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Unwrapping Closed Timelike Curves

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Abstract Closed timelike curves (CTCs) appear in many solutions of the Einstein equation, even with reasonable matter sources. These solutions appear to violate causality and so are considered problematic. Since CTCs reflect the global properties of a spacetime, one can attempt to extend a local CTC-free patch of such a spacetime in a way that does not give rise to CTCs. One such procedure is informally known as unwrapping. However, changes in global identifications tend to lead to local effects, and unwrapping is no exception, as it introduces a special kind of singularity, called quasi-regular. This "unwrapping" singularity is similar to the string singularities. We define an unwrapping of a (locally) axisymmetric spacetime as the universal cover of the spacetime after one or more of the local axes of symmetry is removed. We give two examples of unwrapping of essentially $2+1$ dimensional spacetimes with CTCs, the Gott spacetime and the Gödel spacetime. We show that the unwrapped Gott spacetime, while singular, is at least devoid of CTCs. In contrast, the unwrapped Gödel spacetime still contains CTCs through every point. A "multiple unwrapping" procedure is devised to remove the remaining circular CTCs. We conclude that, based on the given examples, CTCs appearing in the solutions of the Einstein equation are not simply a mathematical artifact of coordinate identifications. Alternative extensions of spacetimes with CTCs tend to lead to other pathologies, such as naked quasi-regular singularities.

Keywords Closed timelike curves · Gödel universe · Gott spacetime

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1 Introduction

Closed timelike curves (CTCs) are closed curves in a spacetime that a timelike test observer can trace [1]. Although spacetimes with CTCs cannot be constructed by evolution, they are still solutions of the Einstein equation, just not of its initial value formulation. CTCs are considered problematic because their presence appears to lead to causality violations. Spacetimes with CTCs are usually dealt with in one of two ways: either a spacetime with CTCs is declared not physically relevant or it is modified globally in such a way that the CTCs are absent. In many cases the CTCs are manifest isometries of the spacetime and follow coordinate curves of a periodically identified coordinate. If this periodic identification is removed, the global structure of the spacetime changes. This global modification is informally known as unwrapping. One is then tempted to declare the unwrapped spacetime to be "more natural" than the original one. This is the content of the claim of Cooperstock and Tieu in [2], who declare that "the imposition of periodicity in a timelike coordinate is the actual source of CTCs, rather than the physics of general relativity". We investigate this claim in detail.

Two natural questions that arise are: Do spacetimes extended by unwrapping contain any CTCs not explicitly removed by unwrapping? Do any other pathologies arise as a result of unwrapping CTCs? To answer these questions we consider the two spacetimes where CTCs are concluded to be artificial by Cooperstock and Tieu, the Gödel universe [3] and the Gott spacetime [4]. These spacetimes serve as nice toy models, as they are highly symmetric and essentially $(2+1)$-dimensional (each is a direct products of a $(2+1)$-dimensional spacetime with a spacelike real line).

Implicit in the claim of [2] seems to be the requirement that the metric of the spacetime without the periodic identifications is locally the same as of the original spacetime with identifications. In particular, if the original spacetime is regular everywhere, the spacetime where the CTCs are unwrapped should be as well. We find that in axisymmetric spacetimes, where the axis of symmetry is a regular $(n-2)$-dimensional subspace, such as the Gödel spacetime, there is an obstacle to having this correspondence everywhere, which manifests itself as a singularity in the unwrapped spacetime. This singularity is of the type known in literature as a quasi-regular singularity [5], where the spacetime curvature is bounded along each incomplete curve. This result also holds for the Gott spacetime.

This "unwrapping" singularity is similar to the one describing infinitely thin straight cosmic strings in $3+1$ dimensions, where strings are $(1+1)$-dimensional timelike singularities. If a $(3+1)$-dimensional spacetime is a direct product of a $(2+1)$-dimensional spacetime and a real line, the third spatial dimension can be projected out, transforming the string into a point particle. The main difference between the unwrapping singularity and a point particle singularity is topological: there are no closed curves winding around the unwrapping singularity.

We show that, while unwrapping the Gott spacetime results in a (singular) spacetime with no CTCs, unwrapping the Gödel space does not remove all of the CTCs.

In the case of the Gödel spacetime, where the CTCs still persist after unwrapping, we investigate a possibility of a "multiple unwrapping", where multiple families of CTCs are unwrapped all at once. In this procedure multiple "strings" are removed, and the resulting multiply connected spacetime is then unwrapped by constructing its universal cover in order to get rid of the CTCs winding around each removed string. This removes all of the circular CTCs winding around each such string and we conjecture that no other CTCs remain, either.

Our example of extending a locally Gödel chart in a way that apparently does not give rise to CTCs results in a spacetime with pervasive quasi-regular naked singularities instead. Similarly, a CTC-free extension of the Gott spacetime, even though it is locally Minkowski almost everywhere, also results in a naked quasi-regular singularity. These examples support the view that attempts to get rid of CTCs by alternate extensions, even if successful, are likely to result in other pathologies.

This paper is organized as follows. In Sect. 2 we give a brief review of some of the spacetimes admitting CTCs and describe how the Gödel and Gott solutions fit into the picture. In Sect. 3 we define what we mean by unwrapping and we investigate the nature of the resulting singularities. In Sect. 4 we unwrap the Gott spacetime and show that there are no CTCs in the unwrapped (singular) spacetime. In Sect. 5 we unwrap the Gödel space and show that the unwrapped space is singular, and moreover, that CTCs are still present. In Sect. 6 we improve the unwrapping procedure in order to remove the remaining circular CTCs and discuss the properties of the resulting space.

2 Spacetimes with CTCs

Spacetimes with CTCs can arise in a variety of ways. In some cases, such as the van Stockum cylinder, the Gödel universe and the Kerr blackhole, CTCs are produced by the "frame dragging" effect of the rotating matter. In other cases, such as the spinning string, they are due to coordinate identifications. In yet other cases the CTCs arise due to the non-trivial topology of the spacetime itself (wormholes). We give a brief overview of some of these spacetimes in this section, with the emphasis on the Gödel spacetime.

The first spacetime where the CTCs are manifest, the van Stockum cylinder, was constructed by Lanczos in 1924, then rediscovered by van Stockum [6] and analyzed by Tipler [7]. This spacetime is stationary and axisymmetric (it admits two commuting Killing vectors, one timelike and one spacelike with closed orbits and a regular $1+1$-dimensional axis), its metric is of the Weyl-Papapetrou type [8]:

$$ds^2 = -A(r)dt^2 + B(r)dt\,d\phi + C(r)d\phi^2 + H(r)(dr^2 + dz^2). \qquad (1)$$

Here ϕ is 2π-periodic, and so the CTCs appear whenever $C(r) < 0$. Conventionally, the solution consists of a spinning dust cylinder matched to an external vacuum solution. For certain values of the cylinder size and angular momentum the CTCs occur in the external vacuum only. See [7] for details.

One of the most famous solutions of the Einstein equation which admit CTCs was obtained in 1949 by Gödel [3]. Its geodesics were computed in [9] and its properties are discussed in [1]. Following [1] Sect. 5.7, we write the metric of the Gödel universe as

$$ds^2 = -dt^2 + dx^2 - \frac{1}{2}e^{2\sqrt{2}\omega x}dy^2 - 2e^{\sqrt{2}\omega x}dt\,dy + dz^2. \qquad (2)$$

Here $\omega = \text{const}$ and (t, x, y, z) take all real values. In the following we will call this coordinate system Cartesian. The manifold of the Gödel metric is $\mathbb{R}^4$ and the spacetime is homogeneous. The matter source in this space can be written as

$$T_{ab} = \rho u_a u_b + \frac{1}{2}\rho g_{ab}, \tag{3}$$

where $\rho = 2\omega^2$ is the energy density in the units where $8\pi G = 1$ and $c = 1$, so that the Einstein equation in the presence of the cosmological constant reads $G_{ab} + \Lambda g_{ab} = T_{ab}$. We use this convention throughout this paper. Here u_a is the timelike unit vector field tangent to the coordinate curves of t in (2). If the second term in (3) is associated with a negative cosmological constant $\Lambda = -\frac{1}{2}\rho$, then the matter content of the Gödel universe is rotating dust (pressureless perfect fluid) with density ρ, and ω is the magnitude of its vorticity flow. This spacetime is a direct product of a three-dimensional spacetime with a real line $\mathbb{R}$, parameterized by the coordinate z, which does not add any interesting features and can be safely ignored. In the following we set $\omega = 1/\sqrt{2}$, so that $\rho = 1$. This is equivalent to rescaling the coordinates, up to a constant overall factor in the metric.

The Gödel space is highly symmetric, admitting 4 out of a possible 6 Killing vectors in the (t, x, y) subspace. These are ∂_t, ∂_y, $\partial_x - y\partial_y$, $-2e^{-x}\partial_t + y\partial_x + (e^{-2x} - y^2/2)\partial_y$. The first one of these commutes with the rest, the last three form an $SO(2,1)$ Lie algebra.

The rotating matter in the Gödel space leads to CTCs, some of which are manifest after a coordinate transformation to a cylindrical-like chart (τ, r, ϕ), where $r > 0$, τ can take any real values and ϕ is a 2π-periodic angular coordinate:

$$\begin{aligned} e^x &= \cosh 2r + \cos\phi \sinh 2r, \\ y e^x &= \sqrt{2}\sin\phi \sinh 2r, \\ \tan\frac{1}{2}\left(\phi + \frac{t}{\sqrt{2}} - \sqrt{2}\tau\right) &= e^{-2r}\tan\frac{1}{2}\phi. \end{aligned} \tag{4}$$

The new metric, after omitting the irrelevant z-coordinate, is

$$ds^2 = -d\tau^2 + dr^2 + \sinh^2 r(1 - \sinh^2 r)d\phi^2 - 2\sqrt{2}\sinh^2 r\, d\tau\, d\phi. \tag{5}$$

The validity of imposing periodicity on ϕ follows from the fact that the last of (4) is 2π-periodic in ϕ, and from the regularity condition on the axis. Specifically, a spacetime admitting an axial ($U(1)$) Killing vector ξ^a, parameterized by a 2π-periodic coordinate ϕ is regular on the rotation axis (a set of fixed points of ξ^a) if and only if the following "elementary flatness" condition holds:

$$\frac{(\nabla_a(\xi^c\xi_c))(\nabla^a(\xi^c\xi_c))}{4\xi^c\xi_c} \to 1, \tag{6}$$

where the limit corresponds to the rotation axis [8]. This is further discussed in Sect. 3. This condition holds on the axis $r = 0$ of (5). The tangent to the coordinate curve of ϕ is future pointing when it is timelike, resulting in CTCs. This occurs for $\sinh r > 1$.

The Gödel spacetime has generally been discarded as pathological, however other CTC-admitting solutions have been harder to dismiss.

There are several known examples of non-simply connected spacetimes with matter sources satisfying the Null, Weak and Dominant energy conditions[1] where CTCs are present. In such cases Carter, in his investigation of the spinning black hole metric [10], makes a distinction between the "trivial" CTCs (those that are not homotopic to zero, i.e. non-contractible) and the "non-trivial" (contractible) ones, such as those present in the Gödel spacetime. The trivial CTCs can be removed by going from a given non-simply connected spacetime to its universal cover, without changing the metric locally. The non-trivial ones obviously persist even in that case.

A simple example of a spacetime admitting trivial CTCs is the Minkowski spacetime

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{7}$$

with the periodically identified timelike coordinate ($t \sim t + T$). This spacetime is homeomorphic to a cylinder $\mathbb{S} \times \mathbb{R}^3$, all CTCs are trivial and can be removed by going to its universal covering space, the usual $\mathbb{R}^4$ manifold with Minkowski metric.

Another example, relevant to some of the unwrapping constructs below, is the spacetime of an infinite spinning cosmic string. A single straight non-spinning cosmic string along the z-direction is described by the metric

$$ds^2 = -dt^2 + dr^2 + \left(1 - \frac{m}{2\pi}\right)^2 r^2 d\phi^2 + dz^2, \tag{8}$$

$0 \le \phi \le 2\pi$, $m \neq 0$. This was first analyzed by Marder [11], who described the conical singularity at $r = 0$, without assigning any physical meaning to it. For the spinning string the metric is

$$ds^2 = -\left(dt + \frac{a}{2\pi} d\phi\right)^2 + dr^2 + \left(1 - \frac{m}{2\pi}\right)^2 r^2 d\phi^2 + dz^2, \tag{9}$$

$0 \le \phi \le 2\pi$, where a is the angular momentum per unit length, and m is the mass per unit length (string tension), respectively. This spacetime can be obtained from that of a non-rotating string (8) by replacing the usual identification $(t, r, \phi, z) \sim (t, r, \phi + 2\pi, z)$ with $(t, r, \phi, z) \sim (t + a, r, \phi + 2\pi, z)$ and substituting $t \to t + a\phi/2\pi$ (see e.g. [12]). This is an example of "topological frame dragging", where an observer on the Killing horizon (corresponding to the zeros of the norm of ∂_ϕ) appears to be rotating relative to an observer at infinity [13, 14]. The manifold of (9) is regular and flat everywhere except at $r = 0$, where there is a conical singularity due to the mass term (the deficit angle is equal to m). The coordinate curves of ϕ are closed and become timelike sufficiently close to the string ($r < \frac{a}{2\pi - m}$). The conical singularity can be smeared out by a suitable matter distribution [15], in which case the CTCs become contractible.

[1]The Null Energy Condition holds if $T_{ab}k^a k^b \ge 0$ for all null k^a, the Weak Energy Condition holds if $T_{ab}k^a k^b \ge 0$ for all timelike k^a, and the Dominant Energy Condition holds if $T_{ab}k^a$ is future-pointing for all non-spacelike k^a. One or more of these conditions are satisfied by all known classical matter sources.

Instead of a single rotating string one can produce CTCs with a pair of non-rotating strings moving with respect to each other with a non-zero impact parameter, as discovered by Gott [4]. The coordinate chart in the vicinity of each string is just (8), and the two charts can be smoothly connected, such that there is a boost along the junction, as discussed in detail in Sect. 4.1. This causal structure of this spacetime is, in a sense, an opposite of the spinning string one, as the CTCs in it have minimum size and extend all the way to infinity [16]. Thus the Gott spacetime violates the "physical acceptability conditions" outlined in [17]. The CTCs of this spacetime are discussed in detail in Sect. 4.2.

In all of the above examples the CTCs exist for all times, and so such spacetimes are usually considered unphysical, since they do not admit a Cauchy surface from which such a spacetime could evolve.

Another well-known example of a vacuum spacetime with CTCs is the Kerr black hole. In this metric there exist both the CTCs that wrap around the ring singularity and those that do not [10]. Both kinds are hidden from an external observer by the event horizon of the black hole.

A different class of spacetimes admitting CTCs are those with non-trivial topology (wormholes) and "exotic" matter sources (those violating the energy conditions), but without singularities. Exotic matter is required to keep the wormholes traversable. The first traversable wormhole metric was given in [18]. An example of such a metric is

$$ds^2 = -dt^2 + dr^2 + (r^2 + R(r)^2)\big(d\theta^2 + \sin^2\theta \mathrm{d}\phi^2\big), \tag{10}$$

where $R(r)$ has compact support, $r \in \mathbb{R}$. Positive and negative values of r correspond to two different flat asymptotic regions. These can be patched together far enough from the wormhole throats, where $R(r) = 0$. To do that we can identify a 3-plane (e.g. $x = r\sin\theta\cos\phi = \text{const}$) in one of these asymptotic regions with a corresponding 3-plane in the other. Since the spacetime is flat there, both the intrinsic and extrinsic curvatures of the planes vanish, and so a spacetime with such an identification satisfies the Einstein equation without the need for any additional matter sources. Once a wormhole exists in a given spatial slice $t = \text{const}$, it can be manipulated into producing CTCs by a variety of means, all based on changing the relative rates of time flow between the throats, as seen by an asymptotic observer, until there is a CTC threading through them. This can be achieved using either the special relativistic time dilation effect, as described in the original paper, or the gravitational one [19]. Since all such CTCs result from the underlying spacetime not being simply connected, going to the simply connected universal cover gets rid of the CTCs.

Another evidence of the ubiquity of CTCs is the solution constructed by Ori [20], where a regular Cauchy horizon bounded by a closed null geodesic develops from a regular spatial slice and the matter sources satisfying energy conditions.

There is a number of conjectures and theorems that deal with the CTCs and their appearance. Tipler [21] has shown that CTCs cannot evolve from non-singular initial data in a regular asymptotically flat spacetime. Hawking [22] has advanced the Chronology Protection Conjecture, which states that, if the Null Energy Condition holds, then the Cauchy horizon (the null boundary of the domain of validity of the Cauchy problem, see e.g. [1]) cannot be compactly generated. Moreover, even if the Null Energy Condition is violated, the quantum effects are likely to prevent the Cauchy horizon from appearing.

3 Unwrapping

Since one of our goals is to show that unwrapping contractible CTCs creates singularities, we first review the definition of singularities in General Relativity. Next we discuss the quasi-regular singularities resulting from changing the angular coordinate identifications in Minkowski spacetime, as described in [5]. Finally, we consider one particular kind of quasi-regular singularities, the unwrapping singularity, one that results from unwrapping closed timelike curves in axisymmetric spacetimes with a regular symmetry axis.

3.1 Singular Spacetimes

Intuitively, one tends to think of a singularity in a spacetime as a place in it where the curvature diverges or something else pathological happens. This approach, while satisfactory in other field theories, where the spacetime is provided *a priori*, does not work in General Relativity, because the spacetime is not given in advance. Instead it is determined by solving the Einstein equation, which relates the matter content to the spacetime curvature. This means that the manifold and the metric must be smooth enough to keep the Einstein tensor finite everywhere. Hence, any singular point is not a part of the spacetime manifold and cannot be described as a "place" [1], so a different definition of singularity is required.

A singularity can be detected by the existence of curves of "finite length" which are inextendible in at least one direction. An inextendible curve γ is defined as a continuous map $\gamma : [0, 1) \to \mathcal{M}$ for which there is no end point, i.e. there is no continuous map $\gamma' : [0, 1] \to \mathcal{M}$, $\gamma \subset \gamma'$. However, while "length along a curve" is well-defined for manifolds with Riemannian metric, a Lorentz metric does not give rise to a distance function for an arbitrary curve. For a geodesic curve one can use its affine length [1], but there is no unique or natural prescription for a distance between two points on a general curve.

The definition of singularities through the existence of incomplete inextendible geodesics is successfully used in the proofs of the singularity theorems [1]. These theorems are generally associated with the most familiar type of singularities, the curvature singularities, such as that of the Schwarzschild metric for $r \to 0$. This is, however, not the only type of singularities possible. A commonly accepted classification of singularities is given in [5]. In the case of curvature singularities some of the curvature invariants, such as R or $R_{ab}R^{ab}$, grow unbounded along an incomplete curve. A different case is the parallel-propagated curvature singularities, where some of the components of the Riemann tensor cannot be bounded along some incomplete curves, even though all of the curvature invariants remain finite or even vanish, such as in the case of singularities formed by gravitational waves.

Yet another case, and the one most relevant to the subject of this paper, is that of quasi-regular singularities. There the curvature tensors remain smooth and bounded everywhere along an incomplete curve, yet the curve is still inextendible. To avoid the case where singularities are created artificially by removing a regular point from a given spacetime, only inextendible spacetimes are considered. A spacetime is inextendible if it is not isometric to a proper subset of another regular spacetime of the same dimension.

A classic example of a quasi-regular singularity is the two-dimensional cone. The differentiable structure and the metric on that space can be taken to be induced from its embedding into a three-dimensional flat space. The space is smooth (even flat) everywhere outside the apex of the cone. However, an attempt to include the apex into the space and continue the geodesics through it leads to problems. One can see that there is a curvature singularity at the apex by considering a cone with a spherical cap (as seen from its three-dimensional embedding) and taking a limit where the cap radius goes to zero (see e.g. [23]). The induced differentiable structure breaks down at the apex of the cone. In fact, this space is an example of a conifold, which is a generalization of a manifold, and it allows quasi-regular singularities of the conical type [24]. Cosmic string, described by the metric (8), is another example of a conical singularity. Mars and Senovilla [25] have shown that an axisymmetric spacetime is regular on the axis if and only if the regularity condition (6) holds. It is obviously violated in the case of a cosmic string with non-zero mass (8).

Here we briefly review some of the definitions of a singularity, each tailored to a particular class of problems. For the simple examples of quasi-regular singularities considered in this paper, all the definitions agree. A detailed review of the topic can be found in [26].

One of the first definitions was Geroch's g-boundary [27], defined by existence of incomplete geodesics, whether timelike, spacelike or null. The timelike geodesic incompleteness is the most severe case, as there would exist inertial observers whose existence comes to an end within a finite proper time. This definition does not address the case of singularities that can only be reached by non-geodesic curves.

Another definition of singularity, originally due to Penrose [28], is through the concept of a conformal boundary. There a spacetime is embedded in another "unphysical" Lorentzian manifold conformally, rather than properly, in effect bringing the "infinity" to the finite values of the coordinates. This allows one to attach a conformal boundary to the spacetime. See e.g. [1]. If such a boundary is reached by a geodesic curve with a finite affine parameter value in the original spacetime, the conformal boundary is singular. This idea works well in highly symmetric cases, where the unphysical spacetime is easy to construct.

A famous attempt to assign a causal boundary to a spacetime without resorting to an "external" concept, such as the unphysical conformal spacetime, is that of Geroch, Kronheimer and Penrose [29]. They attach a "causal boundary" to any spacetime subject to certain causality restrictions.

Schmidt [32] generalized the geodesic affine parameter to non-geodesic curves, in order to characterize singularities that cannot be reached by a freely falling observer. The basic idea is to assign a positive-definite distance function to points on an arbitrary curve using a Riemannian metric on a frame bundle parallel-propagated along the curve. See e.g. [1]. This in turn enabled the use of Cauchy completion to assign an end point to an incomplete curve. These endpoints define a so called "b-boundary". Once the b-boundary points are found, one can talk about a "neighborhood of a singularity" and the behavior of tensors, such as the Riemann tensor, along a curve approaching the singularity.

Scott and Szekeres [30] suggested that the boundary definition need not be restricted to using only the objects intrinsic to the Lorentzian manifold. Moreover, it should be definable for any differentiable manifold with an affine connection, thus accommodating theories other than just the Einstein's General Relativity, such as the gauge theories, Einstein-Cartan and others. The abstract boundary approach is based on the idea of an "envelopment", a way to embed a manifold in a "larger" manifold of the same dimension. The boundary points are then regular points of the topological boundary of the embedding. The abstract boundary points and sets are formed by equivalence classes of envelopments that cover each other. The set of all abstract boundary points is called the abstract boundary, or a-boundary. The singularities are classified as either removable, if they can be covered by a non-singular boundary set or essential if they cannot.

For our purposes the timelike geodesic incompleteness provides an adequate definition of singularities. Indeed, any singularity resulting from a change in coordinate identifications appears in place of a formerly regular point. Since there are timelike geodesics passing through any regular point, these geodesics become incomplete after the change in identifications.

In the case of singularities constructed by changing periodic coordinate identifications in an axisymmetric spacetime, the singular boundary coincides with the set of the fixed points of the periodic coordinate before the identification is changed. For example, for the two-dimensional cone with the metric $ds^2 = dr^2 + r^2 d\phi^2$, where $0 \le \phi \le \Phi < 2\pi$, $r = 0$ is its singular boundary, due to the deficit angle. A conical singularity in the 4-dimensional spacetime with the metric (8) is an example of a two-dimensional singular boundary which is a flat $\mathbb{R}^2$ manifold. The $r = 0$ singularity of the Schwarzschild spacetime can be described as a singular b-boundary with a rather peculiar structure [33].

It is worth noting that the singular boundary is different from the usual boundary of an n-dimensional manifold with a boundary. The latter is itself an $(n-1)$-dimensional manifold without boundary. In contrast, the singular boundary can be of any dimension, it may or may not be a manifold itself, and may or may not have a boundary (or a singular boundary). See [5] for examples and counter-examples.

3.2 Quasi-Regular Singularities in a Flat Spacetime

Here we describe the construction of a conical as well as unwrapped singularities in a flat spacetime. Since changing the coordinate identifications does not change the metric at any of the regular points, the unwrapping singularities are always quasi-regular (they are not associated with a curvature divergence). Provided the original spacetime is asymptotically flat and has no event horizon, neither does the unwrapped one, so the resulting quasi-regular singularity is naked, i.e. there are future directed null curves originating arbitrarily close to the singularity that reach the future null infinity. We first describe the unwrapping singularity for the 4-dimensional Minkowski spacetime and then generalize the definition to apply to the spacetimes of interest.

A conical singularity in a 4-dimensional Minkowski spacetime can be obtained as follows [5]:

1. Remove the timelike two-plane $x = y = 0$ in the Cartesian chart. This is precisely where the cylindrical chart is not defined ($r^2 = x^2 + y^2 = 0$). This space is no longer simply connected and has the topology of $\mathbb{S}^1 \times \mathbb{R}^3$.
2. Unwrap the resulting space to obtain its universal covering $(\bar{\mathcal{M}}, \bar{g})$, with the same flat metric in the cylindrical chart, but with the range of the new angular coordinate $\bar{\phi}$ extended to $\bar{\phi} \in \mathbb{R}$, instead of $0 \le \phi \le 2\pi$. A local Cartesian chart (with the non-negative x-axis removed) is obtained by the standard transformation ($x = r\cos\phi$, $y = r\sin\phi$). A two-dimensional slice of this spacetime in the $x - y$ plane ($t = z = 0$) is shown in Fig. 1 and the three-dimensional embedding of this slice into $\mathbb{R}^3$ is shown in Fig. 2. A well-known version of this space is the Riemann surface forming the domain of the complex Log function.
3. Identify the points under translation through an angle $\Phi \neq 2\pi$, taking care to preserve the rotational isometry $r = \text{const}$, i.e. $(t, r, \phi, z) \sim (t, r, \phi + \Phi, z)$.

For $\Phi \neq 2\pi$ we get the metric of the cosmic string (8) with the string tension m equal to the deficit angle $m = 2\pi - \Phi$. If $\Phi > 2\pi$, the string has negative tension. The presence of a conical singularity is reflected in the focusing (or defocusing) of geodesics passing on the opposite sides of the singularity. This can also be seen from the violation of the regularity condition (6). This condition is obviously satisfied for the ordinary Minkowski spacetime, where $\xi^c \xi_c = r^2$ in the cylindrical chart, but not after the identifications where the deficit angle $m \neq 0$, as can be seen by a coordinate transformation where the deficit angle is traded for a constant factor in $\xi^c \xi_c$, such as in (8), breaking (6).

If the last step is omitted, we get the unwrapped space $(\bar{\mathcal{M}}, \bar{g})$, and the singularity is not conical in the usual sense, as there is no closed curve wrapping around it. We will call this type of quasi-regular singularity the unwrapping singularity.

The subspace $x = y = r = 0$ removed in the step 1 forms the "boundary" of each of the three spaces corresponding to the three steps above. (We put the word boundary in quotes to indicate that it is neither the usual boundary of a manifold, nor yet a singular boundary, but rather an artificial "hole" in the spacetime.) After step 1 this boundary is regular, since it can be included back in to form the original inextendible Minkowski spacetime.

After step 2 the boundary is no longer regular. Indeed, if it were regular, we would be able to include it back into the spacetime and show that any neighborhood of a point on the boundary point is homeomorphic to an open ball in $\mathbb{R}^4$. This would in turn imply that there are closed curves around any such point. Furthermore, any such closed curve is homotopic to a closed curve $r = \text{const}$ lying inside the open ball. However, step 2 explicitly removes all such curves, contradicting the regularity assumption. The singularity is quasi-regular, because the Riemann tensor vanishes for all points $r > 0$.

After the step 3 the boundary is still singular, as any arbitrarily small curve around it has the same fixed deficit angle, breaking the regularity condition (6).

The argument that the singular boundary of the unwrapped spacetime is quasi-regular applies to a class of regular spacetimes that is larger than just the Minkowski

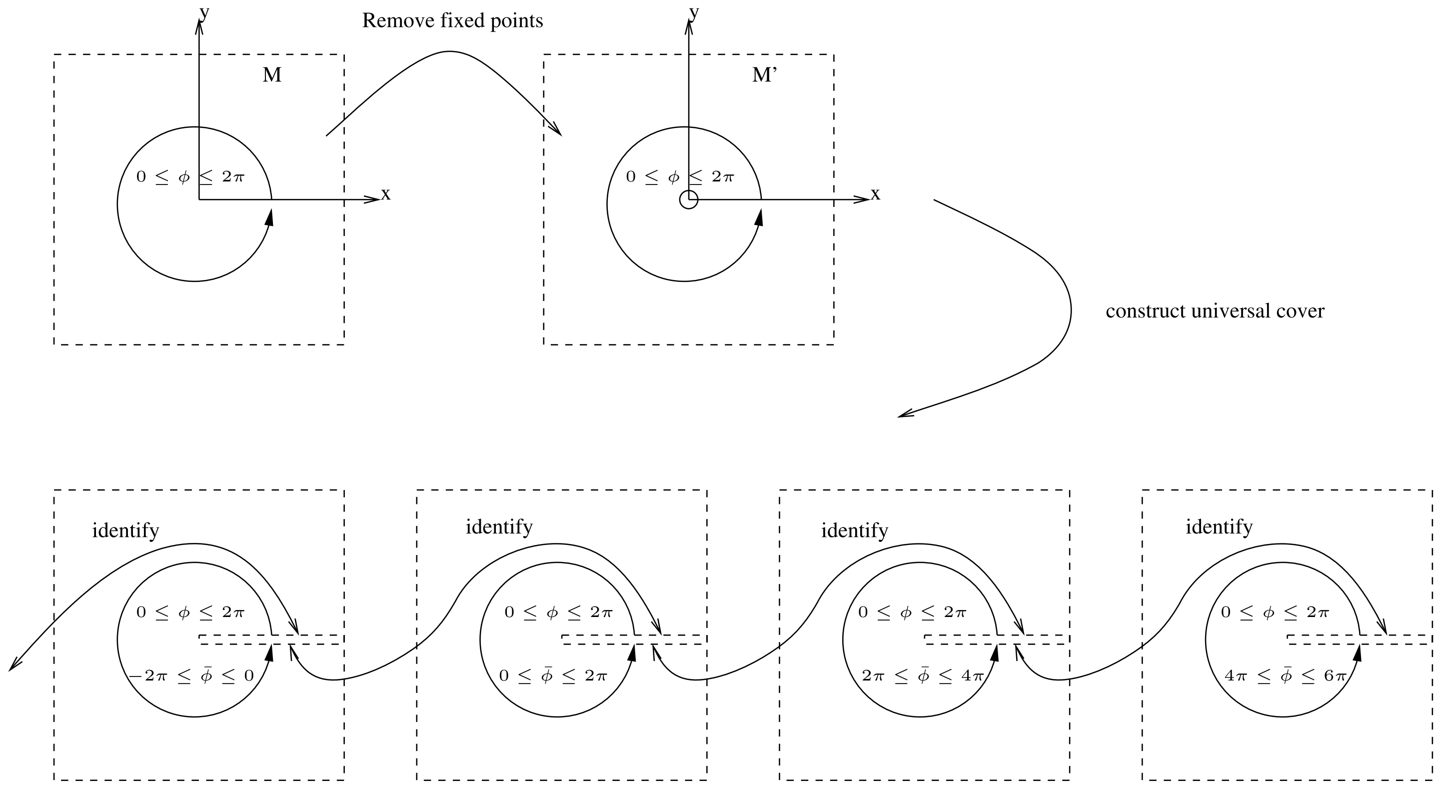

Fig. 1 Unwrapping Minkowski space: (**a**) pick a cylindrical chart, (**b**) remove the fixed points of ϕ at $r = 0$, (**c**) go to the universal cover by extending the range of ϕ to all reals. A $t = z = 0$ slice is shown as a collection of Cartesian charts with $y = 0, x \geq 0$ removed and the surfaces $\phi = 2\pi$ of one chart identified with the surface $\phi = 0$ of the next chart. $\bar{\phi}$ is the global angular coordinate, $\bar{\phi} \in \mathbb{R}$

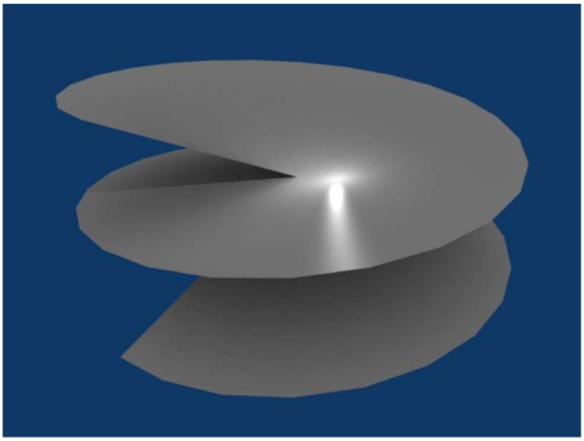

Fig. 2 A visualization of a two-dimensional spatial slice of the unwrapped Minkowski space

space. For the steps 1 and 2 to be applicable for a certain spacetime, it is sufficient to have the spacetime covered by a single Cartesian chart. In particular, it is valid for the Gott and Gödel spacetimes and is generalized in Sect. 3.3 to locally axisymmetric spacetimes.

Moreover, for any future-directed null curve connecting the origin point O in this Cartesian chart with a certain point P in the original spacetime there is a future-directed null curve connecting the boundary in the unwrapped spacetime with one of the infinitely many copies of the point P resulting from unwrapping. Thus, if the unwrapping point is not hidden by an event horizon in the original asymptotically flat spacetime, the quasi-regular singularity in the unwrapped spacetime is necessarily naked.

For completeness, we mention another type of quasi-regular singularity, the Misner singularity, constructed by removing a spacelike two-plane $t = z = 0$ from Minkowski space, then periodically identifying points under a given boost $t^2 - z^2 = \text{const}$, to obtain the 4-dimensional Misner space (a direct product of the two-dimensional Misner space introduced in [34] with the $x - y$ plane). The topology of the resulting space is $\mathbb{S} \times \mathbb{R}^3$, it contains CTCs through every point, and the surface $t = z = 0$ is a quasi-regular singularity. See [5] for detailed examples. If we omit the identification step, we obtain an "unwrapped" Misner space.

One can construct rather complicated quasi-regular singularities by cutting and gluing together spacetime pieces with different properties. For example, Krasnikov [35] describes string-like singularities that are loops or spirals.

3.3 Singularities Created by Unwrapping

We now generalize the unwrapping procedure to a general axisymmetric spacetime and review the properties of the resulting singularity.

Following [8, 25, 36], we define an axisymmetric spacetime $(\mathcal{M}, g)$ as the one which is invariant under the action $\Phi : SO(2) \times \mathcal{M} \to \mathcal{M}$ of the one parameter rotation group $SO(2) \equiv U(1)$, such that the set of the fixed points of Φ is an $(n-2)$-dimensional embedded surface $\mathcal{F}$. If the latter condition is not required to hold, the spacetime is called cyclic. A cyclic, but not axisymmetric, spacetime does not necessarily becomes singular after unwrapping—for example one can unwrap a two-dimensional torus in an obvious way (by going to its universal cover) and get a two-dimensional plane.

Note that the axis of symmetry $\mathcal{F}$ is a set of regular points in $\mathcal{M}$. By this definition, the cosmic string is, while cyclic, not axisymmetric, unless the infinitely thin string is smoothed into the one of finite diameter.

To unwrap an axisymmetric spacetime we follow the same steps as in Sect. 3.2. See also [31, 35]:

- Start with the axisymmetric spacetime $(\mathcal{M}, g)$
- Remove the fixed point set $\mathcal{F}$ to obtain $(\mathcal{M}' = \mathcal{M} \setminus \mathcal{F}, g)$
- Go to the universal covering space $\bar{\mathcal{M}}$ of $\mathcal{M}'$, $\mathbb{Z} : \bar{\mathcal{M}} \to \mathcal{M}'$.

The unwrapped spacetime $(\bar{\mathcal{M}}, g)$ is singular by construction (we excised a set of regular points from its base space), and inextendible, provided $(\mathcal{M}, g)$ is inextendible. We cannot use the regularity criterion (6) to show inextendibility, as it only applies to axisymmetric spacetimes, and $\bar{\mathcal{M}}$ is not axisymmetric, because the rotation Φ is lifted to the translation $\bar{\Phi}$. Instead we mirror the argument from [5] reproduced in Sect. 3.2 for the unwrapped Minkowski spacetime. Suppose there is an extension $\hat{\mathcal{M}}$ of $\bar{\mathcal{M}}$ that includes a point of $\mathcal{F}$ as a regular point. Any compact neighborhood $\hat{\mathcal{O}}$ of a point $p \in \mathcal{F} \subset \hat{\mathcal{M}}$ includes a neighborhood $\bar{\mathcal{O}}$ of the corresponding point on the singular boundary of the unwrapped space $\bar{\mathcal{M}}$. However, $\bar{\mathcal{O}}$ includes a lift $\hat{\Phi}$ of some orbits of the rotation Φ, which are non-compact (and infinitely long, as measured by the generalized affine parameter) in the unwrapped spacetime $\bar{\mathcal{M}}$. Thus $\hat{\mathcal{O}}$ is also non-compact, leading to a contradiction. Any attempt to compactify $\hat{\mathcal{O}}$ while keeping $\hat{\Phi}$ an isometry would make the orbits of $\hat{\Phi}$ closed, thus violating the regularity condition.

The singular boundary of the unwrapped spacetime is again quasi-regular, as there is no curvature divergence anywhere along a lift of the curves from $\mathcal{M}$ to $\bar{\mathcal{M}}$. The singularity is naked, provided the rotation axis is not hidden by the event horizon in $\mathcal{M}$. (A singularity resulting from unwrapping of a BTZ black hole [37] remains shrouded by its (unwrapped) event horizon.)

Finally, to deal with the Gott and Gödel spacetime discussed in the remaining sections, we generalize the unwrapping to "locally axisymmetric" spacetimes. We call a spacetime locally axisymmetric if it admits an isometric embedding of an axisymmetric spacetime of the same dimension. We will only consider the case where removing the axis of symmetry makes the orbits of the isometry non-contractible not only in the embedded spacetime, but also in the full spacetime. Subject to this condition, the spacetime with the local symmetry axis removed can be lifted to its universal cover. This lift again turns the closed orbits of the rotation into open ones and consequently creates a quasi-regular unwrapping singularity with the same properties as before. As long as the CTCs are contractible curves, it is possible to generalize this conclusion to spacetimes without even a local axisymmetry, however we do not address this generalization in this paper. If the full spacetime admits multiple axisymmetric isometrically embedded spacetimes, on can remove all the axes of symmetry first and the lift the resulting spacetime to its universal cover, thus obtaining a "multiply unwrapped" spacetime.

4 Unwrapped Gott Spacetime

The Gott spacetime, discussed in detail in this section, admits CTCs, created by matching two strings Lorentz-boosted in opposite direction. Cooperstock and Tieu suggested that such a matching is artificial and that identification "before the Lorentz boost is applied" is "more natural" [2]. Since their claim is based on a different spacetime (they appear to remove two timelike ribbons from Minkowski space and boost the resulting singularities relative to each other) and relies on a closed curve crossing these ribbon singularities, it is hard to evaluate. Instead, in keeping with the procedure described in Sect. 3.3, we unwrap the Gott spacetime, such that the Gott CTCs correspond to open curves in the unwrapped spacetime. To do that without changing the metric locally, we remove a timelike line from the Gott spacetime and construct the universal cover of the resulting multiply connected (and now singular) spacetime. We show that no CTCs are present in the new spacetime. We also construct some alternative extensions of the Gott spacetime with a timelike line removed and discuss their properties.

4.1 Construction

We first review the way the $(2+1)$-dimensional Gott spacetime is constructed. Following Gott [4], we start with the $(2+1)$-dimensional version of the straight cosmic string spacetime (8), where the string is represented by a point particle:

$$ds^2 = -dt^2 + dr^2 + \left(1 - \frac{m}{2\pi}\right)^2 r^2 d\phi'^2, \tag{11}$$

where $0 \le \phi' \le 2\pi$ and the particle mass m is the deficit angle. The deficit angle can be made explicit by the substitution $\phi' = \phi/(1 - \frac{m}{2\pi})$, where now $0 \le \phi \le 2\pi - m$:

$$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2. \tag{12}$$

The coordinate identification $\phi \sim \phi + 2\pi - m$ corresponds to removing a timelike wedge centered at the particle and identifying the opposite faces of the wedge. The angle ϕ_0 the wedge makes with the horizontal axis corresponds to the remaining coordinate gauge freedom and can be chosen in a way that simplifies a particular calculation. Gott has chosen the wedge angle in a way that identifies the surfaces $\phi_0 = \frac{\pi}{2} - \frac{m}{2} \sim \phi_1 = \frac{\pi}{2} + \frac{m}{2}$, which simplifies his proof of the existence of CTCs. Cutler [38] used the identification $\phi_0 = \frac{\pi}{2} - m \sim \phi_1 = \frac{\pi}{2}$ for one of the strings to show the existence of a timelike cylinder enclosing both strings which no CTCs enter. Carroll et al. [39] identified $\phi_0 = -\frac{m}{2} \sim \phi_1 = \frac{m}{2}$ to visualize the existence of spacelike hypersurfaces through which no CTCs pass. These identification choices are illustrated on Fig. 3. As our goal is to investigate the CTCs in the Gott spacetime, we use the Gott's choice of ϕ_0.

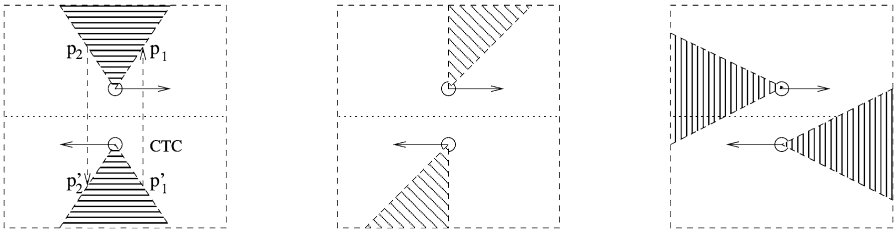

Fig. 3 Conical wedge identification choices in the Gott spacetime. Wedge fill lines indicate the identified points. The strings are shown at the moment of the closest approach. *Left*: Gott identification, where CTCs are manifest (one CTC is shown). *Center*: Cutler identification, used to prove the existence of points not lying on any CTCs. *Right*: Carroll identification, used to visualize the existence of CTC-free spacelike hypersurfaces, as the opposite sides of the wedge are identified at equal times

We write each face of the wedge (denoted by the indices 1 and 2) in the Cartesian coordinate system, expressed in terms of two parameters t and x:

$$\begin{aligned} t_1 &= t, \\ x_1 &= x, \\ y_1 &= x \cot m/2, \\ t_2 &= t, \\ x_2 &= -x, \\ y_2 &= x \cot m/2. \end{aligned} \tag{13}$$

Points corresponding to the same values of (t, x) on both faces are identified.

It is possible to have multiple strings in the same spacetime, as long as the total deficit angle does not exceed 2π, otherwise the topology of the spacetime becomes $\mathbb{S}^2 \times \mathbb{R}$ instead of $\mathbb{R}^3$, where the total deficit angle is equal to 4π [12].

Next step is to boost the wedge in the positive x direction with the velocity $v < c = 1$. The coordinates of the faces are then Lorentz-transformed into the laboratory frame as $t_L = \gamma(t + vx)$, $x_L = \gamma(x + vt)$, $y_L = y$:

$$\begin{aligned} t_{1L} &= \gamma(t + vx), \\ x_{1L} &= \gamma(x + vt), \\ y_{1L} &= x \cot m/2, \\ t_{2L} &= \gamma(t - vx), \\ x_{2L} &= \gamma(-x + vt), \\ y_{2L} &= x \cot m/2. \end{aligned} \tag{14}$$

One can see that the identified points $p_1 = (t_{1L}, x_{1L}, y_{1L})$ and $p_2 = (t_{2L}, x_{2L}, y_{2L})$ have different values of the time coordinate t_L in the laboratory (center of momentum) frame. Specifically, the time difference between the two is

$$\Delta t_L = t_{2L} - t_{1L} = -2\gamma x = v(x_{2L} - x_{1L}), \tag{15}$$

and is always negative. To describe two boosted strings of masses m moving along the x-axis in opposite directions with the velocities v and $-v$ and the impact parameter $2b$, we shift the boosted wedge (14) by b in the positive y direction and introduce a second wedge, $v \to -v'$ and $y \to -y$:

$$\begin{aligned} t_{1L} &= \gamma(t + vx), \\ x_{1L} &= \gamma(x + vt), \\ y_{1L} &= b + x \cot m/2, \\ t_{2L} &= \gamma(t - vx), \\ x_{2L} &= \gamma(-x + vt), \\ y_{2L} &= b + x \cot m/2, \\ t'_{1L} &= \gamma(t' - vx'), \\ x'_{1L} &= \gamma(x' - vt'), \\ y'_{1L} &= -b - x' \cot m/2, \\ t'_{2L} &= \gamma(t' + vx'), \\ x'_{2L} &= \gamma(-x' - vt'), \\ y'_{2L} &= -b - x' \cot m/2, \end{aligned} \tag{16}$$

where primed variables describe the second wedge. We have now constructed the Gott spacetime in a single Cartesian chart (t_L, x_L, y_L), corresponding to the laboratory frame, subject to the two wedge identifications $(t_{1L}, x_{1L}, y_{1L}) \sim (t_{2L}, x_{2L}, y_{2L})$ and $(t'_{1L}, x'_{1L}, y'_{1L}) \sim (t'_{2L}, x'_{2L}, y'_{2L})$. If we choose t' and x' such that $\gamma(t + vx) = \gamma(t' - vx')$ and $\gamma(x + vt) = \gamma(x' - vt')$, then the closest approach of the strings corresponds to $t_L = 0$.

4.2 CTCs of the Gott Spacetime

Following Gott, we now consider the curves composed of two pieces of geodesics with $x_L = \text{const}$, as shown on the Fig. 3 left. The first piece connects the two wedges at the points $p'_1 = (t'_{1L} = -T, x'_{1L} = a, y'_{1L} = -VT)$ and $p_1 = (t_{1L} = T, x_{1L} = a, y_{1L} = VT)$. Here V is the velocity of the observer traveling the geodesics. The travel time along the geodesic T is $T = \frac{b + a\gamma \cot \frac{m}{2}}{V + v\gamma \cot \frac{m}{2}}$. The second piece connects the two wedges at the points $p_2 = (t_{2L} = -T, x_{2L} = -a, y_{2L} = VT)$ and $p'_2 = (t'_{2L} = T, x'_{2L} = -a, y'_{2L} = -VT)$. For the two geodesics to form a closed curve, the initial point of one must be the final point of another: $p_1 \sim p_2$ and $p'_1 \sim p'_2$. In this case the coordinate time $2T$ taken to travel from one wedge to another is balanced exactly by the backward time jump across the wedge between $x_{1L} = a$ and $x_{2L} = -a$. Using (15), we get

$$2T + \Delta t_L = 2T + v(x_{2L} - x_{1L}) = 0, \tag{17}$$

or $T = av$, resulting in the relation $V = \frac{b}{av} + \frac{\cot \frac{m}{2}}{\gamma v}$. The curve is a CTC for $V < 1$, which is possible for large enough a whenever $\gamma v > \cot \frac{m}{2}$. This also sets the lower limit on a for a given boost: $a_{\min} = \frac{\gamma b}{\gamma v - \cot \frac{m}{2}}$ for this choice of geodesics.

These $x_L = \text{const}$ CTCs are not the only ones possible. Cutler [38] has determined the (null) boundary of the region containing CTCs, and it turns out that, while there are closed null curves passing closer to the origin than $a_{\min}$, no CTCs pass through the origin and all CTCs go counter-clockwise around the origin.

Moreover, even when CTCs are present, there always exist a neighborhood of each string free of CTCs. As a consequence, "smoothing out" the strings in a small enough neighborhood does not affect the CTCs, as noted by Cutler.

4.3 Unwrapping Gott Spacetime

The existence of the minimum value for a CTC's distance from the $x = y = 0$ line implies that all CTCs wrap around it in the direction opposite to the relative motion of the strings (they also wrap around both strings at some finite distance from each). The Gott spacetime is flat, and so locally axisymmetric everywhere. We pick the $x = y = 0$ subspace as the symmetry axis. We apply the same unwrapping procedure as in Sect. 3: remove the subspace $x = y = 0$ and construct the universal covering space of the resulting non-simply connected manifold (we assume that the original Gott spacetime is simply connected, as the strings can be smoothed out). Since in the Gott's choice of identifications neither wedge crosses the x-axis, the resulting space can be described using countably many copies of the Cartesian (t, x, y) charts, with the charts n and $n + 1$ joined along the non-negative x-axis of each one, as shown on Fig. 4a.

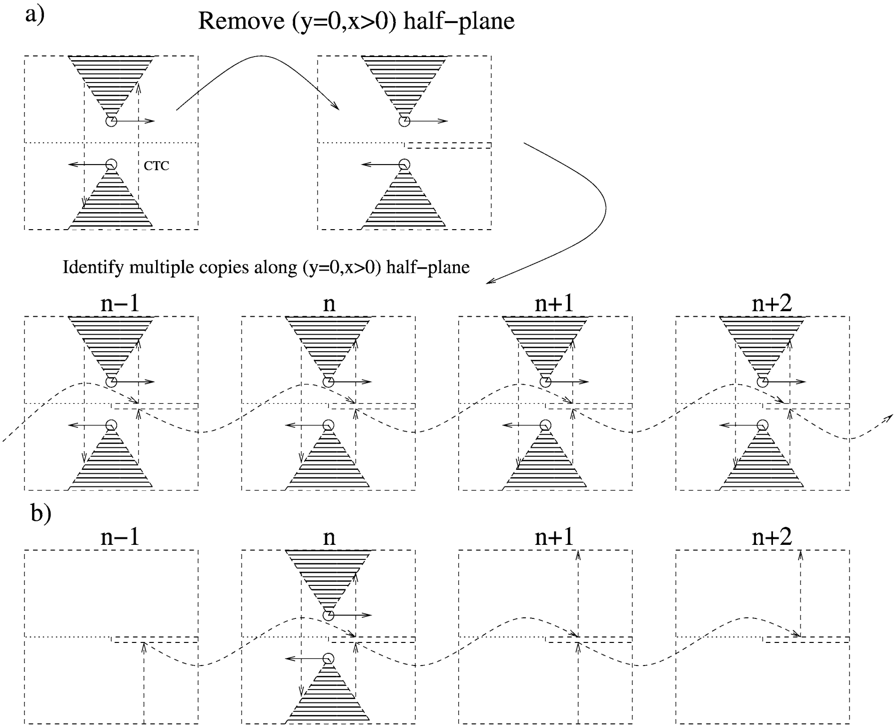

Fig. 4 Unwrapping Gott spacetime. (**a**) The $x = y = 0$ subspace is removed and the universal cover is constructed by patching multiple Cartesian charts together. The former CTCs (indicated by the *vertical arrows*) are now open curves passing from chart to chart. The identifications between charts are indicated by the *wavy arrows*. (**b**) An alternative way to unwrap: a single Gott chart is matched to a collection of Minkowski charts. The former CTCs wrap around just the two strings in the single Gott chart

This unwrapped Gott spacetime is simply connected, has a quasi-regular singularity at $x = y = 0$ and admits no CTCs, no matter how fast the strings are moving relative to each other. It also contains a countable infinity of pairs of boosted strings. The presence of the singularity removes the restriction on the total mass (deficit angle) of all strings, which can now be arbitrarily large, though each string's mass still cannot exceed π.

A timelike or null observer starting on the nth chart and traveling around both strings counter-clockwise ends up on the $(n + 1)$ chart after crossing the positive x-axis. This ensures that no such curve is closed, and so no CTCs are present in the unwrapped Gott spacetime. Instead, what used to be CTCs are now open curves winding around two strings per turn.

It is worth noting that the unwrapped Gott spacetime described above is not the only way to unwrap the CTCs. Since the surface where each two Cartesian charts are joined is locally flat, we do not have to have the two moving strings present on more than one chart. For example, all but one chart can be Minkowski, as shown on Fig. 4b.

We have shown that one can indeed change the identifications in the Gott spacetime such that no CTCs are present. The trade-off for the CTCs removal is introduction of a naked quasi-regular singularity. This singularity is timelike and so it is present in any possible initial data set, making any initial value formulation problematic.

We next turn to another $2 + 1$ dimensional spacetime with CTCs, the Gödel universe, and demonstrate that a straightforward CTCs unwrapping does not work there.

5 Unwrapping Gödel Spacetime

In this section we construct a spacetime which is locally Gödel at every point, but with a different global structure, such that a given set of CTCs in the original spacetime is not longer closed in the unwrapped spacetime. This is not the only way one can get rid of the Gödel CTCs. One obvious way to remove CTCs from the Gödel space is to restrict the radial coordinate in the chart (5), thus creating a boundary where the orbits of ϕ are still spacelike. Examples of this are given in [40] and [41], where a preferred holographic screen is constructed at the radial distance $\sinh r = 1/\sqrt{2}$, where the expansion of the congruence of null geodesics emanating from a point at $r = 0$ is zero. A generalization of this approach to higher-dimensional Taub-NUT-AdS spacetimes is done in [42]. Another approach, considered in [43], is to match the Gödel interior to an exterior spacetime without CTCs. There the metric is explicitly changed locally in the regions where the CTCs used to exist.

5.1 Unwrapping Procedure and Circular CTCs

We now consider a particular example of unwrapping the Gödel space with respect to a given family of CTCs. The metric is given in the (τ, r, ϕ) coordinate chart by the expression (5). This chart is singular at $r = 0$, but this is just a coordinate singularity, as the transformation (4) shows. Since the spacetime is homogeneous, this chart can be constructed using any point in the spacetime as its origin. The CTCs manifest in this chart are the coordinate curves of ϕ when $\sinh(r) > 1$. They are not geodesics, but they are isometries of the spacetime, since the metric in this chart does not depend on ϕ. These CTCs have been extensively studied (see e.g. [1]). Cooperstock and Tieu in [2] "question the continuation of identifying the ϕ values of 0 and 2π when ϕ becomes a timelike coordinate". Since it is impossible to abruptly change the coordinate identification of ϕ only at $\sinh r \geq 1$ without breaking the regularity of the spacetime at $\sinh r = 1$, we will instead remove this identification everywhere.

The Gödel spacetime is axisymmetric, and so the unwrapping procedure is straightforward. We start with the chart (2) of the Gödel spacetime $\mathcal{M}$. In this chart the CTCs that are coordinate curves of ϕ cross the positive x-axis in the counterclockwise direction, like the CTCs of the Gott spacetime. We remove the $x = y = 0$ subspace (corresponding to a single fiber of the Killing vector ∂_t, which also coincides with ∂_τ at $r = 0$) and construct a universal covering space of the resulting non-simply connected manifold $\mathcal{M}'$. The resulting spacetime can be described by either a single global chart (5) with $\phi \in \mathbb{R}$ or by a collection of countably infinitely many Cartesian charts with the charts n and $n + 1$ joined along the positive x-axis of each one. The description using a single cylindrical chart is possible because the subspace $r = 0$, where (5) is not defined, has been removed, and so this chart is valid everywhere in the unwrapped spacetime.

As expected, the removed subspace makes the unwrapped Gödel space singular, and inextendible. Any curve that passed through the subspace $r = 0$ in $\mathcal{M}$ is incomplete in $\mathcal{M}'$ and hence in the unwrapped space, as well.

All circular CTCs cross the positive x-axis in $\mathcal{M}'$ at least once, so their unwrapped-space analogs in the nth Cartesian chart end up on the $(n + 1)$ chart after crossing the axis and so cannot be closed. This corresponds to the orbits of ϕ in the chart (5) being open in the unwrapped space.

5.2 Remaining Circular CTCs

While there are no $r = \text{const} > \sinh^{-1}(1)$ CTCs in the unwrapped Gödel spacetime, this is not the only kind of circular CTC present in the Gödel metric. Since the Gödel spacetime is homogeneous, we can construct a cylindrical chart (5) around any point and obtain CTCs winding around that point. If this new, shifted origin of the cylindrical chart is "far enough" from the old, unshifted one, then the CTCs around it will lie wholly on a single sheet of the space unwrapped around the unshifted origin, and so will remain CTCs even in the unwrapped space.

To demonstrate this, it is convenient to use the quotient space $\tilde{\mathcal{M}}$ of the $(2 + 1)$-dimensional Gödel space $\mathcal{M}$ with the fiber defined by the orbits of the timelike Killing vector y: $y : \mathcal{M} \to \tilde{\mathcal{M}}$. The metric h_{ab} of the quotient space is calculated as $h_{ab} = g_{ab} - \frac{y_a y_b}{y^c y_c}$ (see e.g. [44]). Since y^a is timelike everywhere, with a nonvanishing norm, h_{ab} is nowhere singular, the reduced space is Riemannian, and its line element in the original Cartesian coordinates is just a flat two-dimensional space

$$ds^2 = dx^2 + dt^2. \tag{18}$$

An arbitrary circular CTC in the full space, parameterized by $(0 \leq \phi \leq 2\pi)$, is completely defined by its center (t_0, x_0, y_0) and radius R. It can be written using the equivalent of (4) as

$$
\begin{aligned}
e^{x-x_0} &= \cosh 2R + \cos\phi \sinh 2R,\\
(y-y_0)e^x &= \sqrt{2}\sin\phi \sinh 2R,\\
\tan\frac{1}{2}\left(\phi + \frac{t-t_0}{\sqrt{2}}\right) &= e^{-2R}\tan\frac{1}{2}\phi.
\end{aligned}
\tag{19}
$$

An image of this CTC in the flat quotient space $\tilde{\mathcal{M}}$ is obtained by omitting the coordinate y from (19):

$$
\begin{aligned}
e^{x-x_0} &= \cosh 2R + \cos\phi \sinh 2R,\\
\tan\frac{1}{2}\left(\phi + \frac{t-t_0}{\sqrt{2}}\right) &= e^{-2R}\tan\frac{\phi}{2},
\end{aligned}
\tag{20}
$$

which can be rewritten in an explicit form as

$$
\begin{aligned}
x &= x_0 + \ln\cosh 2R + \cos\phi \sinh 2R,\\
t &= t_0 + 2\sqrt{2}\tan^{-1}\frac{(e^{-2R}-1)\tan\frac{\phi}{2}}{1+e^{-2R}\tan^2\frac{\phi}{2}}.
\end{aligned}
\tag{21}
$$

All CTCs with the same values of x_0 and t_0, but with a different y_0 are mapped into the same closed curve. The image of the singular boundary of the unwrapped space is the point $x = t = 0$ of the quotient space.

We note the following properties of a curve described by (21):

1. It is inscribed in a rectangle centered at (x_0, t_0) and with sides $2R$ and $2\tan^{-1}\sinh R$, and
2. It circumscribes an ellipse inscribed into this rectangle, as described by (22)

$$
\left(\frac{x-x_0}{2R}\right)^2 + \left(\frac{t-t_0}{2\sqrt{2}\tan^{-1}\sinh R}\right)^2 = 1.
\tag{22}
$$

The property 1 follows from the ranges of x and t in (21) for a given R, while the property 2 can be shown by substituting (21) into the left-hand side of (22), finding the four local minima of the resulting function of ϕ and showing that they give exactly the equality (22). At small R the curve (21) tends closer to (22), while at large R it asymptotically approaches the rectangle, as shown on Fig. 5.

We can now show that there exist CTCs that are not unwrapped by the singular boundary resulting from unwrapping around the origin in the chart (5). Since the image of any CTC winding around the boundary has to wrap around the image of the boundary in the reduced space, constructing the image of a CTC that does not wrap the image of the singularity is enough to show the existence of CTCs that are not unwrapped. Since the image of any CTC of radius R with, say, $|x_0| > 2R$ or $|t_0| > 2\sqrt{2}\tan^{-1}\sinh R$ does not wrap around $x = t = 0$, all such curves remain closed after unwrapping.

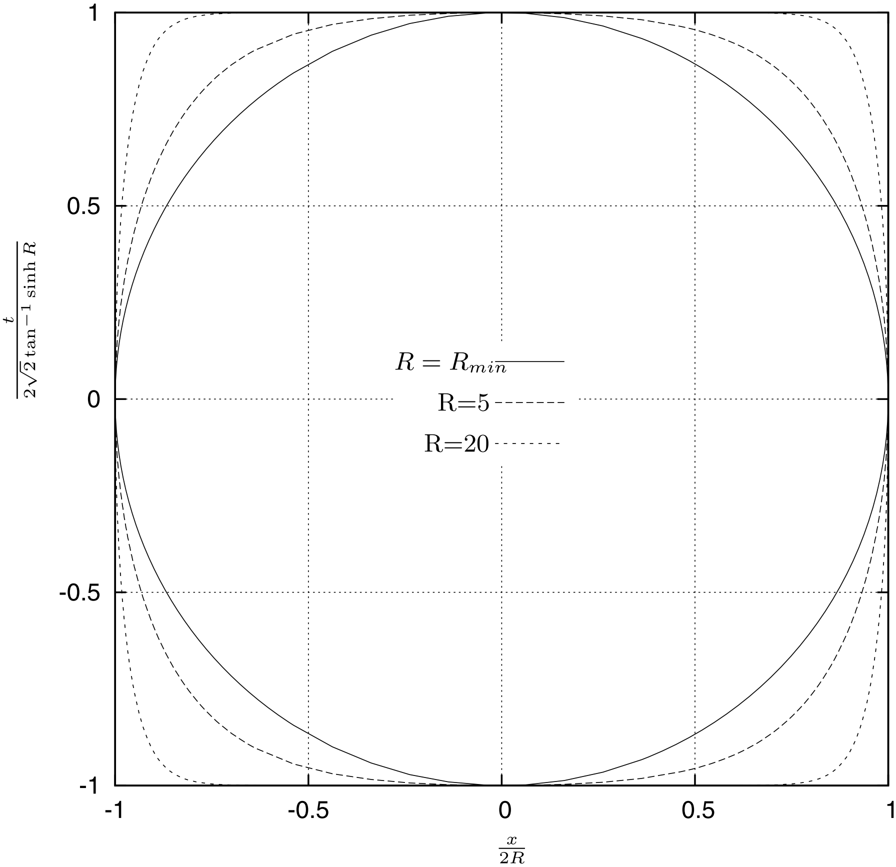

Fig. 5 Normalized concentric CTCs of radius R in the $(x/2R, t/(2\sqrt{2}\tan^{-1}\sinh R))$ coordinates. The CTCs lie inside a square with the side equal to two, but outside of a circle inscribed into it

Thus the naive coordinate identification change of [2] fails to unwrap at least some of the circular CTCs.

6 Multiple Unwrapping of the Gödel Space

By unwrapping the Gödel spacetime we have introduced a quasi-regular singularity into it, yet we did not accomplish the goal of removing all CTCs from it. If one remains intent on also removing circular CTC, one may consider a multiple unwrapping instead, as described at the end of the Sect. 3.3.

6.1 Double Unwrapping

We first consider what happens when we unwrap the CTCs in just two charts. We show that even in this simple case the unwrapped space contains a countably infinite number of quasi-regular singularities.

This "double unwrapping" procedure would look as follows. We consider two different families of circular CTCs, one centered at $x = x_1, y = y_1$ and $x = x_2, t = t_2$ respectively. The subspaces $(x = x_1, t = t_1)$ and $(x = x_2, t = t_2)$ are the fixed points of the corresponding $U(1)$ isometries represented by the CTCs. As before, we first remove these fixed points from the spacetime $\mathcal{M}$, only this time we have to remove both sets, resulting in the singular space $\mathcal{M}''$. Next we construct the universal covering space $\bar{\mathcal{M}}$ of $\mathcal{M}''$ and lift the metric tensor to $\bar{\mathcal{M}}$. Since the original $(2+1)$-Gödel manifold is $\mathbb{R}^3$, we do not need to worry about accidentally removing any topological features unrelated to unwrapping. In this case $\mathcal{M}''$ is homotopy-equivalent to the wedge sum of two circles $\mathbb{S}^1 \vee \mathbb{S}^1$, known as the "figure 8" (see e.g. [45], Chap. 1). This can be shown by explicitly constructing a deformation retraction, an operation that preserves the fundamental group of a manifold. To do that, we first note that, since the orbits of y are open lines, they can be retracted into points by the continuous map $f_s : (t, x, y) \to (t, x, (1-s)y)$. f_0 is the identity map, and f_1 maps $\mathcal{M}''$ into a two-dimensional plane with two points ((x_1, t_1) and (x_2, t_2)) removed. Following [45], we next retract the plane first onto two circles (one around each removed point) connected by a line segment, then contracting the connecting segment into a point. The resulting space is the "figure 8". As a result of the retraction, the singularities now "fill the inside of the circles". The fundamental group of $\mathcal{M}''$ is the fundamental group of the "figure 8", which is just the free product of two copies of $\mathbb{Z}$, $\pi_1 = \mathbb{Z} * \mathbb{Z}$. Each element of the group corresponds to winding around one of the two singularities in $\mathcal{M}''$.

The universal cover of the "figure 8" is well known, it is a tree with countably infinitely many edges and each node connecting four edges (see e.g. [45] for construction). The process of constructing the universal cover of the twice punctured plane is shown schematically on Fig. 6. Each edge of this graph corresponds to a CTC winding around one of the removed subspaces of fixed points. In the unwrapped space this CTC becomes open and corresponds to a given path along the graph. Traversing one edge corresponds to "going around" one of the singularities, so there is a one-to-one correspondence between the singularities in the twice-unwrapped Gödel and the edges in the graph.

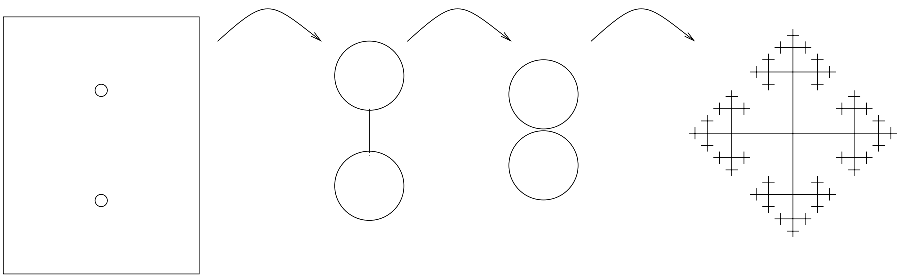

Fig. 6 Constructing the homotopy equivalence of the twice-unwrapped Gödel space. We start with the $(t, x, y = 0)$ subspace with two points removed (*left*), then retract the space onto first two circles connected by a line segment, then the "figure 8", and finally construct the universal cover of the "figure 8" (only the first four levels of nodes are shown). Each edge corresponds to a circle wrapping around one of the two singularities in the original space, so the unwrapped space contains infinitely many singularities

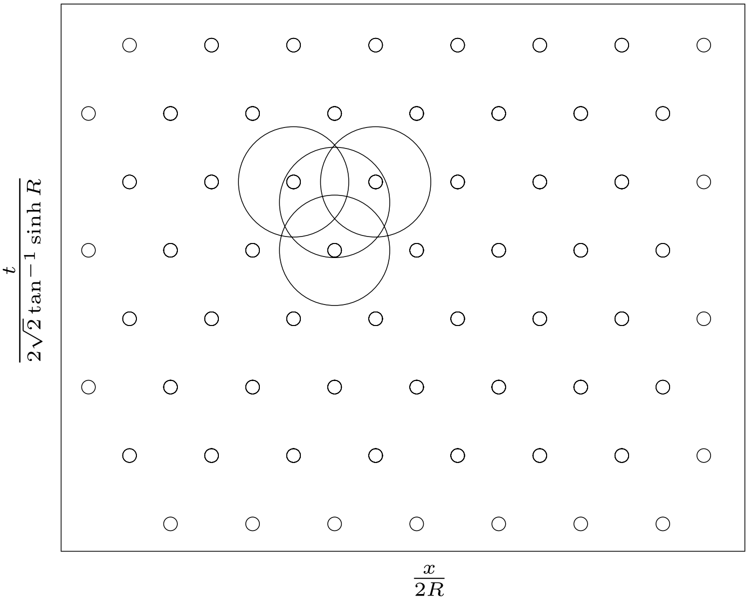

Fig. 7 A tessellation of the two-dimensional quotient space of the Gödel spacetime. Each vertex corresponds to a timelike line in the full spacetime and each circle is an image of a closed null curve $R = R_{\min}$. The tessellation is dense enough to make any circular CTCs wrap around at least one such line, as shown. Once the lines are removed and the resulting non-simply connected spacetime is lifted into the full spacetime and then into its universal cover, no circular CTCs are present in this "multiply unwrapped" spacetime

We can now conclude that unwrapping around two axes at once in a simply connected spacetime results in a spacetime with a countable infinity of quasi-regular singularities of the type discussed in Sect. 3.2.

6.2 Multiple Unwrapping

As discussed in Sect. 5.2, there are infinitely many families of concentric CTCs parameterized by (t_0, x_0, y_0) for which at least some CTCs persist after unwrapping around $(x = 0, t = 0)$. According to the property 2, each CTC lies outside an ellipse (22). All such ellipses are larger than the image of a closed null curve corresponding to $\sinh R = 1$. If we modify the spacetime in a way that transforms any such ellipse into an open curve, the resulting spacetime will not have any circular CTCs. This can be accomplished by first tessellating the quotient space (x, t) with a dense enough triangular lattice, such that there is a vertex inside any ellipse with $\sinh R = 1$, then lifting it into the full 3-dimensional Gödel spacetime using y as the fiber and finally by going to the universal covering space, in a procedure analogous to the one described in Sect. 6.1.

The tessellation is shown schematically on Fig. 7. The fundamental group of the tessellated space is the free group on $\mathbb{Z}$ generators, one for each removed point. A small patch of the unwrapped quotient space is illustrated on Fig. 8. Each helix corresponds to a family of unwrapped concentric circular CTCs. The price to pay for

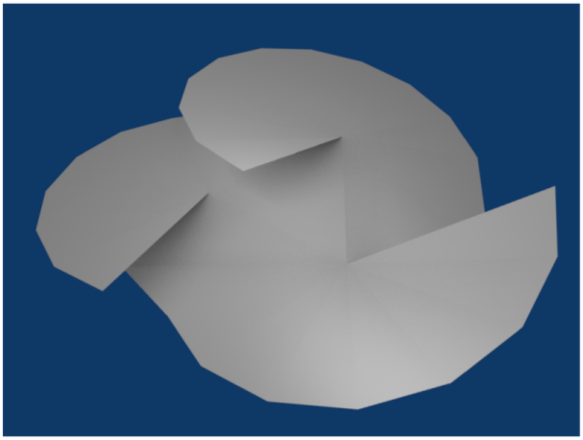

Fig. 8 A visualization of the two-dimensional quotient space of the Gödel spacetime unwrapped at three points at once. Only a small patch of this space is shown, as unwrapping around two or more points results in a countable infinity of singular boundaries

removing all circular CTCs is the introduction of a naked singular boundary consisting of a countable infinity of disjoint pieces.

6.3 Sector-like CTCs in the Gödel Space

One can ask whether any other types of CTCs are present in the multiply unwrapped Gödel spacetime. For example, is it possible to weave one's way in between the vertices and come back to the starting point along a CTC? This seems unlikely and we conjecture that no such CTCs exist. To support this conjecture we describe a different kind of CTCs, we call sector-like CTCs, and show that they, too, are transformed into open curves by the multiple unwrapping procedure of Sect. 6.2.

The idea of constructing a CTC surviving the tessellation of Sect. 6.2 is to exploit the property of the Gödel spacetime where an arc with a larger radius but a smaller angular distance can get us just as far back in time in the chart (5). The hope is then to go far along a timelike curve in a radial direction, then along an arc, then back to the starting point, thus covering a sector instead of a full circle in this chart. If the resulting sector is thin enough, then we can try to fit it in the tessellated space in such a way that no vertices are inside the sector.

To check if this can be done, we calculate the angular and linear distance along the arc required to overcome the time lost traveling forward and back along the two radial directions. Since a closed null sector would be "thinner" than the corresponding timelike sector, we analyze the null sector first.

The (negative) time change in the coordinate τ along a null arc of radius R and angle $\Delta\phi$ in the chart (5) can be calculated as

$$\Delta\tau = (\sinh^2 R(\sqrt{2} - \coth R))\Delta\phi, \tag{23}$$

and it has to compensate for the positive time change of $2R$ along the two radii of the sector, resulting in the total angular change of

$$\Delta\phi = \frac{2R}{\sinh^2 R(\sqrt{2} - \coth R)}. \tag{24}$$

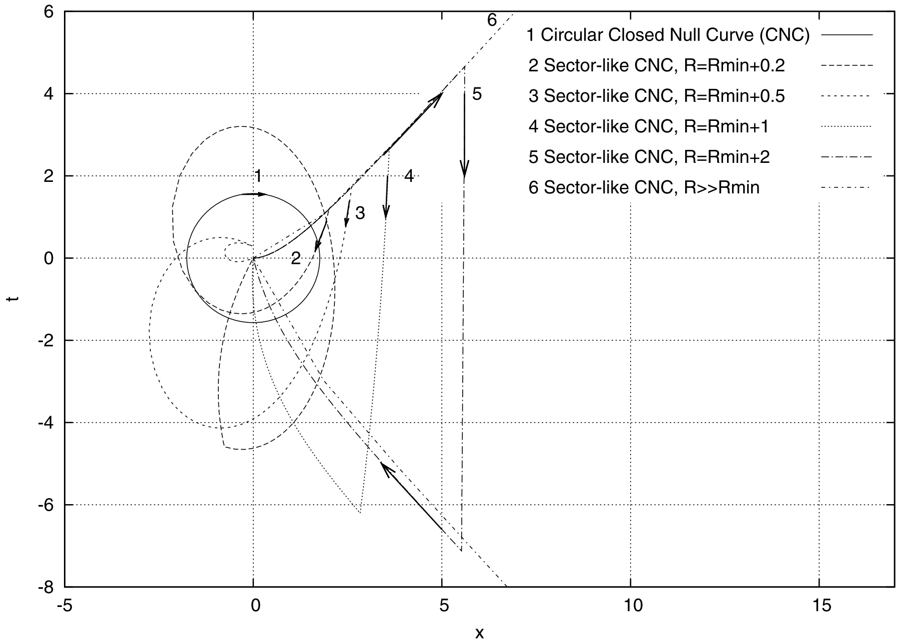

Fig. 9 Images of sector-like closed null curves in the (x,t) coordinates for a range of values R. *Arrows* indicate the future null directions. For large R the curve tends to a triangle with a fixed minimum angle. The angle increases as R goes down, and for R close to the minimum value of $\sinh R_{\min} = 1$ the null curve has to wrap around the origin several times to compensate for the time lost along the radial paths

To see if the sector is thin enough to fit between the vertices of the lattice for large enough R, we project it into the flat quotient space (18). The three legs of the path written in the (τ, r, ϕ) coordinates and parameterized by λ are

$$\begin{aligned}
&(\lambda, \lambda, 0), \quad 0 \le \lambda \le R, \\
&\left(2R - \lambda, R, \frac{\lambda - R}{\sinh^2 R(\sqrt{2} - \coth R)}\right), \quad R \le \lambda \le 3R, \\
&\left(\lambda - 4R, 4R - \lambda, \frac{2R}{\sinh^2 R(\sqrt{2} - \coth R)}\right), \quad 3R \le \lambda \le 4R.
\end{aligned} \tag{25}$$

The same curve in the (x,t) chart of (18) can be described using the explicit coordinate transformation (21). The resulting curves are plotted for several values of R on Fig. 9. For large R the three turning points of the path in the (x,t) chart asymptotically approach $(0,0)$, $(2R, 2\sqrt{2}R)$, $(2R, -2\sqrt{2}R)$. Thus, no matter how large R is, the tessellation dense enough to unwrap circular CTCs also unwraps the sector-like closed null (and therefore timelike) curves. In this sense, the original CTCs described by Gödel appear to have the smallest "footprint" in the flat quotient space.

Whether or not there are other CTCs that persist in the multiply unwrapped spacetime, it is quite clear that removing CTCs from the Gödel spacetime solely by changing the coordinate identification results in a rather contrived space with a countable infinity of naked quasi-regular singularities.

7 Conclusion

We have defined and investigated "unwrapping" CTCs in two $(2+1)$-dimensional toy models, the Gödel spacetime and the Gott spacetime, as a concrete implementation of the claim by Cooperstock and Tieu [2] that the periodic identification of a timelike coordinate is "purely artificial". The procedure requires removing a timelike line from the spacetime and constructing a universal cover of the resulting non-simply connected spacetime. We have demonstrated that such an unwrapping creates a naked quasi-regular singularity, corresponding to the removed timelike line in the original space. The same argument was extended to any locally axisymmetric spacetime where CTC wrap around the axis, as is the case in the Gott spacetime.

While the unwrapped Gott spacetime is devoid of CTCs, the unwrapped Gödel spacetime still contains them. We have defined a "multiple unwrapping" of the Gödel spacetime in order to remove the remaining circular CTCs. As a result, this multiply unwrapped spacetime contains a countably infinite number of singularities. We conjecture that this multiple unwrapping removes all other CTCs as well, and support it by giving an explicit example of a sector-like CTC, which is also removed by the multiple unwrapping.

Our investigation into the ways of removing the CTCs by means of changing coordinate identifications resulting in unwrapping suggests that CTCs appearing in the solutions of the Einstein equation are not a mathematical artifact of arbitrary coordinate identifications, but rather are an unavoidable, if an undesirable, consequence of General Relativity. Different ways to extend the same local coordinate patch of a pathological spacetime may lead to different pathologies, such as CTCs or naked quasi-regular singularities, but are unlikely to result in a physically acceptable regular spacetime.

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