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The Wigner Function as Distribution Function

M. Revzen[1]

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Some entangled states have nonnegative Wigner representative function. The latter allow being viewed as a distribution function of local hidden variables. It is argued herewith that the interpretation of expectation *values using such distribution functions as local hidden variable theory requires restrictions pertaining to the observables under study. The reasoning lead to support the view that violation of Bell's inequalities that is always possible for entangled states hinges not only on the states involved but also whether the dynamical variables have their values defined even when they cannot be measured.*

KEY WORDS: Bell's inequality; local hidden variables; phase space; Wigner function; distribution function; dispersive variables.

1. INTRODUCTION

It is indeed and honour and pleasure to contribute to Santos's Festschrift. I know Emilio for many years (and even had the delightful experience of collaborating with him)—and, like many others, have developed a great affection for him as a man and respect for him as a scientist and thinker. And although his interests are and were wide in scope I think that its fair to say that his foremost (scientific) love is devoted to the so called Foundation of Quantum Mechanics (QM) and in particular the problem of physical reality and locality in nature.

For this reason I thought that I shall use this opportunity to express my own thoughts on this loveable enigma: so our main concern in this note shall be the issue of local hidden variables (LHVs) in QM. In this regard I wish first to distinguish "locality" from "LHVs". Whereas the

[1]Department of Physics, Technion – Israel Institute of Technology, Haifa 32000, Israel. e-mail: revzen@physics.technion.ac.il

first "simply" means Einstein locality—the latter means that our (favourite) theory is underpinned with the notion that its dynamical variables (DVs) (i.e. its observables) can be accounted for in full via specification of local variables. In the following we will deviate from Santos rigorous approach and, rather, consider explicit example wherein one can see the crucial importance of having LHVs to fish out Bell's inequality violation (BIQV). The example we consider is wherein arena of our "hidden variables (HV)" is the phase space, i.e. our "HV" are (q, p) (for one degree of freedom). Our example coincides with the problem posed by Bell[(1)] "Einstein Podolsky and Rosen (EPR) and Eugene Paul Wigner (EPW) problem".[(2)] We shall, so it claimed herewith, clarify the above problem via using it to illustrate the relevance of LHVs theory for the validity of Bell's inequality[(3)] rather than locality. So, as in Ref. 1, we consider the "simple" DV of $A(q, p) = \text{sgn}q$, i.e. we take as our DV the sign of the coordinate of our particle's location, q (or the quadrature sign, if we deal with the Electro–Magnetic field). Of course we must also know the probability of having the particular "HVs": (q, p), viz, we must know the "state" of our system, e.g. the distribution function for our (q, p) which we label by $W(q, p)$. We have of course, $W(q, p) \geqslant 0$, $\int W(q, p) dq\, dp = 1$. Thus with our system in the "state" $W(q, p)$ the expected value for the DV $A(q, p) = \text{sgn}q = \pm 1$ is given by

$$\langle \text{sgn}q \rangle = \int_{\infty}^{\infty} dq\, dp W(q, p) \text{sgn}q. \tag{1}$$

We may now wish to evaluate a "rotated" DV $\text{sgn}q'$ with $q'(q, p) = aq + bp$, here the numbers (a, b) specify the "orientation" of q' in terms of the original HV: (q, p). The expectation value of the "rotated" DV is given by Eq. (1) with $\text{sgn}q$ replaced by $\text{sgn}q'(q, p)$. This illustrates a fundamental property of LHV: LHV give the value of the DV without regard to whether or not it is measureable. In our example we have the value of $\text{sgn}q$ and $\text{sgn}q'$ even though we can measure, i.e. observe its expected value, of only one DV at a time. In the language of EPR,[(4)] all observables have "physical reality" even if, in QM, they are represented by non-commuting operators.

In his article entitled (scented with an impish whiff) "EPR correlations and EPW distributions", Bell[(1)] studied the possibility of underpinning quantum theory with LHVs[(3)] in the case of two spinless particles. He analyzed the correlations arising from measurements of positions of these particles in free space—a situation closer to the original one envisaged by EPR[(4)]—utilizing the fact that Wigner's distribution[(5)] simulates a local "classical" model of such correlations in phase space. Bell suggested[1] that the nonnegativity of the Wigner function for certain quantum-mechanical states would preclude BIQV with such states when one considers the correlations constructed from a dichotomic variable defined as the sign of the coordinates of the particles. I would like to review[2] our study in this and argue that Bell's inequality was derived for LHV theories and thus is only preferially relevant to the problem of locality. Indeed since QM as a theory disallows the assignment of simultenous values to noncommuting DV—which is assumed in the derivation of BIQV is mute as far as the respect or disrespect of locality in QM; rather, it is the LHV underpinning possibility, is what BIQV remove.

We first recall a few properties of the Wigner function.[6] One can show that the expectation value of any operator $\hat{A}$ in a state defined by the density matrix $\hat{\rho}$ can be expressed as

$$\mathrm{Tr}(\hat{\rho}\hat{A}) = \int d\lambda W_{\hat{\rho}}(\lambda) W_{\hat{A}}(\lambda), \tag{2}$$

where $W_{\hat{Q}}(\lambda)$ is the Wigner representative of the quantal operator $\hat{Q}$ defined in Eq. (6), and λ designates the appropriate phase space coordinates, i.e., $\lambda = (\mathbf{q}, \mathbf{p}) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$, n being the number of degrees of freedom. It should be noted that in Bell's considerations of LHVs, the values of the observables obey the so-called Bell's factorization,[3,7] which leaves the value of each observable independent of the "setting" of the other. In the expressions for two-particle correlations in terms of the Wigner representatives, when each of the DVs depends on its own phase-space coordinates, this factorization is satisfied automatically. This is our justification for referring to the description in terms of the Wigner function as *local*.[1]

We illustrate the above considerations using the two-mode squeezed state (TMSS) $|\zeta\rangle$, defined as

$$|\zeta\rangle = \exp^{\zeta(a^{\dagger}b^{\dagger} - ab)} |00\rangle \equiv S(\zeta)|00\rangle, \tag{3}$$

this equation defines the operator S. Here, the operators $a,\ a^{\dagger}$ refer to the beam of channel 1, while $b,\ b^{\dagger}$ refer to those of channel 2 (Ref. 8). In the limit of the squeezing parameter ζ increasing without limit, the state (3) approaches the EPR state[4] $|EPR\rangle = \delta(q_1 - q_2)$ (here the subscripts refer to the channels), as can be readily seen writing the state (3) in the coordinate representation as (we use well known normal ordering formula)[9]

$$\begin{aligned}\langle q_1 q_2 | \zeta \rangle &= \frac{1}{\cosh \zeta} \sum_{n=0} \tanh^n \zeta \, \langle q_1 q_2 | nn \rangle \\ &\overset{\zeta \to \infty}{\longrightarrow} \sim \delta(q_1 - q_2). \end{aligned} \tag{4}$$

Now, the Wigner function, W_ζ, of the TMSS is given by Banaszek and Wodkiewicz[10]

$$\begin{aligned} W_\zeta(q_1, q_2, p_1, p_2) = \frac{1}{\pi^2} \exp \Big[&-\cosh(2\zeta) \left(q_1^2 + q_2^2 + p_1^2 + p_2^2 \right) \\ &-2\sinh(2\zeta)(q_1 q_2 - p_1 p_2) \Big]. \end{aligned} \tag{5}$$

It is clearly *nonnegative* for all qs and ps, and thus may be considered as a distribution in phase space (q_1, q_2, p_1, p_2) associated with the state $|\zeta\rangle$. Thus we may refer to the variables (q_1, q_2, p_1, p_2) as LHV's, and correlations weighed with $W_\zeta(q_1, q_2, p_1, p_2)$ should preclude BIQV for DVs for which this may be a legitimate view.[11]

As was mentioned above, Bell suggested[1] that the nonnegativity of the Wigner function of the EPR state would preclude BIQV with this state when one considers the correlations of a dichotomic variable defined as the sign of the coordinates of the particles. The correlations considered in that work are those that are involved in the CHSH[12] inequality, i.e., the inequality that is often studied in terms of the Bell operator.[13] (In the present paper, Bell's inequality and BIQV refer to this CHSH inequality.) Bell's original argument that nonnegativity of Wigner's function suffices to preclude BIQV was shown[14] to be inaccurate. Difficulties in handling normalization of the EPR state considered by Bell were shown to involve a misleading factor.

The TMSS's were studied extensively since the early 1980s in connection with BIQV in general and, in particular, for their connection to the EPR state.[15–21] These studies focused on the polarization as the observable DV. Banaszek and Wodkiewicz[10] noted that *while the Wigner function of the TMSS is nonnegative, it allows for BIQV*, when the DV involved in the correlations is the parity. Their study was extended by Chen *et al.*[22] who showed, by using appropriately defined spin-like variables (which, together with the parity operator, close an $SU(2)$ algebra), that the TMSS, $|\zeta\rangle$, allows the maximal possible[23,24] BIQV for $\zeta \to \infty$, i.e., when it is maximally entangled[25] and, as stated above, it tends to the EPR state. An alternative parametrization (termed configurational) for spin-like operators was given in Ref. 26. This choice of DVs is more convenient for our analysis as it involves the DVs considered by Bell and admits a simple interpretation.

Our study aims at clarifying the relation between the nonnegative Wigner function of the TMSS, $|\zeta\rangle$, for all values of ζ, the DVs involved in the CHSH inequality[12,13] and the possibility of BIQV. The latter, by Bell's theorem,[3,7] prohibits the underpinning of the theory with a LHVs theory. Note that this attribute (nonnegativity) of the Wigner function depends on the variables over which it is defined.[27]

The paper is organized as follows. In the next section, we describe the properties that should be required of a QM problem in order that its translation in terms of Wigner representatives can be legitimately considered as a LHVs theory. We then divide the problem indicated in the last paragraph into three levels. The first level, which the works hitherto addressed, is to consider BIQV with the TMSS, viz., with a state having nonnegative Wigner function. In this connection we give, in Sec. 3, a brief review of Chen *et al.*[22] considerations and those of Ref. 26. We argue that the former approach[22] involves, exclusively, DVs whose Wigner representatives are physically unsuitable for allowing a LHVs theory underpinning (in addition, they do not fulfil the property of boundedness, a mathematical condition that enters the derivation of Bell's inequality). Such DVs that are ineligible for a LHVs theory in phase space (the domain of Wigner's function[27]) are termed *improper* or *dispersive* DVs—the definition of these terms and their justification is also included in Sec. 2. We then consider the next level of the problem, viz., where in addition to having the nonnegative Wigner function of $|\zeta\rangle$, we have a DV that is proper (or nondispersive), i.e., one that can be accounted for by the LHVs that the phase space provides (indeed it is the very one considered by Bell[1]: the sign of the coordinate of the particle). However, we show that its mates, i.e., its rotated (we use here the spin analogy) partner(s) which, with it, must be present in the Bell operator,[13] are dispersive (they are also not bounded) and hence, again, no LHVs theory can be sustained here. We also discuss the alternative approach of retaining the original DVs and rotating the wave function and show that in this case it leads to a *non* nonnegative Wigner function. In Sec. 4, we finally study the last level which is the one considered by Bell. In addition to having the nonnegative Wigner function and the proper DV—its "rotated" mates are now obtained by time evolution with a "free" Hamiltonian. For this case we show that the evolved DV remains nondispersive, or alternatively (perhaps less surprising), the "rotated" wave function continues to give rise to a nonnegative Wigner function. We thus arrive at the conclusion that Bell's expectation[1] that the EPR state will not allow BIQV is confirmed. However, our approach underscores the importance of the perhaps not sufficiently stressed assumption involved in the derivation of Bell's inequalities,[3,12] viz., that the LHVs theory be such that the DVs are defined simultaneously, even when they cannot be measured simultaneously. This point was noted before.[28–33] Indeed, such a requirement is tantamount to having the LHVs endowing physical reality (in the EPR sense[4]) to the DVs measurable attributes.

To remain close to the formalism as discussed by Bell,[1] we shall throughout refer to changes in the DVs as "evolution". This retains complete generality, since to define the evolution we can choose a Hamiltonian leading to the required change.

2. HIDDEN VARIABLES AND WIGNER'S TRANSFORM

We now discuss a specific way of implementing the above LHVs program in terms of the theory of Wigner's transforms, We define the Wigner representative $W_{\hat{Q}}(q, p)$ of the quantal operator $\hat{Q}$ (for one degree of freedom) as[34]

$$W_{\hat{Q}}(q, p) = \int e^{-ip\cdot y} \left\langle q + \frac{1}{2}y \left| \hat{Q} \right| q - \frac{1}{2}y \right\rangle dy \tag{6}$$

while the Wigner function for the density operator is defined with an extra factor of $\frac{1}{2\pi}$ for each degree of freedom, i.e., for one degree of freedom:

$$W_{\hat{\rho}}(q, p) = \frac{1}{2\pi} \int e^{-ip\cdot y} \left\langle q + \frac{1}{2}y \left| \hat{\rho} \right| q - \frac{1}{2}y \right\rangle dy. \tag{7}$$

Then one can prove that the expectation value of an operator $\hat{A}$ with the density matrix $\hat{\rho}$ is[34]

$$\mathrm{Tr}(\hat{\rho}\hat{A}) = \int W_{\hat{\rho}}(q, p) W_{\hat{A}}(q, p) dq\, dp. \tag{8}$$

It can be shown[35] that the only wave function whose Wigner transform is nonnegative is a Gaussian: in this case, the associated Wigner transform is apparently interpretable as a probability density in phase space (see Eq. (8)). The TMSS of Eq. (3) is an example where this interpretation is indeed feasible. If, in addition, the Wigner representatives of the DVs under study are of the proper, or nondispersive, nature required above, we have a candidate for a LHVs theory, where the LHVs are represented by the canonical variables q and p. It seems clear from the outset that it will be rather exceptional for a DV to fall into this category. It is the purpose of the discussion that follows in the present section to identify a class of operators $\hat{A}$ that do correspond to proper DVs. Although the analysis is certainly not exhaustive, it serves the purpose of indicating a number of sufficient conditions leading to proper DVs. For simplicity, the analysis will be restricted to systems with only one degree of freedom.

Consider a function $f(x)$, where $-\infty \leqslant x \leqslant \infty$, bounded as $|f(x)| \leqslant 1$.

1. We define the operator $\hat{A}_1 = f(\hat{q})$ through its spectral representation as

$$\hat{A}_1 = f(\hat{q}) = \int_{-\infty}^{\infty} |q'\rangle f(q')\langle q'| dq'. \tag{9}$$

The eigenvalues of this operator are $f(x)$, so that its spectrum lies in the interval $[-1, 1]$. For instance:

(a) $f(x) = \tanh x$ gives a continuous spectrum in the interval $[-1, 1]$.

(b) $f(x) = \mathrm{sgn} x$ (where the sgn function takes on the value 1 for $x > 0$ and -1 for $x < 0$) has a discrete spectrum, consisting of the two values 1 and -1.

One can easily show that the Wigner transform of the operator $f(\hat{q})$ of Eq. (9) is

$$W_{f(\hat{q})}(q', p') = f(q') \tag{10}$$

a function which takes on, as its values, precisely the eigenvalues of the operator $f(\hat{q})$. According to our nomenclature, we are thus dealing with a proper dynamical variables. In these examples we see the nondispersive property explicitly, since

$$W_{[f(\hat{q})]^k}(q', p') = \left[W_{f(\hat{q})}(q', p')\right]^k. \tag{11}$$

2. Similar considerations apply to the operator $\hat{A}_2 = f(\hat{p})$.

3. Another case, which is very relevant for our future considerations, is that of the operator

$$\hat{A}_3 = f(\hat{\hat{q}}), \tag{12}$$

where

$$\hat{\hat{q}} = a\hat{q} + b\hat{p} \tag{13}$$

(a and b being numerical constants) is a linear combination of the position and momentum operators $\hat{q}$ and $\hat{p}$. If we add, to Eq. (13), the following one:

$$\hat{\bar{p}} = c\hat{q} + d\hat{p}, \tag{14}$$

c and d being numerical constants satisfying the condition

$$ad - bc = 1, \tag{15}$$

then the pair of Eqs. (13) and (14) can be considered as a transformation from the canonical position and momentum operators $\hat{q}$ and $\hat{p}$ to the new ones $\hat{\bar{q}}$ and $\hat{\bar{p}}$. Thanks to the condition (15), the commutator $[\hat{q}, \hat{p}] = [\hat{\bar{q}}, \hat{\bar{p}}] = i$ is preserved and the transformation is canonical: it is the quantum-mechanical counterpart[36] of the classical linear canonical transformation obtained from Eqs. (13) and (14) by removing the "hats" and considering the $q, p, \bar{q}$ and $\bar{p}$ as c-number canonical variables; in the classical problem it is the Poisson bracket that is preserved by the transformation.

We find

$$W_{f(a\hat{q}+b\hat{p})}(q', p') = W_{f(\hat{q})}(aq' + bp', cq' + dp') \tag{16}$$

and, using Eq. (10) for the right-hand side, we finally obtain

$$W_{f(a\hat{q}+b\hat{p})}(q', p') = f(aq' + bp'), \tag{17}$$

which clearly reduces to Eq. (10) when $a = 1$ and $b = 0$.

Right after Eq. (??) we identified the spectrum of $f(a\hat{q}+b\hat{p})$ as $f(x)$. Now, Eq. (17) tells us that the Wigner transform of this operator takes on, as its values, exactly the eigenvalues of the quantum-mechanical operator: we are thus dealing with a proper DV. As a result, *we have found a class of observables*, i.e., $f(a\hat{q} + b\hat{p})$ which, together with their Wigner transforms, i.e., $f(aq' + bp')$, *may be termed proper DVs*.

3. THE EPR–EPW PROBLEM

As outlined in the Introduction, we consider the so-called EPR–EPW problem[1,14] in successive levels. The first level is: Given a state, $|\zeta\rangle$ in our case, whose Wigner representative function is nonnegative, does such a state allow BIQV?

The answer to this was shown[10,22] to be in the affirmative. The DV considered was the parity, S_z ($\hat{N}$ being the number operator),

$$S_z \equiv \sum_{n=0}^{\infty}[|2n + 1\rangle\langle 2n + 1| - |2n\rangle\langle 2n|] = -(-1)^{\hat{N}}. \tag{18}$$

In Ref. 20, “rotated” parity operators were introduced:

$$S_x = \sum_{n=0}^{\infty}[|2n+1\langle 2n| + |2n\rangle\langle 2n+1|], \tag{19}$$

$$S_y = i\sum_{n=0}^{\infty}[|2n\rangle\langle 2n+1| - |2n+1\rangle\langle 2n|]. \tag{20}$$

These operators close an $su(2)$ algebra and are viewed as components of a three-dimensional vector operator. We may thus consider a “rotation” in parity space by, e.g.,

$$S'_x(\vartheta) = e^{\frac{i\vartheta}{2}S_z}S_x e^{\frac{-i\vartheta}{2}S_z} = S_x\cos\vartheta - S_y\sin\vartheta = \mathbf{S}\cdot\mathbf{n} \tag{21}$$

with $\mathbf{n}$ a unit vector which, in this case, is in the “$x - y$” plane of the parity space. It will be convenient for us later to refer to the above as the “time evolution” of S_x under the “Hamiltonian” S_z in Eq. (21): in this way we refer to the “rotation” angle, ϑ, as the time, t. Sticking to the geometric notation, the Bell operator[13] is (the superscripts refer to the channels, $a, a^\dagger$ being channel 1 and $b, b^\dagger$ channel 2)

$$\begin{aligned}\hat{\mathcal{B}} = \mathbf{S^1}\cdot\mathbf{n}\ \mathbf{S^2}\cdot\mathbf{m} + \mathbf{S^1}\cdot\mathbf{n}'\ \mathbf{S^2}\cdot\mathbf{m} \\ + \mathbf{S^1}\cdot\mathbf{n}\ \mathbf{S^2}\cdot\mathbf{m}' - \mathbf{S^1}\cdot\mathbf{n}'\ \mathbf{S^2}\cdot\mathbf{m}'\end{aligned} \tag{22}$$

and the Bell inequality we study is

$$|\langle\hat{\mathcal{B}}\rangle| \leqslant 2. \tag{23}$$

Varying $\mathbf{n}$, $\mathbf{n}'$ and $\mathbf{m}$, $\mathbf{m}'$ to maximize $|\langle\hat{\boldsymbol{\mathcal{B}}}\rangle|$ for the state $|\zeta\rangle$ we get[26]

$$|\langle\zeta|\hat{\mathcal{B}}|\zeta\rangle| = 2\sqrt{1+F^2(\zeta)}, \tag{24}$$

$$F(\zeta) = \langle\zeta|S_x^1 S_x^2|\zeta\rangle = \tanh 2\zeta. \tag{25}$$

Thus the state $|\zeta\rangle$ allows BIQV, even though the Wigner function of the corresponding density operator may be viewed as a probability density of LHVs (the phase space coordinates). However, as was stressed in Sec. 1, this does not violate Bell’s theorem which prohibits BIQV for a

LHVs theory. Thus the correlations appearing in the Bell operator have the structure[6]

$$\langle\zeta|S_z^1 S_z^2|\zeta\rangle = \int_{-\infty}^{\infty} dp_1\, dq_1\, dp_2\, dq_2 W_\zeta(p_1, q_1, p_2, q_2) \times W_{S_z^1}(p_1, q_1) W_{S_z^2(p_2,q_2)}. \tag{26}$$

Here, the factorization of the Wigner function of the two channels is automatic. As explained in detail in Sec. 2, for the right-hand side of Eq. (26) to be interpretable as a LHVs theory, aside from a nonnegative Wigner function for the state, W_ζ, we require that the Wigner representatives of the DVs, the S_z's in this case, give the observable values of these DVs, viz., the eigenvalues of the quantal parity operator (for the phase point: q, p). As already indicated, we refer to a DV with this property as a *proper* or *non*dispersive DV.[37] This is not the case for any of the parity operators, $S_i (i = x, y, z)$; in fact, e.g., we can easily verify that

$$W_{S_z}(q, p) = -\pi\delta(\alpha) = -\pi\delta(q)\delta(p), \quad \alpha = q + ip. \tag{27}$$

This clearly is not an eigenvalue of the parity operator (which is ± 1). Thus in this case this DV is *im*proper or dispersive.[37] Therefore, we are not dealing here with a LHVs theory. (In addition, Eq. (27) makes clear the assertion made in Sec. 1 that the Wigner representative of $\hat{S}_z$ violates the property of boundedness.)

We have thus completed the discussion of the first level of the EPR–EPW problem: nothing new was gained but we considered examples that will serve us below.

The second level of the EPR–EPW problem is when, in addition to having a nonnegative Wigner function for the state, we have a DV whose Wigner representative is the value of the DV—i.e., it is a proper or nondispersive DV. Would this situation allow BIQV? Would it conform to Bell's theorem? Recently,[11,26] an alternative configuration was discussed for the parity operators. In this alternative configuration the operators are given in the q representation. Denoting the operators in this configuration by $\Pi_i (i = x, y, z)$, we have

$$\Pi_z \equiv -\int_0^{\infty} dq\, [|\mathcal{E}\rangle\langle\mathcal{E}| - |\mathcal{O}\rangle\langle\mathcal{O}|] = S_z, \tag{28}$$

here,

$$|\mathcal{E}\rangle = \frac{1}{\sqrt{2}}[|q\rangle + |-q\rangle], \quad |\mathcal{O}\rangle = \frac{1}{\sqrt{2}}[|q\rangle - |-q\rangle], \tag{29}$$

so that

$$\Pi_z = -\int_{-\infty}^{\infty} dq\,[|q\rangle\langle -q|]\,. \tag{30}$$

The equality $\langle n|\Pi_z|n'\rangle = \langle n|S_z|n'\rangle$ is easily verifiable. The natural vectorial operators that close an $su(2)$ algebra with Π_z are

$$\Pi_x = \int_0^{\infty} dq\,[|\mathcal{E}\rangle\langle\mathcal{O}| + |\mathcal{O}\rangle\langle\mathcal{E}|]\,, \tag{31}$$

$$\Pi_y = \int_0^{\infty} dq\,[|\mathcal{E}\rangle\langle\mathcal{O}| - |\mathcal{O}\rangle\langle\mathcal{E}|]\,. \tag{32}$$

We note that Π_x is diagonal in q, i,e.,

$$\Pi_x = \int_0^{\infty} dq\,[|q\rangle\langle q| - |-q\rangle\langle -q|] = \mathrm{sgn}(\hat{q}) \tag{33}$$

is the spectral representation of Π_x. Its Wigner function is

$$W_{\Pi_x}(q,p) = \ \mathrm{sgn}\ q, \tag{34}$$

i.e., it gives the eigenvalues (± 1) of the operator and hence is a proper (nondispersive) DV, just as in the discussion of Eq. (9), case (b), of Sec. 2. In this case, with Π_i, much like in the previous case (with the $S_i, i = x, y, z$) it is easy to get BIQV by selecting the appropriate orientational parameters. For convenience, while retaining complete generality, we consider the choice of the orientational parameters by choosing the times (for both channels) of the evolution of $\Pi_x^1(t_1), \Pi_x^2(t_2)$ under the Hamiltonian $H = \Pi_z$. (we note that Bell[1] considered the same case with $\zeta \to \infty$, i.e. the EPR state, but with the free Hamiltonian, $H = p^2/2$).

Direct calculations show that by appropriate choices of the times (t_1, t_1' and t_2, t_2') we get, for our case,

$$\langle\hat{B}\rangle = 2\sqrt{2}\bar{F}(2\zeta), \qquad \bar{F} = \frac{2}{\pi}\arctan\,(\sinh 2\zeta). \tag{35}$$

Thus we see that in this case, where seemingly the quantal description may be given a LHVs underpinning, we get a BIQV which, we are told, is an impossibility. However, the present Bell operator involves not only the "proper" DV, Π_x, but also Π_y which evolves via our Hamiltonian, $H = \Pi_z$. The latter, i.e., π_y, is *not* a proper DV. In fact, its Wigner representative is given by

$$W_{\Pi_y}(q, p) = -\delta(q)\mathcal{P}\frac{1}{p}, \tag{36}$$

where $\mathcal{P}$ stands for the "principal value". Thus, once again, no LHVs underpinning for the correlation involved in Eq. (35) is possible, after all. (The boundedness condition for the Wigner representatives is violated as well.)

We may attempt to consider the problem in a Schrödinger-like manner by applying the time evolution operator to the state $|\zeta\rangle$; this, however, leads to a new state, $|\zeta'\rangle$, whose Wigner function is no longer nonnegative over all phase space. This can be proven most readily by considering an alternative expression for the state $|\zeta\rangle$ obtained in Ref. 26, i.e.,

$$|\zeta\rangle = \int_0^\infty \int_0^\infty dq\, dq'[(g_+ + g_-)|\mathcal{E}\mathcal{E}'\rangle + (g_+ - g_-)|\mathcal{O}\mathcal{O}'\rangle], \tag{37}$$

where

$$\begin{aligned} g \pm (q, q, ; \zeta) &= \langle qq'|S(\pm\zeta)|00\rangle \\ &= \frac{1}{\sqrt{\pi}} \exp\left\{ -\frac{1}{2}[q^2 + q'^2 \pm 2qq'\tanh(2\zeta)] \right. \\ &\qquad \left. \times \cosh(2\zeta)\right\}. \end{aligned} \tag{38}$$

Using this expression for $|\zeta\rangle$, we have directly

$$e^{-i\gamma\Pi_z}|\zeta\rangle = |\zeta'\rangle = \cos\gamma|\zeta\rangle + \sin\gamma| - \zeta\rangle \tag{39}$$

and the Wigner function of $|\zeta'\rangle$ is no longer nonnegative.[35]

4. BILINEAR HAMILTONIANS

Level 3 of our EPR–EPW problem is the study of cases wherein: (1) The states have nonnegative Wigner representatives which, at some limit, reduce to the EPR state—our $|\zeta\rangle$ is such a state. (2) There is a DV (= observable) that is nondispersive (= proper), i.e., such that the Wigner representative of its quantal version gives its eigenvalues in terms of our LHVs: p, q—our Π_x is such a DV. We inquire for possible BIQV when this DV evolves via Hamiltonians which leave the Wigner representative of the state under study nonnegative. Alternatively, we inquire for BIQV when our DVs evolve by Hamiltonians which allow the initially proper DV to remain so. In the next paragraphs we study the relationship between these two alternatives.

The only nonnegative Wigner functions are gaussians.[35] Since gaussians remain gaussians under linear transformations, it follows that single-channel Hamiltonians that leave the Wigner function non-negative are bilinear ones. We will consider:

$$H_0(i) = \frac{1}{2}(\hat{p}_i^2 + w_i^2 \hat{q}_i^2), \tag{40}$$

where the subscript $i = 1, 2$ denotes the channel. For simplicity we shall consider, in H_0, the frequency $w_i = 1$ for both channels.

We consider the harmonic oscillator Hamiltonian H_0 first. Evolution of the state $|\zeta\rangle$, Eq. (3), under H_0, during a time t_1 for channel 1 and t_2 for channel 2, gives:

$$|\zeta(t_1, t_2)\rangle = |\zeta'\rangle = \exp^{-\zeta(a^\dagger b^\dagger e^{-i\theta} - abe^{i\theta})} |00\rangle, \tag{41}$$

where $\theta = t_1 + t_2$. The corresponding Wigner function can be obtained either directly from the state (41), or from Eq. (5), applying Eq. (??) with $a = \cos t_i$, $b = \sin t_i$, $c = -\sin t_i$ and $d = \cos t_i$, with the result

$$\begin{aligned} W_{\zeta(\theta)} = \frac{1}{\pi^2} \exp\big\{ & -\cosh(2\zeta)\big(q_1^2 + q_2^2 + p_1^2 + p_2^2\big) \\ & -2 \sinh(2\zeta)[(q_1 q_2 - p_1 p_2) \cos\theta \\ & -(q_1 p_2 + q_2 p_1) \sin\theta]\big\}. \end{aligned} \tag{42}$$

Direct evaluation of

$$E(t_1, t_2) = \int_{-\infty}^{\infty} dq\, dp W_{\zeta(\theta)}(q, p) \Pi_x^1 \Pi_x^2 \tag{43}$$

$dq\, dp = dq_1 dq_2 dp_1 dp_2$ gives (see Appendix)

$$E(t_1, t_2) = \frac{\chi}{\pi}, \quad \cos\chi = \tanh(2\zeta)\cos\theta. \tag{44}$$

We have used the notation of Ref. 1

$$E(t_1, t_2) = P_{++}(\theta) + P_{--}(\theta) - P_{-+}(\theta) - P_{+-}(\theta). \tag{45}$$

The first subscript refers to the eigenvalue (i.e., ± 1) of Π_x^1 and the second subscript refers to that of the second channel Π_x^2: i.e., P_{++} is the integral of $W_{\zeta(\theta)}(q, p)$ (see Eq. (43)) over the region $q_1 > 0$, $q_2 > 0$, etc. The alternative view, i.e., allowing Π_x^i to evolve in time, while keeping W_ζ fixed, is readily done (see Appendix) by noting that $\Pi_x^i(t_i) = \text{sgn}(\hat{q}_i \cos t_i + \hat{p}_i \sin t_i)$ and computing the resulting integral for $E(t_1, t_2)$

$$E(t_1, t_2) = \int_{-\infty}^{\infty} dq\, dp W_\zeta(q, p) \Pi_x^1(t_1) \Pi_x^2(t_2) \tag{46}$$

for this case upon the change of variables: $\bar{q}_i = q_i \cos t_i + p_i \sin t_i$ and $\bar{p}_i = -q_i \sin t_i + p_i \cos t_i$. We obviously obtain the same answer at the end. Perhaps more elegantly, one can find the Wigner representative of the time evolution of Π_x^i by applying the general result (17) of Sec. 2, with $a = \cos t_i, b = \sin t_i, c = -\sin t_i$ and $d = \cos t_i$.

It is easily shown (cf Ref. 1) that, in case the time dependence occurs only in the combination $t_1 + t_2$ (which is the case in the present situation (Eq. (41)), the CHSH inequality[12] implies the following inequality

$$3P_{+-}(\theta) - P_{+-}(3\theta) \geqslant 0. \tag{47}$$

In the $\zeta \to \infty$ limit, i.e., when the state $|\zeta\rangle$ is maximally entangled and approaches the EPR state, $\tanh(2\zeta) \to 1$. In this limit $\chi \to \cos^{-1}(\cos\theta) = \theta$ (cf. Appendix) and $P_{+-}(\theta) = \frac{1}{2\pi}\theta$; thus the inequality is saturated. It can be shown that for finite ζ the inequality is always satisfied. Bell suggested that correlations of observables of the type of $\Pi_x^{1,2}$ (cf. Eq. (43)) for the EPR state, evolving under the free Hamiltonian, would not allow for BIQV; we observe that this indeed occurs for the harmonic oscillator Hamiltonian used here.

However, his reasoning perhaps was somewhat misleading: the reason is that it is not only the nonnegativity of the relevant Wigner function that matters, but also the type of evolution induced in the observables by the Hamiltonian in question. The fulfillment of the CHSH inequality in the present case, in which the evolution is induced by the harmonic oscillator Hamiltonian, is consistent with the discussion given in Sec. 2, Eq. (17). It is apt to notice that the present evolution is not analogous to rotation of the spins in the Bohm EPR version. The latter involves what was termed[26] orientational variation, which leads (depending on the preferred viewpoint) either to improper (dispersive) DVs even when one starts with proper DVs, or, alternatively, to a *non* nonnegative Wigner function. In either case, BIQV's do not contradict Bell's theorem.

5. CONCLUSIONS AND REMARKS

In this study we took the Clauser *et al.*[12] inequality as the representative of the so-called Bell's inequalities. Indeed this inequality is the often analyzed and experimentally tested one and is the one used by Bell himself in his study of the subject of this work: the relation of the nonnegative Wigner function of the EPR state to possible BIQVs. In some sense, our results are mundane: no violation is possible when such is not to be allowed by Bell's inequality. We subjected the reader to a lengthy derivation and explanation of what we considered points worthy of clarification. These were the delineation of what is meant by proper and improper DVs in the context of the Wigner function as a probability distribution in phase space, the canonical variables of the latter playing the role of the LHVs, and showed that a proper observable (= DV) is nondispersive. Thus only proper DVs can be considered as accountable for by a LHV theory with the phase space variables (q, p) being the LHVs. A proper DV is one whose Wigner function representative gives the eigenvalues of the corresponding quantal DV which the LHV theory aims at underpinning.

Now, although the word "local" was repeated several times, locality as such was not an issue in the present discussion: Bell's locality condition is automatically fulfilled as the Wigner function of any DVs that depend on distinct phase space coordinates factorizes. Thus our discussion underscores a tacit assumption in the derivation of the Bell inequality we consider: viz., the DVs must all have a definite value even though they are not or even cannot be measured simultaneously. This point was noted in the past.[28–33] In point of fact, two often quoted examples for underpinning noncommuting DVs with LHVs—Bell's[3] and Wigner's[28]—are manifestly so, although these examples are, perhaps, somewhat artificial. In the present work—which in its essence follows Bell's suggestion[1]—we outlined a canonical theory which automatically abides by the locality requirement (the phase space variables are local), and BIQ is abided by in cases where the DVs are proper ones, even when they are noncommuting.

Our main conclusion is that the validity of Bell's inequality that we have considered hinges on the assumption of having definite values for all the DVs—thus endowing them with physical reality—and not the issue of locality. Such view warrents, it seems, counterfactual reasoning. Of course one might ponder what would one mean by a LHVs theory without a definite value for all the DVs; however this is a separate issue.

APPENDIX: EVALUATION OF $E(t_1, t_2)$ FOR THE HARMONIC HAMILTONIAN

We first evaluate $P_{-+}(t_1, t_2)$, cf. Eq. (61). The integral, Eq. (59), *after* the integration over the p's and letting $q_1 \to -q_1$, is

$$P_{-+}(t_1,t_2) = \frac{1}{\pi\cosh(2\zeta)\sqrt{(1-\tanh^2(2\zeta)\cos^2\theta)}} \times \int_0^\infty dq_1\,dq_2 \exp\big[-\cosh(2\zeta)\Gamma(\theta,\zeta) \times (q_1^2+q_2^2-2q_1q_2\tanh(2\zeta)\cos\theta)\big]. \tag{48}$$

Here $\theta = (t_1 + t_2)$ and $\Gamma(\theta,\zeta) = (1-\tanh(2\zeta))/(1-\tanh(2\zeta)\cos^2\theta)$. This integral is evaluated directly to give

$$P_{-+}(t_1,t_2) = \frac{1}{2\pi}\left[\frac{\pi}{2} - \arctan\left(\frac{\tanh(2\zeta)\cos\theta}{\sqrt{(1-\tanh^2(2\zeta)\cos^2\theta)}}\right)\right].$$

Similar calculation gives

$$P_{++}(t_1,t_2) = \frac{1}{2\pi}\left[\frac{\pi}{2} + \arctan\left(\frac{\tanh(2\zeta)\cos\theta}{\sqrt{(1-\tanh^2(2\zeta)\cos^2\theta)}}\right)\right].$$

The equality $P_{++}(t,t') = P_{--}(t,t')$ and $P_{+-}(t,t') = P_{-+}(t,t')$ is easily verifiable, hence we have

$$E(t_1,t_2) = 2P_{++}(\theta) - 2P_{-+}(\theta) = 2\frac{\chi}{\pi} \tag{49}$$

with $\tanh(2\zeta)\cos\theta \equiv \cos\chi, \theta = t_1 + t_2$.

REFERENCES

1. J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, Cambridge, 1987), p. 196.
2. M. Revzen, P. Mello, A. Mann, and L. Johanson, *Phys. Rev. A* **71**, 022103 1–11 (2005).
3. J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, Cambridge, 1987), p. 14.
4. A. Einstein, B. Podolsky, and N. Rosen, *Phys. Rev*. **47**, 777 (1935).

5. E. P. Wigner, *Phys. Rev*. **40**, 749 (1932).
6. U. Leonhardt, *Measuring the Quantum State of Light* (Cambridge University Press, Cambridge, 1997).
7. J. F. Clauser and A. Shimony, *Rep. Prog. Phys*. **41**, 1881 (1978).
8. F. C. Khanna, A. Mann, M. Revzen, and S. Roy, *Phys. Lett. A* **294**, 1 (2002).
9. B. Yurke and M. Potasek, *Phys. Rev. A* **36**, 3464 (1987).
10. K. Banaszek and K. Wodkiewicz, *Phys. Rev. A* **58**, 4345 (1998); *Phys. Rev. Lett*. **82**, 2009 (1999).
11. M. Revzen, in *Recent Developments in Quantum Physics*, Feb. 1–2 (Conference in honor of Asher Peres's 70th birthday, Technion, Haifa, Israel, 2004).
12. J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, *Phys. Rev. Lett*. **23**, 880 (1969).
13. S. L. Braunstein, A. Mann, and M. Revzen, *Phys. Rev. Lett*. **68**, 3259 (1992).
14. L. M. Johansen, *Phys. Lett. A* **236**, 173 (1997).
15. M. D. Reid and D. F. Walls, *Phys. Rev. A* **34**, 1260 (1986).
16. D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer, Berlin, 1994).
17. P. Grangier, M. J. Potasek, and B. Yurke, *Phys. Rev. A* **38**, 3132 (1988).
18. A. Kuzmich, I. A. Walmsley, and L. Mandel, *Phys. Rev. Lett*. **85**, 1349 (2002).
19. Z. Y. Ou, S. F. Pereira, and H. J. Kimble, *Phys. Rev. Lett*. **68**, 3663 (1992).
20. Z. Y. Ou, S. F. Pereira, and H. J. Kimble, *App. Phys. B* **55**, 265 (1992).
21. A. Casado, T. W. Marshall, and E. Santos, *J. Opt. Soc. Am. B* **15**, 1572 (1998).
22. Zeng-Bing Chen, Jian-Wei Pan Guang Hou, and Yong-De Zhang, *Phys. Rev. Lett*. **88**, 040406 (2002).
23. B. S. Cirelson, *Lett. Math. Phys*. **4**, 93 (1980).
24. L. J. Landau, *Phys. Lett. A* **120**, 54 (1987).
25. In this limit the probability of an even parity state equals that of an odd parity state, cf. Eq. (37). It is for such a state that maximal BIQV is attainable; see, e.g., N. Gisin, *Phys. Lett. A* **154**, 201 (1991).
26. G. Gour, F. C. Khanna, A. Mann, and M. Revzen, *Phys. Lett. A* **324**, 415 (2004).
27. Examples of alternative domains were considered in J. A. Vaccaro and D. T. Pegg, *Phys. Rev. A* **41**, 5156 (1990); G. S. Agarwal, D. Home, and W. Schleich, *Phys. Lett. A* **170**, 359 (1992).
28. E. P. Wigner, *Am. J. Phys*. **38**, 1005 (1970).
29. J. J. Sakurai, *Modern Quantum Mechanics, Revised Edition* (Addison–Wesley, Reading, MA, 1985), p. 227.
30. W. De Baere, A. Mann, and M. Revzen, *Found. Phys*. **29**, 67 (1999).
31. M. Revzen, M. Lokajicek, and A. Mann, *Quant. Semiclas. Opt*. **9**, 501 (1997).
32. H. Stapp, *Am. J. Phys*. **72**, 30 (2004).
33. F. Laloë, *Am. J. Phys*. **69**, 655 (2001).
34. W. Schleich, *Quantum Optics in Phase Space* (Wiley-VCH, Berlin, 2001), p. 89.
35. R. L. Hudson, *Rep. Math. Phys*. **6**, 249 (1974); W. Schleich, *Quantum Optics in Phase Space* (Wiley-VCH, Berlin, 2001), p. 74.
36. M. Moshinsky and C. Quesne, *J. Math. Phys*. **12**, 1772 (1971); J. G. Krüger and A. Poffyn, *Physica A* **91**, 99 (1978).
37. A nondispersive dynamical variable is not necessarily a proper dynamical variable. The latter should be used as the requirement for hidden variable underpinning whenever distinction arises.