat{\Pi}_i, \quad \sigma_p \subset \sigma.$$

A state of the system is represented by the statistical operator $\widehat{\rho}$, which is self adjoint, positive and with trace equal to one. In the state represented by $\widehat{\rho}$, the probability for each property p of a context is given by the Born rule

$$\Pr(p) = \mathrm{Tr}(\widehat{\rho} \, \widehat{\Pi}_p).$$

This rule gives a well defined probability (i.e. it is positive, normalized, and satisfy the additivity property) when it is applied to the properties of a given context.

Moreover, the probability of a property b conditional to a property a, *defined* by

$$\Pr(b|a) = \frac{\Pr(b \wedge a)}{\Pr(a)}, \tag{7}$$

is also a well defined probability within a context. The conditional probability can be used to give a statistical meaning to the implication relation. It is not difficult to prove that a property a implies a property b $(a \Rightarrow b)$ *if and only if* $\Pr(b|a) = 1$ for *all* the states of the system.

We emphasize that the conjunction and the disjunction of the logical structure of the quantum properties are obtained from the previous notion of implication, defined by the inclusion of subspaces. On this lattice of properties, the Born rule is applied to obtain the probabilities, which are only meaningful for subsets of properties within a context. Properties belonging to different contexts do not have simultaneous physical meaning.

In the next section we are going to present an extended notion of context to deal with descriptions involving time dependent properties.

4 Time Translations and the Lattice of Time Dependent Properties

The time enters quantum theory through the Schrödinger equation generating the evolution of the state of an isolated system. The evolution of a vector of the Hilbert space $\mathcal{H}$, representing a pure state, is given by

$$|\varphi_{t'}\rangle = \widehat{U}(t', t) |\varphi_t\rangle, \quad \widehat{U}(t', t) = e^{-i\widehat{H}(t'-t)/\hbar}, \tag{8}$$

where $\widehat{H}$ is the Hamiltonian operator of the system.

Let us consider the physical system at the time t, in a pure state represented by the vector $|\varphi_t\rangle$. Moreover, assume that $|\varphi_t\rangle$ is an eigenvector with eigenvalue one of the projector $\widehat{\Pi}_p$ corresponding to the property p

$$\widehat{\Pi}_p|\varphi_t\rangle = |\varphi_t\rangle. \tag{9}$$

We can say that the physical system *has* the property p at the time t, because it has probability equal to one, according to the Born rule:

$$\mathrm{Pr}(p) = \langle\varphi_t|\widehat{\Pi}_p|\varphi_t\rangle = \langle\varphi_t|\varphi_t\rangle = 1.$$

At a later time t', the time evolved state is represented by the vector $|\varphi_{t'}\rangle$ given by (8). Using (8) and (9) it is easy to prove that

$$\widehat{\Pi}_{p'}|\varphi_{t'}\rangle = |\varphi_{t'}\rangle, \quad \widehat{\Pi}_{p'} = \widehat{U}(t',t)\widehat{\Pi}_p\widehat{U}^{-1}(t',t).$$

Therefore, if the system has the property represented by the projector $\widehat{\Pi}_p$ at time t, it also has the property represented by the projector $\widehat{\Pi}_{p'}$ at time t'. We have obtained in this way a procedure for the *time translation of properties*.

It is easily proved that the obtained relation between p at time t and p' at time t' is transitive, reflexive and symmetric. Therefore, it is an equivalence relation, that we shall indicate by the expression $(\widehat{\Pi}_p, t) \smile (\widehat{\Pi}_{p'}, t')$.

The expression $(\widehat{\Pi}_p, t)$ is a symbol indicating the *property p at time t*. We shall also indicate by $[\widehat{\Pi}_p, t]$ the *class of properties equivalent to the property p at time t*.

If a property p at time t is equivalent to p' at time t', the Born rule gives for them the same probability,

$$\mathrm{Pr}(\widehat{\Pi}_{p'}, t') = \mathrm{Tr}(\widehat{\rho}_{t'}\widehat{\Pi}_{p'}) = \mathrm{Tr}(\widehat{U}(t',t)\widehat{\rho}_t\widehat{U}^{-1}(t',t)\widehat{\Pi}_{p'}) = \mathrm{Tr}(\widehat{\rho}_t\widehat{\Pi}_p) = \mathrm{Pr}(\widehat{\Pi}_p, t), \tag{10}$$

and therefore a single probability is obtained for the properties of the same class of equivalence. In physical terms, all the properties of a given class are essentially the same property, as they can be obtained one from the other trough time evolution.

The just considered time translation, and the implication of ordinary quantum mechanics presented in the previous section, suggest that we *define* that the equivalence class $[\widehat{\Pi}_1, t_1]$ *implies* the equivalence class $[\widehat{\Pi}_2, t_2]$ if the representative elements at a common time t_0 verify the implication of the usual formalism of quantum mechanics, i.e.

$$\widehat{\Pi}_{1,0}\,\mathcal{H} \subset \widehat{\Pi}_{2,0}\mathcal{H},$$
$$\widehat{\Pi}_{1,0} \equiv \widehat{U}(t_0,t_1)\widehat{\Pi}_1\widehat{U}^{-1}(t_0,t_1), \quad \widehat{\Pi}_{2,0} \equiv \widehat{U}(t_0,t_2)\widehat{\Pi}_2\widehat{U}^{-1}(t_0,t_2)\,\mathcal{H}.$$

It is not difficult to prove that if two projectors $\widehat{\Pi}_1$ and $\widehat{\Pi}_2$ verify this condition for a given time t_0, they verify the condition for all possible values of t_0. The implication relation is transitive, reflexive and antisymmetric, and therefore it is a well defined *order relation* on the equivalence classes.

Having defined an order relation on the equivalence classes, the *conjunction* (*disjunction*) of two classes $[\widehat{\Pi}, t]$ and $[\widehat{\Pi}', t']$ can be obtained as the greatest lower (least upper) bound, i.e.

$$[\widehat{\Pi}, t] \wedge [\widehat{\Pi}', t'] = \text{Inf}\{[\widehat{\Pi}, t]; [\widehat{\Pi}', t']\} = \left[\lim_{n \to \infty} (\widehat{\Pi}_0 \widehat{\Pi}'_0)^n, t_0 \right], \qquad (11)$$

$$[\widehat{\Pi}, t] \vee [\widehat{\Pi}', t'] = \text{Sup}\{[\widehat{\Pi}, t]; [\widehat{\Pi}', t']\} = \left[\left(\widehat{I} - \lim_{n \to \infty} \{ (\widehat{I} - \widehat{\Pi}_0)(\widehat{I} - \widehat{\Pi}'_0) \}^n \right), t_0 \right], \qquad (12)$$

where $\widehat{\Pi}_0 = \widehat{U}(t_0, t)\, \widehat{\Pi}\, \widehat{U}^{-1}(t_0, t)$ and $\widehat{\Pi}'_0 = \widehat{U}(t_0, t')\, \widehat{\Pi}'\, \widehat{U}^{-1}(t_0, t')$ are the translations of the properties $\widehat{\Pi}$ and $\widehat{\Pi}'$ to the time t_0. The projectors $\lim_{n \to \infty} (\widehat{\Pi}_0 \widehat{\Pi}'_0)^n$ and $(\widehat{I} - \lim_{n \to \infty} \{ (\widehat{I} - \widehat{\Pi}_0)(\widehat{I} - \widehat{\Pi}'_0) \}^n)$ generate the greatest lower and the least upper bounds of the subspaces generated by $\widehat{\Pi}_0$ and $\widehat{\Pi}'_0$ [12].

The *negation* of an equivalence class $[\widehat{\Pi}, t]$ is defined by

$$\overline{[\widehat{\Pi}, t]} = [\widehat{\overline{\Pi}}, t] = [(\widehat{I} - \widehat{\Pi}), t].$$

With the implication, disjunction, conjunction and negation previously obtained, the set of equivalent classes has the structure of an orthocomplemented nondistributive lattice.

5 The Generalized Contexts

The usual concept of context was briefly reviewed in Sect. 3 as a subset of all possible simultaneous properties which can be organized as a meaningful description of a quantum system at a given time, and endowed with a boolean logic with well defined probabilities.

The definitions and notations given in the previous section will be useful to our purpose of representing valid *descriptions involving properties at different times*, which we are going to call *generalized contexts*. In what follows, we shall only consider descriptions involving properties at two times t_1 and t_2, but our formalism has an immediate extension to cases involving more than two times.

Let us consider a context of properties at time t_1, generated by atomic properties $p_j^{(1)}$ represented by projectors $\widehat{\Pi}_j^{(1)}$ verifying

$$\widehat{\Pi}_i^{(1)} \widehat{\Pi}_j^{(1)} = \delta_{ij} \widehat{\Pi}_i^{(1)}, \quad \sum_{j \in \sigma^{(1)}} \widehat{\Pi}_j^{(1)} = \widehat{I}, \ i, j \in \sigma^{(1)}. \qquad (13)$$

Let us also consider a context of properties at time t_2, generated by atomic properties $p_\mu^{(2)}$ represented by projectors $\widehat{\Pi}_\mu^{(2)}$ verifying

$$\widehat{\Pi}_\mu^{(2)} \widehat{\Pi}_\nu^{(2)} = \delta_{\mu\nu} \widehat{\Pi}_\mu^{(2)}, \quad \sum_{\mu \in \sigma^{(2)}} \widehat{\Pi}_\mu^{(2)} = \widehat{I}, \ \mu, \nu \in \sigma^{(2)}. \qquad (14)$$

We wish to represent with our formalism a universe of discourse capable to incorporate expressions like "the property $p_j^{(1)}$ at time t_1 *and* the property $p_\mu^{(2)}$ at time t_2". With this purpose, we note that the properties associated to different times t_1 and t_2 can be translated to a common time t_0, by using the equivalence relations previously defined

$$
\begin{aligned}
(\widehat{\Pi}_i^{(1)}, t_1) \smile (\widehat{\Pi}_i^{(1,0)}, t_0), \qquad & \widehat{\Pi}_i^{(1,0)} \equiv \widehat{U}(t_0, t_1) \widehat{\Pi}_i^{(1)} \widehat{U}^{-1}(t_0, t_1), \\
(\widehat{\Pi}_\mu^{(2)}, t_2) \smile (\widehat{\Pi}_\mu^{(2,0)}, t_0), \qquad & \widehat{\Pi}_\mu^{(2,0)} \equiv \widehat{U}(t_0, t_2) \widehat{\Pi}_\mu^{(2)} \widehat{U}^{-1}(t_0, t_2).
\end{aligned}
\tag{15}
$$

The conjunction of the equivalence classes $[\widehat{\Pi}_i^{(1)}, t_1]$ and $[\widehat{\Pi}_\mu^{(2)}, t_2]$ can be obtained applying (11)

$$
[\widehat{\Pi}_i^{(1)}, t_1] \wedge [\widehat{\Pi}_\mu^{(2)}, t_2] = [\widehat{\Pi}_i^{(1,0)}, t_0] \wedge [\widehat{\Pi}_\mu^{(2,0)}, t_0] = \left[\lim_{n \to \infty} (\widehat{\Pi}_i^{(1,0)} \widehat{\Pi}_\mu^{(2,0)})^n, t_0 \right].
$$

The conjunction of the classes with representative elements $\widehat{\Pi}_i^{(1)}$ at t_1 and $\widehat{\Pi}_\mu^{(2)}$ at t_2, is also the conjunction of the classes with representative elements $\widehat{\Pi}_i^{(1,0)}$ and $\widehat{\Pi}_\mu^{(2,0)}$ at the common time t_0. Moreover, the conjunction is a class with the representative element $\lim_{n \to \infty} (\widehat{\Pi}_i^{(1,0)} \widehat{\Pi}_\mu^{(2,0)})^n$ at the time t_0. The conjunction of properties at the same time is already defined in quantum mechanics, *for the particular case in which they are represented by commuting projectors*. The usual quantum theory do not give any meaning to the conjunction of simultaneous properties represented by non commuting operators.

To make contact with the usual formalism of quantum theory, it seems natural to consider quantum descriptions of a system, involving the properties generated by the projectors $\widehat{\Pi}_i^{(1)}$ at the time t_1 and $\widehat{\Pi}_\mu^{(2)}$ at the time t_2, *only for the cases* in which the projectors $\widehat{\Pi}_i^{(1)}$ and $\widehat{\Pi}_\mu^{(2)}$ commute when translated to a common time t_0, i.e.

$$
\widehat{\Pi}_i^{(1,0)} \widehat{\Pi}_\mu^{(2,0)} - \widehat{\Pi}_\mu^{(2,0)} \widehat{\Pi}_i^{(1,0)} = 0.
\tag{16}
$$

If this is the case, we have $\lim_{n \to \infty} (\widehat{\Pi}_i^{(1,0)} \widehat{\Pi}_\mu^{(2,0)})^n = \widehat{\Pi}_i^{(1,0)} \widehat{\Pi}_\mu^{(2,0)}$, and for the equivalence class of composed properties $h_{i\mu}$, representing "the property $p_j^{(1)}$ at time t_1 *and* the property $p_\mu^{(2)}$ at time t_2" we obtain

$$
h_{i\mu} = [\widehat{\Pi}_i^{(1)}, t_1] \wedge [\widehat{\Pi}_\mu^{(2)}, t_2] = [\widehat{\Pi}_i^{(1,0)} \widehat{\Pi}_\mu^{(2,0)}, t_0] = [\widehat{\Pi}_{i\mu}^{(0)}, t_0], \quad \widehat{\Pi}_{i\mu}^{(0)} \equiv \widehat{\Pi}_i^{(1,0)} \widehat{\Pi}_\mu^{(2,0)}.
$$

As we can see, the conjunction of properties at different times t_1 and t_2 is equivalent to a single property represented by the projector $\widehat{\Pi}_{i\mu}^{(0)}$ at the single time t_0.

If the different contexts at times t_1 and t_2 produce commuting projectors $\widehat{\Pi}_i^{(1,0)}$ and $\widehat{\Pi}_\mu^{(2,0)}$ at the common time t_0, it is easy to prove that

$$
\widehat{\Pi}_{i\mu}^{(0)} \widehat{\Pi}_{j\nu}^{(0)} = \delta_{ij} \delta_{\mu\nu} \widehat{\Pi}_{i\mu}^{(0)}, \quad \sum_{i\mu} \widehat{\Pi}_{i\mu}^{(0)} = \widehat{I}.
\tag{17}
$$

Therefore, we realize that the composed properties $h_{i\mu}$, represented at the time t_0 by the complete and exclusive set of projectors $\widehat{\Pi}_{i\mu}^{(0)}$, can be interpreted as the atomic properties generating a usual context in the sense already described in the previous section. More general properties are obtained from the atomic ones using the disjunction operation defined in (12). For example, taking into account the commutation relation (16), we obtain

$$h_{i\mu} \vee h_{j\nu} = [\widehat{\Pi}_{i\mu}^{(0)} + \widehat{\Pi}_{j\nu}^{(0)}, t_0].$$

More generally, we can represent the property $p_j^{(1)}$ at time t_1 *and* the property $p_\mu^{(2)}$ at time t_2, with j and μ having any value in the subsets $\Delta^{(1)} \subset \sigma^{(1)}$ and $\Delta^{(2)} \subset \sigma^{(2)}$, in the form

$$h_{\Delta^{(1)},\Delta^{(2)}} = \left[\sum_{i \in \Delta^{(1)}} \sum_{\mu \in \Delta^{(2)}} \widehat{\Pi}_{i\mu}^{(0)}, t_0 \right]. \tag{18}$$

As a consequence of (17), the set of properties obtained in this way is an orthocomplemented and distributive lattice.

As we proved in (10), the Born rule defines a single probability to all elements of an equivalence class. If the state of the system at time t_0 is represented by $\widehat{\rho}_{t_0}$, the probability of the class of properties $h_{\Delta^{(1)},\Delta^{(2)}}$ has the following expression

$$\Pr(h_{\Delta^{(1)},\Delta^{(2)}}) = \sum_{i \in \Delta^{(1)}} \sum_{\mu \in \Delta^{(2)}} \mathrm{Tr}(\widehat{\rho}_{t_0} \widehat{\Pi}_{i\mu}^{(0)}). \tag{19}$$

As we already mentioned in the previous section, a description of a physical system should not involve properties belonging to different contexts. As a natural extension of the notion of context, we *postulate* that a description of a physical system involving properties at two different times t_1 and t_2 is valid if these properties are represented by commuting projectors when they are translated to a single time t_0. We will call *generalized context* to each of these valid descriptions. On each generalized context, the probabilities given by the Born rule are well defined (i.e. they are positive, normalized and additive), and therefore they may have a meaning in terms of frequencies.

In summary, our formalism is based on the notion of time translation, allowing to transform the properties at a sequence of different times into properties at a single common time. A usual context of properties is first considered for each time of the sequence. If the projectors representing the atomic properties of each context commute when they are translated to a common time, the contexts at different times can be organized forming a generalized context of properties. A generalized context of properties is a distributive and orthocomplemented lattice, a boolean logic with well defined implication, negation, conjunction and disjunction. This logic can be used for speaking and reasoning about the selected properties of the system at different times. Well defined probabilities on the elements of the lattice of properties are obtained using the well known Born rule.

6 Comparison of the Generalized Contexts with the Sets of Consistent Histories

In our opinion, the generalized contexts seem to be a natural generalization of the usual contexts of quantum mechanics. They are suitable to deal with the logic of properties at different times. But to be a "natural generalization" may have no any scientific value, and perhaps may only reflects our confidence in the usual form of quantum theory. Therefore, it is necessary to compare our new approach with the theory of consistent histories, designed to deal with the same kind of problems, and also to apply the new formalism to physically relevant situations. The main relation between both theories is given by the following Theorem:

A generalized context obtained with our formalism, is also a consistent set of histories, with the same probabilities.

We give the proof for a generalized context with two times. The probability for the property $p_j^{(1)}$ at time t_1 *and* the property $p_\mu^{(2)}$ at time t_2, is given by (19)

$$\Pr((p_j^{(1)}, t_1) \wedge (p_\mu^{(2)}, t_2)) = \text{Tr}(\widehat{\rho}_{t_0} \widehat{\Pi}_{j\mu}^{(0)}) = \text{Tr}(\widehat{\rho}_{t_0} \widehat{\Pi}_j^{(1,0)} \widehat{\Pi}_\mu^{(2,0)})$$
$$= \text{Tr}(\widehat{\Pi}_\mu^{(2,0)} \widehat{\Pi}_j^{(1,0)} \widehat{\rho}_{t_0} \widehat{\Pi}_j^{(1,0)} \widehat{\Pi}_\mu^{(2,0)}),$$

where the last equality is a consequence of the commutation relation (16) and the cyclic permutation of the operators in the trace. Taking into account the definitions (15), we obtain for the probability the same expression which is obtained with (2) for consistent histories. Moreover, the consistency conditions $\text{Tr}(\widehat{C}_a \widehat{\rho}_{t_0} \widehat{C}_b^\dagger) = 0$, for $a \neq b$, are satisfied due to the commutation relations (16). A simple generalization of this proof is obtained for a generalized context with n times. In simple words, the meaning of this theorem is that our formalism put more restrictions than the theory of consistent histories on the number of valid descriptions of a physical system.

A search of the sets of consistent histories which are forbidden by our formalism, and their physical relevance, is unavoidable.

We can analyze with our formalism the spin $\frac{1}{2}$ system, already described using the theory of consistent histories at the end of Sect. 2. Once again we consider a description including the two possible values of the spin along the z axis for the time t_2, and we ask which properties can also be considered at the time t_1 ($t_0 < t_1 < t_2$), in such a way that they are compatible with the properties chosen at the time t_2.

The atomic properties for the time t_2 are represented by the projectors $\widehat{E}_{z+} = |z+\rangle\langle z+|$ and $\widehat{E}_{z-} = |z-\rangle\langle z-|$, while the atomic properties at t_1 are represented by $\widehat{E}_{\overline{n}_1+} = |\overline{n}_1+\rangle\langle\overline{n}_1+|$ and $\widehat{E}_{\overline{n}_1-} = |\overline{n}_1-\rangle\langle\overline{n}_1-|$, for an unknown direction $\overline{n}_1$ of the spin. These projectors are invariant under time translations, due to the vanishing of the Hamiltonian. Therefore, they are invariant when translated to any common time. We may choose this common time as t_0 ($t_0 < t_1 < t_2$), where the initial state $\widehat{\rho}_{t_0}$ is given. If the commutation conditions (16) are satisfied, we should have $\widehat{E}_{\overline{n}_1\pm}\widehat{E}_{z\pm} - \widehat{E}_{z\pm}\widehat{E}_{\overline{n}_1\pm} = \widehat{E}_{\overline{n}_1\mp}\widehat{E}_{z\pm} - \widehat{E}_{z\pm}\widehat{E}_{\overline{n}_1\mp} = 0$, which gives the z direction ($\overline{n}_1 = (0, 0, 1)$) as the only possibility. The z components of the spin at time t_1 is the only choice compatible with the z components at the time t_2, and it corresponds to the case (ii) obtained with consistent histories in Sect. 2. Moreover, this is the only possible choice *for any initial state* $\widehat{\rho}_{t_0}$.

The case (i) for the Gelmann and Hartle condition, and all the possibilities for the Griffiths condition are ruled out by our formalism of generalized contexts. Only the case (ii), which is time continuous with respect to property ascriptions, remains.

It is also necessary to verify if the postulated compatibility condition for time translated properties is successful to give a good description of well established physical processes.

We only mention here what are the results for the well known double slit experiment. R. Omnès [13] proved that with no measurement instruments there is no place for a set of consistent histories including in its universe of discourse through which slit passed the particle before reaching a zone in front of the slits. Therefore, as a consequence of the theorem given at the beginning of this section, there is also no room for such a description with our generalized contexts. For the case of the double slit with a measurement instruments recording through which slit passed the particle, and another instrument recording the particle in different zones of a plane in front of the double slit, we found with our approach the existence of a generalized context suitable for the description of the registration of the instruments (but not of the particle positions). As a consequence, the theorem of the beginning of this section can be used to deduce that such a description has also a place in a set of consistent histories. These are preliminary results which will be included in a forthcoming paper.

The version of the theory of consistent histories given by R. Omnès [7–9] emphasizes its role as a logical construction, i.e. as a tool for obtaining valid descriptions and reasonings about properties of the system. As this is also the case in our formalism, it is interesting to compare both logical structures.

As we briefly summarized in Sect. 2, in the theory of consistent histories there are ordinary contexts on each time of the sequence. The conjunction, disjunction and negation of properties at different times are defined through the union, intersection and complement of the corresponding spectrums, as shown for the two times case in the paragraph below (5). In this theory, a history a implies a history b when $\Pr(b|a) \equiv \Pr(b \wedge a)/\Pr(a) = 1$. As the probabilities depend on the state, the implication of the theory is also state dependent. If the set of histories verify the state dependent consistency conditions given by (3), (4) or (5), it is named a set of consistent histories, and within this set the conventional axioms of formal logic are satisfied. Therefore, the possible universes of discourse provided by this theory have a very special entanglement with the state of the system.

This situation is not entirely satisfactory, because in the usual axiomatic theories of quantum mechanics the state is considered as a functional on the space of observables, and it appears after these observables in a somehow subordinate position. The importance of the notion of state functionals acting on a previously defined space of observables was stressed by one of us in references [5] and [6]. In our approach of Sects. 4 and 5, the logical structure of the properties is an orthocomplemented lattice defined independently of the state of the system and of the probability definition. Moreover, the conditions to have a generalized context are commutation relations, also state independent (see the condition given by (16) for the two times case). Probability is later on introduced on the already constructed logical structure, trough the usual Born rule.

7 Conclusions

We have introduced in this paper a formalism suitable to deal with descriptions and reasonings about physical systems involving quantum properties at different times. The dynamic generated by the Schrödinger equation provides a natural definition for the time translation of quantum properties. Time translations generate a partition in equivalence classes of the set of properties and times. From a physical point of view, properties at different times which are connected by a time translation are essentially the same property, on which the Born rule gives the same probability value.

Time translation also provide the possibility to define an implication between classes. We used this implication to obtain through infimum and supremum the definitions of conjunction and disjunctions of classes. The orthogonal complement of Hilbert spaces is immediately generalized to obtain the negation of a class. In this way we construct a non distributive orthocomplemented lattice of classes of properties and times, and we obtain what in our opinion is a natural extension of the logical structure of quantum mechanics given by Birkhoff and J. von Neumann [10], one of the standard approaches to quantum logic.

As the lattice is non distributive, the Born rule do not provide a well defined probability on the whole of it. Therefore, we extended the usual notion of context to the notion of generalized context, which is a subset of the whole set of classes, organized in a distributive and orthocomplemented lattice. On each generalized context, the Born rule provides a well defined probability. A generalized context is a boolean logic which can be used for speaking and reasoning about properties of the system at different times. It is interesting to note that our formalism allows to define the logic of quantum properties without referring to any state of the system under consideration.

Our approach impose more restrictions than the theory of consistent histories on the possible valid descriptions of a physical system. For a spin system, we proved that our more restrictive conditions eliminate the sets of consistent histories which do not satisfy time continuity for the property ascriptions. This continuity is in our opinion a desirable property, and it is a direct consequence of the fact that our formalism only allows quantum properties represented by commuting projectors when translated to a common time.

We also obtained good preliminary results with our approach describing the double slit experiment with and without measurement instruments detecting the particle passing trough the slits. This open the possibility to apply our formalism to the description of the measurement process and to the classical limit, and moreover to explore in this framework the role of the environment induced decoherence. The work in this direction is in progress.

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