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The Vacuum Electromagnetic Fields and the Schrödinger Equation

A.J. Faria · H.M. França · G.G. Gomes · R.C. Sponchiado

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Abstract We consider the simple case of a nonrelativistic charged harmonic oscillator in one dimension, to investigate how to take into account the radiation reaction and vacuum fluctuation forces within the Schrödinger equation. The effects of both *zero-point* and *thermal* classical electromagnetic vacuum fields, characteristic of stochastic electrodynamics, are separately considered. Our study confirms that the zero-point electromagnetic fluctuations are dynamically related to the momentum operator $p = -i\hbar\partial/\partial x$ used in the Schrödinger equation.

Keywords Foundations of quantum mechanics · Zero-point radiation · Thermal radiation · Stochastic electrodynamics · Quantum electrodynamics

1 Introduction

In this work we shall discuss the relation between the vacuum zero-point electromagnetic fields and the momentum operator $p = -i\hbar\partial/\partial x$ used in the Schrödinger equation. We start by indicating the importance of the radiation reaction and the vacuum zero-point electromagnetic field to understand the atomic transitions and the atomic stability, using the Heisenberg picture and quantum electrodynamics.

Consider a physical system like the hydrogen atom. Its Hamiltonian is $H_S = \vec{p}^2/2m - e^2/r$, and the atomic states are such that $H_S|\mathrm{vac}, a\rangle = \epsilon_a|\mathrm{vac}, a\rangle$, where ϵ_a is the energy of the atom and $|\mathrm{vac}, a\rangle \equiv |\mathrm{vac}\rangle|a\rangle$ denotes the state in which the atom is in the stationary state $|a\rangle$ and the field is in its vacuum state $|\mathrm{vac}\rangle$ of no photons. Considering the above system, Dalibard, Dupont-Roc and Cohen-Tannoudji [1, 2] have discussed the role of the *vacuum zero-point fluctuations and the radiation reaction* forces, *with the identification of their respective contributions*, in the

A.J. Faria · H.M. França (✉) · G.G. Gomes · R.C. Sponchiado
Instituto de Física, Universidade de São Paulo, C.P. 66318, 05315-970 Sao Paulo, SP, Brazil
e-mail: hfranca@if.usp.br

domain of the atomic transitions with emission of electromagnetic radiation. Considering the conceptual importance of this paper to our work, we summarize their main conclusion.

Using a perturbative calculation based on the Heisenberg picture, Dalibard et al. concluded that the time variation of the atomic system energy is such that

$$\begin{aligned}&\langle \mathrm{vac}, a|\frac{dH_S}{dt}|\mathrm{vac}, a\rangle\\&\quad= -\frac{2}{3}\frac{e^2}{c^3}\langle a|(\ddot{\vec{r}})^2|a\rangle\\&\qquad+\frac{2}{3}\frac{e^2}{c^3}\left[\sum_{b(\epsilon_b>\epsilon_a)}\langle a|\ddot{\vec{r}}|b\rangle\cdot\langle b|\ddot{\vec{r}}|a\rangle-\sum_{b(\epsilon_b<\epsilon_a)}\langle a|\ddot{\vec{r}}|b\rangle\cdot\langle b|\ddot{\vec{r}}|a\rangle\right].\end{aligned}\tag{1}$$

The first term in the right side of (1) is the contribution of the radiation reaction whereas the second and third terms are the contribution of the vacuum fluctuation forces. It is straightforward to show that (1) can be written as

$$\langle \mathrm{vac}, a|\frac{dH_S}{dt}|\mathrm{vac}, a\rangle = -\frac{4}{3}\frac{e^2}{c^3}\sum_{b(\epsilon_b<\epsilon_a)}\langle a|\ddot{\vec{r}}|b\rangle\cdot\langle b|\ddot{\vec{r}}|a\rangle.\tag{2}$$

Notice that the factor $2/3$ in the first term of (1) was replaced by $4/3$ in (2).

We also note that, "if self reaction was alone (see the first term in (1)), the atomic ground state would not be stable, since the square of the acceleration has a nonzero average value in such a state. Moreover, such a result is extremely simple and exactly coincides with what is found in classical radiation theory. The complete result (see (2)), which includes the vacuum forces, is even more satisfactory because the electron in the vacuum can only lose energy by cascading downwards to lower energy levels. The ground state cannot be stable in the absence of vacuum fluctuations which exactly balance the energy loss due to self reaction" [1]. See Ref. [3] which illustrates this point, presenting the interesting example of an electron spiraling around the proton in H-atom. In other words, *"if self reaction was alone, the ground state would collapse and the atomic commutation relation* $[x, p] = i\hbar$ *would not hold"* [1].

As stated in Refs. [1–4], "all self reaction effects, which are independent of $\hbar$, are strictly identical to those derived from classical radiation theory. All zero-point vacuum fluctuation effects, which are proportional to $\hbar$ can be interpreted by considering the vibration of the electron induced by a *random* field having a spectral power density equal to $\hbar\omega/2$ per mode". Therefore, in several situations, the zero-point and thermal vacuum fields can be successfully replaced by *classical random* fields [4–6], so that the electric and magnetic fields can be considered as fluctuating sources of energy. A quantum approach, which has some similarity with the QED approach discussed by Dalibard, Dupont-Roc and Cohen-Tannoudji [1, 2], was presented by Barone and Caldeira in 1991 [7]. We also suggest the paper by Senitzky [8] as a complementary reading.

We organize the presentation of our paper as follows. In order to better clarify the features of the interaction between the atom and the vacuum fluctuating fields, we shall study, in Sect. 2, the statistical properties of a charged harmonic oscillator interacting with these electromagnetic fields, using the Schrödinger equation. We consider separately the effects of each kind of fluctuating field. The effects of the *zero-point* radiation are analyzed in Sect. 3, and the *thermal* field are studied within Sect. 4. The effects of the radiation reaction are considered in both cases. Our conclusions are summarized in the final section of the paper.

2 Charged Harmonic Oscillator According to the Schrödinger Picture

For clarity and to simplify some calculations, we shall consider the case of a simple one-dimensional harmonic oscillator immersed in a classical radiation bath. The classical vacuum fields to be considered here are the random *zero-point* and *thermal* electromagnetic fields of *classical* stochastic electrodynamics (SED) [4, 5]. An excellent review of SED is given in the book by de la Peña and Cetto [6].

We shall assume that the charged oscillator motion is nonrelativistic so that the dipole approximation will be used [5]. We shall see that this approximation is consistent with the calculations presented in Sects. 3 and 4. Following the notation of Boyer [5], the x component of the zero-point electric field, acting on the bounded charge moving close to the coordinate system origin, is

$$E_0(\vec{r},t) = \sum_{\lambda=1}^{2} \int d^3k \epsilon_x(\vec{k},\lambda) \frac{\sqrt{\hbar\omega/2}}{2\pi} [e^{i\theta(\vec{k},\lambda)} e^{-i\omega t} e^{i\vec{k}\cdot\vec{r}} + \text{c.c.}]. \tag{3}$$

In the *long wavelength approximation*, one can write this expression as a function of the time t only, that is,

$$E_0(t) \simeq \sum_{\lambda=1}^{2} \int d^3k \epsilon_x(\vec{k},\lambda) \frac{\sqrt{\hbar\omega/2}}{2\pi} [e^{i\theta(\vec{k},\lambda)} e^{-i\omega t} + \text{c.c.}], \tag{4}$$

because the factor $e^{i\vec{k}\cdot\vec{r}} \simeq 1$.

In (3) and (4), $\theta(\vec{k},\lambda)$ are statistically independent random phases and uniformly distributed in the interval $[0, 2\pi]$, $\vec{k}$ is the wave vector such that $|\vec{k}| = \omega/c$, and $\epsilon_x(\vec{k},\lambda)$ is the polarization vector projected in the x axis, with $\lambda = 1, 2$. The zero-point radiation spectral density associated with $E_0(t)$ is such that [4, 5]

$$\rho_0(\omega) = \frac{\hbar\omega^3}{2\pi^2 c^3} \equiv \frac{\omega^2}{\pi^2 c^3}\left(\frac{\hbar\omega}{2}\right). \tag{5}$$

The thermal electric field $E_T(t)$ is also random and is, by assumption, statistically independent from $E_0(t)$. It can be written in a manner similar to (3), namely

$$E_T(t) \simeq \sum_{\lambda=1}^{2} \int d^3k \epsilon_x(\vec{k},\lambda) \frac{h(\omega,T)}{2\pi} [e^{i\theta(\vec{k},\lambda)} e^{-i\omega t} + \text{c.c.}], \tag{6}$$

where T is the absolute temperature and the function $h(\omega, T)$ is given by

$$h(\omega, T) = \sqrt{\frac{\hbar\omega}{2}\left[\coth\left(\frac{\hbar\omega}{2k_B T}\right) - 1\right]}. \tag{7}$$

In (7) the zero-point energy (see (4)) was subtracted. The phases $\theta(\vec{k}, \lambda)$ are another set of statistically independent random phases. Notice that $h(\omega, T) = 0$ if $T = 0$. The thermal radiation spectral density associated with $E_T(t)$ is such that

$$\rho_T(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3}\left(\frac{1}{e^{\hbar\omega/k_B T} - 1}\right). \tag{8}$$

We recall that the charged harmonic oscillator has a natural frequency ω_0 and mass m, with $mc^2 \gg \hbar\omega_0$.

We shall consider firstly the effect of the zero-point electric field $E_0(t)$, given in (4), and the radiation reaction force. The electric field associated with the radiation reaction will be denoted by $E_{RR}(t)$ and will be obtained later. Since the particle is charged, the above fields must be introduced in the Schrödinger equation through the vector potential $A_x(t)$ such that

$$-\frac{1}{c}\frac{\partial A_x}{\partial t} = E_0(t) + E_{RR}(t), \tag{9}$$

when the *zero-point* field and the radiation reaction are considered, or

$$-\frac{1}{c}\frac{\partial A_x}{\partial t} = E_T(t) + E_{RR}(t), \tag{10}$$

in the case of considering *thermal* radiation and the radiation reaction.

3 The Effects of the Zero-Point Fields and the Radiation Reaction in the Schrödinger Equation

For reader convenience we shall obtain, in what follows, a nonperturbative solution of the Schrödinger equation by using the same method already presented in reference [9]. By considering the dipole (or long wavelength) approximation, the one dimensional Schrödinger equation of the charged oscillator coupled to a classical electromagnetic field takes the form [10]

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = \left[\frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x} - \frac{e}{c}A_x(t)\right)^2 + \frac{m\omega_0^2 x^2}{2}\right]\psi(x,t), \tag{11}$$

where $A_x(t)$ is the x component of the vector potential acting on the charged particle (e is the electron charge). At this point the exact analytical form of $A_x(t)$ is not known, because the radiation reaction field $E_{RR}(t)$ was not determined (see (9) and (10)). For the moment we shall simply assume that $A_x(t)$ is a c-number that varies with t and is *independent* of x. It should be noticed that this assumption is valid provided that the motion is nonrelativistic.

The time independent Schrödinger equation has a ground state solution $\phi_0(x)$ such that

$$\phi_0(x) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega_0 x^2}{2\hbar}\right). \tag{12}$$

Moreover, we know that

$$\int_{-\infty}^{\infty} dx \phi_0^2(x) x^2 = \frac{\hbar}{2m\omega_0}. \tag{13}$$

The time dependent equation (11) has an exact solution that can be written as

$$\psi(x,t) = \phi_0(x - q_c(t)) \exp\left\{\frac{i}{\hbar}\left[\left(p_c(t) + \frac{e}{c}A_x(t)\right)x - g(t)\right]\right\}, \tag{14}$$

where the functions $q_c(t)$, $p_c(t)$ and $g(t)$ are unknown c-numbers that will be determined by substitution of (14) into (11). With the above substitution, we get the following equations [11]:

$$p_c(t) = m\dot{q}_c(t), \tag{15}$$

and

$$\dot{p}_c(t) = -m\omega_0^2 q_c(t) - \frac{e}{c}\frac{\partial A_x(t)}{\partial t}, \tag{16}$$

and also the equation $2\dot{g}(t) = \hbar\omega_0 + m\dot{q}_c^2(t) - m\omega_0^2 q_c^2(t)$.

One can combine (15) and (16) to obtain the differential equation

$$m\ddot{q}_c(t) = -m\omega_0^2 q_c(t) + eE_x(t), \tag{17}$$

where we have used the fact that $cE_x(t) = -\partial A_x(t)/\partial t$. Notice that, by assumption, every term in (17) is a c-number.

According to our definition, the total electric field is given by

$$E_x(t) = E_0(t) + E_{RR}(t), \tag{18}$$

where $E_0(t)$ (see (4)) is the classical zero-point electric field and $E_{RR}(t)$ is the classical radiation reaction electric field (the particle is *charged*, therefore the radiation reaction field must contribute to $E_x(t)$).

The correct expression for the classical radiation reaction force $eE_{RR}(t)$ is more difficult to obtain because, according to the Schrödinger picture, *the charged particle does not have a precise location*. One can only say that

$$|\psi(x,t)|^2 = \left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac{1}{2}} \exp\left[-\frac{m\omega_0(x - q_c(t))^2}{\hbar}\right] \tag{19}$$

is the time dependent probability density. Notice that, in order to obtain (19), one must solve (17) which depends on the still undefined radiation reaction force $eE_{RR}(t)$. This force, however, can be precisely defined in the case of large mass. In this case, one can safely consider that (see also Ref. [12])

$$eE_{RR}(t) \simeq \frac{2e^2}{3c^3} \dddot{q}_c(t) \tag{20}$$

is a very good approximation because the Gaussian (19) is so narrow that the harmonically bound particle has a *trajectory*. Based on these considerations we conclude that the expression (20) is valid in the case $mc^2 \gg \hbar\omega_0$, which is consistent with the *long wavelength* approximation. Therefore (17) can be written as

$$\ddot{q}_c(t) + \omega_0^2 q_c(t) \simeq \frac{e}{m} E_0(t) + \frac{2e^2}{3mc^3} \dddot{q}_c(t), \tag{21}$$

where $E_0(t)$ is given by (4). The last term in (21) is responsible for the excited states decay of the oscillator.

The stationary solution of (21) is given by [4, 5]

$$q_c(t) = \frac{e}{m} \sum_{\lambda=1}^{2} \int d^3k \epsilon_x(\vec{k}, \lambda) \frac{\sqrt{\hbar\omega/2}}{2\pi} \left[\frac{e^{i\theta(\vec{k},\lambda)} e^{-i\omega t}}{\omega_0^2 - \omega^2 - i \frac{2e^2}{3mc^3} \omega^3} + \text{c.c.} \right], \tag{22}$$

which is a random real function of the time.

According to the Born statistical interpretation, the expectation value of x^2, namely $\overline{x^2(t)}$, is given by

$$\overline{x^2(t)} = \int_{-\infty}^{\infty} dx |\psi(x,t)|^2 x^2. \tag{23}$$

Taking into account expressions (14) and (12), one can show

$$\begin{aligned} \overline{x^2(t)} &= \int_{-\infty}^{\infty} dx \phi_0^2(x - q_c(t))[(x - q_c(t))^2 + q_c^2(t)] \\ &= \frac{\hbar}{2m\omega_0} + q_c^2(t). \end{aligned} \tag{24}$$

We recall that $q_c^2(t)$ depends on the random phases $\theta(\vec{k}, \lambda)$, as we see from (22).

From result (24) we can calculate the particle position mean square. This quantity is obtained by averaging over the random phases $\theta(\vec{k}, \lambda)$ present in (22). The average over the random variables (indicated by the symbol $\langle\rangle$) is such that [5]

$$\begin{aligned} \langle e^{i\theta(\vec{k},\lambda)} e^{i\theta(\vec{k}',\lambda')} \rangle &= \langle e^{-i\theta(\vec{k},\lambda)} e^{-i\theta(\vec{k}',\lambda')} \rangle = 0, \\ \langle e^{i\theta(\vec{k},\lambda)} e^{-i\theta(\vec{k}',\lambda')} \rangle &= \delta_{\lambda,\lambda'} \delta_{\vec{k},\vec{k}'}. \end{aligned} \tag{25}$$

Here $\delta_{\vec{k},\vec{k}'} = 1$ for $\vec{k} = \vec{k}'$ and $\delta_{\vec{k},\vec{k}'} = 0$ for $\vec{k} \neq \vec{k}'$. Hence, applying the random phases average to expression (24), we obtain

$$\langle \overline{x^2} \rangle = \frac{\hbar}{2m\omega_0} + \langle q_c^2(t) \rangle. \tag{26}$$

According to stationary solution (22), the average of $q_c^2(t)$ over the random phases is such that [4, 5]

$$\langle q_c^2(t)\rangle = \frac{2e^2}{3\pi m^2 c^3}\int_0^\infty d\omega \frac{\hbar\omega^3}{(\omega^2-\omega_0^2)^2 + (\frac{2e^2}{3mc^3})^2\omega^6}. \tag{27}$$

Since $(\frac{2}{3}\frac{e^2}{\hbar c}\frac{\hbar\omega_0}{mc^2})^2 \ll 1$, the integrand of (27) has a very sharp peak at $\omega \approx \omega_0$. Therefore, this integral can be approximated by

$$\langle q_c^2(t)\rangle = \frac{\hbar\gamma}{4\pi m\omega_0}\int_0^\infty \frac{d\omega}{(\omega-\omega_0)^2 + (\gamma/2)^2}, \tag{28}$$

where $\gamma \equiv \frac{2}{3}\frac{e^2\omega_0^2}{mc^3}$, with $\gamma/\omega_0 \ll 1$. This is a standard integral and the result is

$$\langle q_c^2(t)\rangle = \frac{\hbar}{2\pi m\omega_0}\left[\frac{\pi}{2} + \arctan\left(\frac{2\omega_0}{\gamma}\right)\right]. \tag{29}$$

It is possible to show that $\langle q_c^2(t)\rangle$ is charge independent because $\frac{\gamma}{\omega_0} \ll 1$. An expansion of (29) in powers of the small constant γ/ω_0 gives

$$\langle q_c^2(t)\rangle = \frac{\hbar}{2m\omega_0}\left[1 - \frac{1}{\pi}\left(\frac{\gamma}{2\omega_0}\right) + \frac{1}{3\pi}\left(\frac{\gamma}{2\omega_0}\right)^3 + \cdots\right]. \tag{30}$$

Notice that $\gamma/\omega_0 \approx 10^{-7}$ for an atomic oscillator (m is the electron mass and $\hbar\omega_0 \simeq$ 10eV). Substituting result (30) in expression (26), we get

$$\langle\overline{x^2}\rangle = \frac{\hbar}{m\omega_0}\left(1 - \frac{1}{\pi}\frac{\gamma}{2\omega_0} + \cdots\right), \tag{31}$$

where the term $\gamma/2\pi\omega_0$ is the first radiative correction. Since $\gamma/\omega_0 \ll 1$ we get $\langle\overline{x^2}\rangle = \frac{\hbar}{m\omega_0}$ which corresponds to a ground state energy that is *twice* the correct value! This discrepancy was disclosed by some of us a few years ago [12].

In the following section, we shall show that neither the thermal electromagnetic field nor the radiation reaction force are responsible for this incorrect result. The reason for the discrepancy are the zero-point electromagnetic fluctuations that were introduced into the Schrödinger equation by two channels. We shall explain this in the next section.

4 The Effects of the Thermal Electromagnetic Fields and the Radiation Reaction in the Schrödinger Equation

Our first observation refers to the momentum operator used in Schrödinger equation (11), namely, $p = -i\hbar\frac{\partial}{\partial x}$. This operator is related to the vacuum zero-point electromagnetic field. This conclusion was pointed out by Sokolov and Tumanov a long time ago [13]. These authors used the Heisenberg picture and QED. We recommend this paper to the interested reader.

We shall discuss, in what follows, the more recent work of Milonni [14]. According to Milonni the commutator between the position operator $x(t)$ and the canonical momentum operator $p(t) = m\dot{x} + \frac{e}{c}A_x(t)$ can be calculated, within the Heisenberg picture of QED. The result is (see [14], Sect. 2.6, Eqs. (2.88) and (2.89))

$$\left[x(t), m\dot{x} + \frac{e}{c}A_x(t)\right] = [x(t), m\dot{x}] = \frac{ie^2}{m}\frac{8\pi}{3}\int_0^\infty \frac{\rho_0(\omega)\omega d\omega}{(\omega^2-\omega_0^2)^2 + (\frac{2e^2}{3mc^3}\omega^3)^2}, \quad (32)$$

where $\rho_0(\omega)$ is the zero-point spectral density (see (5)). Recall that *only* $\rho_0(\omega)$ depends on $\hbar$ in the above expression. The calculation of the integral (32) is similar to the calculation of $\langle q_c^2(t)\rangle$ presented within Sect. 3. The result is (see (28), (29) and (30))

$$[x(t), p(t)] \simeq i\hbar\left[1 - \frac{1}{\pi}\frac{\gamma}{\omega_0} + O\left(\frac{\gamma^3}{\omega_0^3}\right)\right] \simeq i\hbar. \quad (33)$$

The conclusion is that, according to the Heisenberg picture, the constant $\hbar$ appearing in (33) *has its origin in the zero-point radiation with spectral distribution* $\rho_0(\omega) = \hbar\omega^3/2\pi^2c^3$. This finding is, in our opinion, responsible for the wrong result obtained in the equation (31). This point deserves a detailed discussion.

If we replace the zero-point field $E_0(t)$, used in (21), by the thermal random field $E_T(t)$ given by (6), we shall get a *different* result for $\langle\overline{x^2}\rangle$. Therefore, the discussion of the thermal field effect is an interesting clarifying example.

The total electric field acting on the charged particle will be $E_x(t) = E_T(t) + E_{RR}(t)$, where $E_{RR}(t)$ is the radiation reaction field. As before, the vector potential $A_x(t)$ is such that $E_x(t) = -\frac{1}{c}\frac{\partial A_x}{\partial t}$. Following the calculations explained in Sect. 3. The function $q_c(t)$ will be given by (22), with the replacement of $\sqrt{\hbar\omega/2}$ by $h(\omega, T)$ introduced in (7). In this case one can show that $\langle q_c^2(t)\rangle$ will be given by

$$\langle q_c^2(t)\rangle = \frac{2e^2}{3\pi m^2c^3}\int_0^\infty d\omega \frac{\hbar\omega^3[\coth(\frac{\hbar\omega}{2k_BT}) - 1]}{(\omega^2-\omega_0^2)^2 + (\frac{2e^2}{3mc^3})^2\omega^6}. \quad (34)$$

See (27) for comparison. Notice that, according the relation $\coth(\frac{\hbar\omega}{2k_BT}) - 1 = (e^{\frac{\hbar\omega}{k_BT}} - 1)^{-1}$ appearing in the numerator of (34), *the zero-point spectral distribution* ρ_0 *was subtracted*. This occurred simply with the replacement of $E_0(t)$ by $E_T(t)$.

Introducing again the small constant $\frac{\gamma}{\omega_0}$, the above integral can be calculated with the same approximations used previously (Sect. 3). With this procedure we get

$$\begin{aligned}\langle q_c^2(t)\rangle &= \frac{\hbar}{m\omega_0}\left(\frac{1}{e^{\hbar\omega_0/k_BT} - 1}\right)\left[1 - \frac{1}{\pi}\left(\frac{\gamma}{2\omega_0}\right) + \frac{1}{3\pi}\left(\frac{\gamma}{2\omega_0}\right)^3 + \cdots\right] \\ &\simeq \frac{\hbar}{m\omega_0}\left(\frac{1}{e^{\hbar\omega_0/k_BT} - 1}\right).\end{aligned} \quad (35)$$

This new result combined with our previously expression (26) gives

$$\langle\overline{x^2}\rangle = \frac{\hbar}{2m\omega_0}\left(1 + \frac{2}{e^{\hbar\omega_0/k_BT} - 1}\right)\left(1 - \frac{1}{\pi}\frac{\gamma}{2\omega_0} + \cdots\right)$$

$$\simeq \frac{\hbar}{2m\omega_0}\left(1+\frac{2}{e^{\hbar\omega_0/k_BT}-1}\right). \tag{36}$$

This is the *correct* value of $\langle\overline{x^2}\rangle$ for an *arbitrary* temperature T. *Notice that we get* $\langle\overline{x^2}\rangle = \frac{\hbar}{2m\omega_0}$ *when* $T=0$!

This is expected on physical grounds and is in agreement with the calculations based on Heisenberg picture (see Ref. [5], Sect. IV). One can conclude (see (32) and (33)) that, in the relation $p=-i\hbar\partial/\partial x$, $\hbar$ *has its origin in the zero-point electromagnetic field* with spectral distribution $\rho_0(\omega)=\hbar\omega^3/2\pi^2c^3$.

5 Conclusion

We have studied the problem of a charged harmonic oscillator, under the influence of the radiation reaction and the vacuum fluctuating forces, using the Schrödinger equation. If the radiation reaction, together with the classical zero-point electromagnetic fields, characteristic of SED [4–7], are introduced into the Schrödinger equation, by means of the vector potential $A_x(t)$ (see (9)), a discrepancy is generated in the result (31) for $\langle\overline{x^2}\rangle$. This happens because the zero-point fluctuations were introduced into the Schrödinger equations (11) by two channels: the vector potential $A_x(t)$ given by (9), and the momentum operator $p=-i\hbar\partial/\partial x$. We have explained the discrepancy between $\langle\overline{x^2}\rangle$ given by (31) and $\langle\overline{x^2}\rangle$ given by (36). The error is corrected if one uses only the *radiation reaction* and the *thermal* electromagnetic fields (see $A_x(t)$ in (10)).

We want to conclude by mentioning that we have studied the introduction of the thermal and zero-point classical electromagnetic fluctuating fields into the Schrödinger equation for the harmonic oscillator. This fundamental discussion was possible to be treated within the realm of SED. We also clarified a relevant question, that is, the *quantitative* relation between the momentum operator and the zero-point electromagnetic fluctuations. This point still has other minor unsolved problems (higher order radiative corrections as shown in (33)) which deserve further investigations.

Finally we would like to say that the SED zero-point fluctuations show their effects mainly in linear physical systems [5, 6]. However, an interesting nonlinear phenomenon, namely, the "tunneling" from a potential well with barrier, was successfully explained as a zero-point fluctuation effect [15].

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